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Endomorphism and automorphism graphs of finite groups

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Abstract

Let G be a group. The directed endomorphism graph, $\overrightarrow{\mathrm{EG}}(G)$ of G is a directed graph with vertex set G having a directed edge from the vertex a to the vertex b if $a \neq b$ and there exists an endomorphism on G mapping a to b. The endomorphism graph, $\mathrm{EG}(G)$ is the corresponding undirected simple graph. The automorphism graph of G is similarly defined for automorphisms: it is a disjoint union of complete graphs on the orbits of $\mathrm{Aut}(G)$. The endomorphism digraph is a special case of a digraph associated with a transformation monoid, and we begin by introducing this. We have explored graph theoretic properties like size, planarity, girth etc. and tried finding out for which types of groups these graphs are complete, diconnected, trees, bipartite and so on, as well as computing these graphs for some special groups. We conclude with examples showing that things are not always simple.

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1 Introduction

The subject of graphs on groups is of undoubted importance. The oldest such graphs, Cayley graphs [6], date from the 19th Century, and are very significant in both algebraic combinatorics and geometric group theory, the latter in the context of hyperbolic groups as defined by Gromov [13]. They are also related to other areas of mathematics such as regular maps [8], and are used as classifiers for data mining [17].

More relevant to the present study are graphs which are defined just in terms of the group structure; the vertices are group elements and edges are defined by a group-theoretic property. The oldest of these, from 1955, is the commuting graph, in which the vertices are group elements, two vertices joined if they commute. This was introduced by Brauer and Fowler [3] in 1955 in their proof that there are only finitely many finite simple groups with a given involution centralizer (a result which could be regarded as the first step towards the classification of the finite simple groups). Several other graphs, including the generating graph [20] and the power graph [1, 7, 18], have been defined and studied. The power graph makes a brief appearance below; we will define it there.

The graphs studied here are defined on a group G in terms of the endomorphisms and automorphisms of G.

Definition 1.1. An **endomorphism** of a group G is a map $f : G \to G$ such that $(x * y)^f = x^f * y^f$. (We write maps on the right of their argument so that they compose left-to-right.)

The **endomorphism digraph** $\overrightarrow{\mathrm{EG}}(G)$ of a group G takes the vertex set to be G with an arc from x to y if some endomorphism of G maps x to y. The **endomorphism graph** $\mathrm{EG}(G)$ is obtained by ignoring the directions and suppressing double edges that result.

There are **compressed** versions of these, obtained by deleting the identity and shrinking each automorphism class to a single vertex. We denote these by $\overrightarrow{\mathrm{EG}}_{-}(G)$ and $\mathrm{EG}_{-}(G)$ respectively.

A point basis in a digraph is a set such that every vertex receives an incoming arc from at least one vertex in the set (or is part of the set itself). A Hamiltonian cycle in a digraph is a directed path that visits every vertex exactly once and returns to the starting vertex. A digraph which contains a Hamiltonian cycle is a Hamiltonian digraph. A vertex v is reachable from a vertex u in a digraph if there is a directed path from u to v. A digraph is diconnected if every pair of vertices u, v in its vertex set satisfies: u is reachable from v and v is reachable from u [2, 14, 16].

In this paper, we first introduce these graphs as a special case of graphs associated with transformation monoids, which enables us to conclude that the undirected graphs are perfect. Then we explore properties such as planarity and girth, and compute the graphs for certain special groups.

Our notation is fairly standard. The endomorphisms of G form a transformation monoid on G called the **endomorphism monoid** $\mathrm{End}(G)$ of G; its units are the automorphisms of G and form the **automorphism group** $\mathrm{Aut}(G)$ of G. The identity element of G is denoted by e_G (or just e if the group is clear). Transformation monoids form a natural context for our work, so we look at these first before concentrating on endomorphism graphs. Then we examine properties of endomorphism graphs such as planarity and girth and calculate their structure for specific groups such as dihedral, dicyclic and symmetric groups and abelian p-groups.

2 Transformation monoids, preorders, and digraphs

The general context for our construction is that of a **transformation monoid** on a set X; this is a collection of maps from X to X closed under composition and containing the identity map. Thus, it is a monoid (a semigroup with identity). If X is a finite set, then the units of a transformation monoid are the permutations it contains; these form a permutation group, which defines an orbit partition on X.

Let M be a transformation monoid on X. Define a relation $\to$ by the rule that $x \to y$ if there is an element of M which maps x to y. (Here we allow $x = y$). This relation is reflexive and transitive; thus it is a **partial preorder**. It is also called a **preferential arrangement**, since it describes a situation where someone has preferences among a set of choices but may be indifferent to some pairs in the sense that neither is preferred to the other.

Let $\to$ be a partial preorder on X. Define a relation $\equiv$ on X by $x \equiv y$ if both $x \to y$ and $y \to x$ hold. Then $\equiv$ is an equivalence relation on X, and so defines a partition of X; the equivalence classes are called **indifference classes**, and the partial preorder induces a partial order on the set of equivalence classes.

A partial preorder gives rise to a directed graph $\vec{\Gamma}$ on X, whose arcs are the pairs (x, y) with $x \to y$ and $x \neq y$. We also obtain an undirected graph by ignoring the directions (and placing a single edge $\{x, y\}$ if $x \to y$ and $y \to x$). The graph of a partial preorder is a **perfect graph** (that is, all its induced subgraphs have clique number equal to chromatic number). This is because we can turn a partial preorder into a partial order by simply imposing a total order on each indifference class; then Γ is the comparability graph of this partial order, and it is perfect by Mirsky's theorem [21].

