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# Comments on "Classification of groups with toroidal coprime graphs"

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**Abstract**

In this note, we comment on Theorem 2.11 of Selvakumar and Subajini [Classification of groups with toroidal coprime graphs, *Australas. J. Combin.* 69(2) (2017), 174–183]. Specifically, we show that the equality "if $G$ is cyclic, then $\alpha(\Gamma_G) = \max\{\, |N_{p_i}| : i = 1, \dots, k \,\}$" asserted in Theorem 2.11 does not hold in general, by providing a counterexample. Furthermore, we establish a general formula for the independence number of the coprime graph of a finite cyclic group.

The *coprime graph* $\Gamma_G$ of a finite group $G$ is the simple graph whose vertex set is $G$, where two distinct elements $x, y \in G$ are adjacent if the orders of $x$ and $y$ are coprime. Selvakumar and Subajini [2] investigated the independence number $\alpha(\Gamma_G)$ of the coprime graph $\Gamma_G$ and proved the following theorem.

**Theorem 1.** *[2, Theorem 2.11] Let $G$ be a group of order $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where the $p_i$ are distinct primes and $k \geq 1$. Then $\alpha(\Gamma_G) \geq \max\{\, |N_{p_i}| : i = 1, \dots, k \,\}$, where $N_{p_i} = \{\, x \in G : p_i$ divides $|x| \,\}$. Moreover, if $G$ is cyclic, then $\alpha(\Gamma_G) = \max\{\, |N_{p_i}| : i = 1, \dots, k \,\}$.*

Here $|x|$ denotes the order of $x \in G$. However, the equality "if $G$ is cyclic, then $\alpha(\Gamma_G) = \max\{\, |N_{p_i}| : i = 1, \dots, k \,\}$" asserted in Theorem 1 does not hold in general, as demonstrated by the following counterexample.

**Example 2.** Let $G$ be the cyclic group of order $105 = 3 \cdot 5 \cdot 7$. Then $|N_3| = 70$, $|N_5| = 84$, and $|N_7| = 90$. Now consider $I = \big\{\, g \in G : |g| \in \{15, 21, 35, 105\} \big\}$. Clearly $I$ is an independent set of $\Gamma_G$, since the orders of any two distinct elements of $I$ are not coprime. Moreover, $|I| = 92$, since a cyclic group of order divisible by $d$ contains exactly $\varphi(d)$ elements of order $d$, where $\varphi$ denotes Euler's totient function. Hence $\alpha(\Gamma_G) \geq 92$. This contradicts the equality asserted in Theorem 1, which states $\alpha(\Gamma_G) = \max\{|N_3|, |N_5|, |N_7|\} = 90$.

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The further purpose of this note is to establish a formula for the independence number of the coprime graph associated with a finite cyclic group. We first introduce the necessary definitions and notation to be used in the sequel. Throughout the paper, let $n > 1$ denote an integer and let $G$ denote a cyclic group of order $n$. We denote by $\pi(n)$ the set of prime divisors of $n$, by $\nu_p(n)$ the exponent of the prime $p$ in the prime factorization of $n$, and by $(a, b)$ the greatest common divisor of positive integers $a$ and $b$. For a set $X$, we denote by $\mathcal{P}(X)$ the power set of $X$. For sets $A$ and $B$, let $A \setminus B = \{x \in A \colon x \notin B\}$. Let $\Gamma$ be a graph. For disjoint nonempty subsets $A, B \subseteq V(\Gamma)$, we denote $A \sim B$ if every vertex of $A$ is adjacent to every vertex of $B$, and $A \nsim B$ if no vertex of $A$ is adjacent to any vertex of $B$. For terminology and notation not introduced in this note, we refer the reader to [1, 2].

For each nonempty subset $A \subseteq \pi(n)$, define

$$S_A = \{g \in G : \pi(|g|) = A\}.$$

By construction, each subset $S_A$ of $G$ is an independent set of $\Gamma_G$.

**Lemma 3.** *Let $A, B \subseteq \pi(n)$ be distinct nonempty subsets.*

(i) *If $A \cap B = \varnothing$, then $S_A \sim S_B$.*
(ii) *If $A \cap B \neq \varnothing$, then $S_A \nsim S_B$.*

***Proof.*** Observe that $S_A \cap S_B = \varnothing$, since $A$ and $B$ are distinct.

(i) Assume $A \cap B = \varnothing$. Let $x \in S_A$ and $y \in S_B$. Since $\pi(|x|) = A$ and $\pi(|y|) = B$, we obtain $\pi(|x|) \cap \pi(|y|) = A \cap B = \varnothing$. Thus $(|x|, |y|) = 1$, which implies that $x$ and $y$ are adjacent. Consequently, $S_A \sim S_B$.
(ii) Assume $A \cap B \neq \varnothing$. Let $x \in S_A$ and $y \in S_B$. Since $\pi(|x|) = A$ and $\pi(|y|) = B$, we obtain $\pi(|x|) \cap \pi(|y|) = A \cap B \neq \varnothing$. Thus $(|x|, |y|) > 1$, which implies $x$ and $y$ are non-adjacent. Consequently, $S_A \nsim S_B$.

□

The following theorem establishes a complete characterization of the maximal independent sets of $\Gamma_G$ and the independence number $\alpha(\Gamma_G)$.

**Theorem 4.** *Every maximal independent set of $\Gamma_G$ is precisely of the form*

$$\bigcup_{A \in \mathcal{F}} S_A,$$

*where $\mathcal{F} \subseteq \mathcal{P}(\pi(n))$ is a maximal family of subsets with pairwise nonempty intersections. Consequently,*

$$\alpha(\Gamma_G) = \max_{\mathcal{F} \in \mathfrak{M}(\pi(n))} \sum_{A \in \mathcal{F}} |S_A|,$$

*where $\mathfrak{M}(\pi(n))$ denotes the collection of all maximal families of subsets of $\pi(n)$ with pairwise nonempty intersections.*

***Proof.*** Let $I$ be any maximal independent set of $\Gamma_G$. For each $g \in I$, define $A_g := \pi(|g|) \subseteq \pi(n)$ and set $\mathcal{F} = \{ A_g : g \in I \} \subseteq \mathcal{P}(\pi(n))$. By construction, $I \subseteq \bigcup_{A_g \in \mathcal{F}} S_{A_g}$, since for each $g \in I$ we have $g \in S_{A_g}$. For the reverse inclusion, let $h \in \bigcup_{A_g \in \mathcal{F}} S_{A_g}$. Then $h \in S_{A_g}$ for some $g \in I$, and so $\pi(|h|) = \pi(|g|)$. Since $I$ is independent, for any $x \in I$ we obtain $\pi(|x|) \cap \pi(|h|) = \pi(|x|) \cap \pi(|g|) \neq \varnothing$, which implies $(|x|, |h|) > 1$. Therefore $h$ is non-adjacent to every vertex of $I$. Moreover, since $I$ is maximal, we have $h \in I$. Hence,

$$I = \bigcup_{A_g \in \mathcal{F}} S_{A_g}.$$

To show that the sets in $\mathcal{F}$ intersect pairwise, let $A_g, A_h \in \mathcal{F}$ be distinct. Since $g, h \in I$ and $I$ is independent, we have $(|g|, |h|) > 1$. Therefore $A_g \cap A_h = \pi(|g|) \cap \pi(|h|) \neq \varnothing$. Finally, we claim that $\mathcal{F}$ is maximal with respect to pairwise nonempty intersections. Suppose, for contradiction, that there exists $B \in \mathcal{P}(\pi(n)) \setminus \mathcal{F}$ such that the enlarged family $\mathcal{F} \cup \{B\}$ remains pairwise intersecting. Then for each $A_g \in \mathcal{F}$, we have $A_g \cap B \neq \varnothing$. Choose $h \in G \setminus I$ with $\pi(|h|) = B$. Since $B$ intersects every $A_g$, it follows that $(|h|, |g|) > 1$ for all $g \in I$. This implies $h$ is non-adjacent to every vertex of $I$, and therefore $I \cup \{h\}$ is an independent set of $\Gamma_G$. This contradicts the maximality of $I$. Hence $\mathcal{F} \subseteq \mathcal{P}(\pi(n))$ is maximal with pairwise nonempty intersections.

Conversely, suppose that $\mathcal{F} \subseteq \mathcal{P}(\pi(n))$ is a maximal family with pairwise nonempty intersections. For each $A \in \mathcal{F}$, recall that the set $S_A$ is independent in $\Gamma_G$. By Lemma 3(ii), it follows that

$$J := \bigcup_{A \in \mathcal{F}} S_A$$

is an independent set of $\Gamma_G$. To show that $J$ is maximal, let $x \in G \setminus J$ with $C = \pi(|x|)$. Since $\mathcal{F}$ is maximal with pairwise nonempty intersections, we have $C \cap D = \varnothing$ for all $D \in \mathcal{F}$. By Lemma 3(i), it follows that $S_C \sim S_D$. Thus, since $x \in S_C$, the vertex $x$ is adjacent to every vertex of $J$. Hence $J$ is maximal. □

In order to obtain an explicit formula for the independence number $\alpha(\Gamma_G)$ in the cases where $|\pi(n)| \in \{1, 2, 3\}$, we first require the following lemma.

**Lemma 5.** *Let $A$ be a nonempty subset of $\pi(n)$. Then*

$$|S_A| = \prod_{p \in A} \left(p^{\nu_p(n)} - 1\right).$$

***Proof.*** Since $G$ is cyclic of order $n$, for each divisor $d$ of $n$ with $\pi(d) = A$ there are precisely $\varphi(d)$ elements of order $d$. Consequently,

$$|S_A| = \sum_{\substack{d|n \\ \pi(d)=A}} \varphi(d).$$

Now let $A = \{p_1, \ldots, p_t\} \subseteq \pi(n)$. Observe that any divisor $d$ of $n$ with $\pi(d) = A$ can be expressed as

$$d = p_1^{a_1} \cdots p_t^{a_t}, \quad 1 \leq a_i \leq \nu_{p_i}(n) \ \text{ for each } \ i = 1, \ldots, t.$$

Since $\varphi$ is multiplicative and satisfies $\varphi(p^k) = p^k - p^{k-1}$ for all primes $p$ and integers $k \geq 1$, it follows that

$$\varphi(d) = \prod_{i=1}^{t} \big(p_i^{a_i} - p_i^{a_i-1}\big).$$

Summing over all exponent choices, we obtain

$$\sum_{\substack{d|n \\ \pi(d)=A}} \varphi(d) = \sum_{\substack{1 \leq a_i \leq \nu_{p_i}(n) \\ i=1,\dots,t}} \prod_{i=1}^{t} \big(p_i^{a_i} - p_i^{a_i-1}\big) = \prod_{i=1}^{t} \left( \sum_{a_i=1}^{\nu_{p_i}(n)} \big(p_i^{a_i} - p_i^{a_i-1}\big) \right) = \prod_{i=1}^{t} \big(p_i^{\nu_{p_i}(n)} - 1\big).$$

Hence,

$$|S_A| = \prod_{i=1}^{t} \big(p_i^{\nu_{p_i}(n)} - 1\big).$$

□

**Corollary 6.** *If $\pi(n) = \{p\}$ for some prime $p$, then $\alpha(\Gamma_G) = n - 1$.*

***Proof.*** By Theorem 4, the unique maximal independent set of $\Gamma_G$ is $S_{\{p\}}$. Consequently, $\alpha(\Gamma_G) = |S_{\{p\}}|$. By Lemma 5, we obtain $|S_{\{p\}}| = n - 1$. Therefore $\alpha(\Gamma_G) = n - 1$. □

**Corollary 7.** *If $\pi(n) = \{p, q\}$ for distinct primes $p, q$, then*

$$\alpha(\Gamma_G) = n - \min\{p^{\nu_p(n)},\, q^{\nu_q(n)}\}.$$

***Proof.*** By Theorem 4, the maximal independent sets of $\Gamma_G$ are precisely $S_{\{p\}} \cup S_{\{p,q\}}$ and $S_{\{q\}} \cup S_{\{p,q\}}$. Consequently, $\alpha(\Gamma_G) = \max\{|S_{\{p\}} \cup S_{\{p,q\}}|, |S_{\{q\}} \cup S_{\{p,q\}}|\}$. By Lemma 5, we simply obtain $|S_{\{p\}} \cup S_{\{p,q\}}| = n - q^{\nu_q(n)}$ and $|S_{\{q\}} \cup S_{\{p,q\}}| = n - p^{\nu_p(n)}$. Therefore $\alpha(\Gamma_G) = n - \min\{p^{\nu_p(n)}, q^{\nu_q(n)}\}$. □

**Corollary 8.** *If $\pi(n) = \{p, q, r\}$ for distinct primes $p, q, r$, then*

$$\alpha(\Gamma_G) = n - \min\big\{p^{\nu_p(n)}q^{\nu_q(n)},\, p^{\nu_p(n)}r^{\nu_r(n)},\, q^{\nu_q(n)}r^{\nu_r(n)},\, p^{\nu_p(n)} + q^{\nu_q(n)} + r^{\nu_r(n)} - 2\big\}.$$

***Proof.*** By Theorem 4, the maximal independent sets of $\Gamma_G$ are precisely $I_1 = S_{\{p\}} \cup S_{\{p,q\}} \cup S_{\{p,r\}} \cup S_{\{p,q,r\}}$, $I_2 = S_{\{q\}} \cup S_{\{p,q\}} \cup S_{\{q,r\}} \cup S_{\{p,q,r\}}$, $I_3 = S_{\{r\}} \cup S_{\{p,r\}} \cup S_{\{q,r\}} \cup S_{\{p,q,r\}}$, and $I_4 = S_{\{p,q\}} \cup S_{\{p,r\}} \cup S_{\{q,r\}} \cup S_{\{p,q,r\}}$. Consequently, $\alpha(\Gamma_G) = \max\{|I_1|, |I_2|, |I_3|, |I_4|\}$. By Lemma 5, we simply obtain

$$|I_1| = n - q^{\nu_q(n)}r^{\nu_r(n)}, \quad |I_2| = n - p^{\nu_p(n)}r^{\nu_r(n)}, \quad |I_3| = n - p^{\nu_p(n)}q^{\nu_q(n)},$$

and $|I_4| = n - \big(p^{\nu_p(n)} + q^{\nu_q(n)} + r^{\nu_r(n)} - 2\big)$. Therefore,

$$\alpha(\Gamma_G) = n - \min\big\{p^{\nu_p(n)}q^{\nu_q(n)},\, p^{\nu_p(n)}r^{\nu_r(n)},\, q^{\nu_q(n)}r^{\nu_r(n)},\, p^{\nu_p(n)} + q^{\nu_q(n)} + r^{\nu_r(n)} - 2\big\}.$$

□

## References

[1] D. B. West, *Introduction to Graph Theory*, 2nd ed., Prentice Hall, Upper Saddle River, NJ, 2001.

[2] K. Selvakumar and M. Subajini, Classification of groups with toroidal coprime graphs, *Australas. J. Combin.* 69(2) (2017), 174–183.

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