In the case of a transformation monoid M, the orbit partition of the group of units refines the partition into indifference classes for the partial preorder, though they are not equal in general.

If A and B are indifference classes for M, and $a \to b$ for some $a \in A$ and $b \in B$, then $a' \to b'$ holds for all $a' \in A$ and all $b' \in B$. For there exist elements of M carrying a' to a, a to b, and b to b'; their composition maps a' to b'. The same holds if we replace indifference classes by unit group orbits. This suggests that we study the simpler digraph and graph obtained by contracting the indifference classes or the unit group orbits to single orbits.

3 The endomorphism digraph of a group

The set $\operatorname{End} G$ of endomorphisms of a group G forms a transformation monoid on the set of elements of G; the directed endomorphism graph $\overrightarrow{\mathrm{EG}}(G)$ is just the digraph attached to the transformation monoid in the preceding section. We also denote the digraph whose edges come from automorphisms of G by $\overrightarrow{\mathrm{AG}}(G)$.

In a group, every element can be mapped to the identity e by an endomorphism, but e cannot be mapped to any non-identity element. So there are arrows $a \to e$ but no arrows $e \to a$ for every $a \neq e$. Accordingly, we lose no information by deleting the identity element of the group. Let $[a]_e$ denote the indifference class of the element a when the transformation monoid under consideration is the endomorphism monoid of a group. If $a \to b$ in the endomorphism digraph, then $a' \to b'$ for every $a' \in [a]_e$ and $b' \in [b]_e$. This justifies our compression procedure. We will call the indifference classes of the preorder the **endomorphism classes**: they form a partition of G such that the **automorphism classes** (the orbits of the automorphism group) form a refinement (in the sense that each automorphism class is contained in a single endomorphism class). Let $[a]$ denote the automorphism class of a.

Note that any isomorphism from G to H induces isomorphisms from the various endomorphism digraphs on G to those on H. An isomorphism between compressed endomorphism digraphs $\overrightarrow{\mathrm{EG}}_{-}(G)$ and $\overrightarrow{\mathrm{EG}}_{-}(H)$ is said to be **strong** if it maps each automorphism class in G to an automorphism class of the same size in H. Note that strong isomorphisms are precisely those induced by isomorphisms of the uncompressed digraphs. This suggests three questions:

Question 3.1. (a) Can there be an isomorphism from $\overrightarrow{\mathrm{EG}}_{-}(G)$ to $\overrightarrow{\mathrm{EG}}_{-}(H)$ which is not strong? Can this happen if $|G| = |H|$?

(b) Can non-isomorphic groups have isomorphic directed endomorphism graphs?

(c) Is there a group for which endomorphism and automorphism classes do not coincide?

The first part of the first question has an easy negative answer: take cyclic groups of different prime orders; the compressed isomorphism digraphs each have just a single vertex. We will discuss the stronger form, and the answer to the other two questions, in Section 9.

4 Direct products

The **strong product** of two digraphs (or graphs) Γ and Δ, $\Gamma \boxtimes \Delta$ on vertex sets X and Y is the graph whose vertex set is the Cartesian product $X \times Y$, with an arc (or edge) from (x, y) to (x', y') if one of the following holds:

(a) $x \to x'$ (or $x \sim x'$) in Γ, $y = y'$;

(b) $x = x'$, $y \to y'$ (or $y \sim y'$) in Δ;

(c) $x \to x'$ (or $x \sim x'$) in Γ, $y \to y'$ (or $y \sim y'$) in Δ.

Observation 4.1. Note that, if we take the strong product of digraphs Γ and Δ and then symmetrise it by ignoring directions, we do not obtain the strong product of the symmetrisations of Γ and Δ. (If Γ and Δ are the single arcs $x \to x'$ and $y \to y'$, the strong product has no arc between (x, y') and (x', y), but the strong product of the symmetrisations is the complete graph on four vertices.)

Proposition 4.2. *Let G and H be groups with coprime orders. Then the endomorphism monoid of $G \times H$ is $\mathrm{End}(G) \times \mathrm{End}(H)$, and the endomorphism digraph of $G \times H$ is the strong product of $\overrightarrow{\mathrm{EG}}(G)$ and $\overrightarrow{\mathrm{EG}}(H)$.*

For the proof, note that, if f is an endomorphism of a group G, then the order of g^f divides the order of g. So any endomorphism of $G \times H$ must map G and H to themselves, and so induce endomorphisms of G and H. It is easy to see that this map is a bijection, so the result about endomorphisms is proved. From this it follows that the endomorphism digraph of $G \times H$ is the strong product of the endomorphism digraphs of G and H.

Observation 4.3. The procedure of collapsing automorphism classes commutes with this isomorphism, but it does not follow that $\overrightarrow{\mathrm{EG}}_{-}(G \times H)$ is isomorphic to $\overrightarrow{\mathrm{EG}}_{-}(G) \boxtimes \overrightarrow{\mathrm{EG}}_{-}(H)$: for automorphism classes $([e_G], [h])$ and $([g], [e_H])$ occur as vertices in $\overrightarrow{\mathrm{EG}}_{-}(G \times H)$ but not in the strong product. However, it is true that if $\overrightarrow{\mathrm{EG}}(G_1) \simeq \overrightarrow{\mathrm{EG}}(G_2)$ and $\overrightarrow{\mathrm{EG}}(H_1) \simeq \overrightarrow{\mathrm{EG}}(H_2)$, then $\overrightarrow{\mathrm{EG}}(G_1 \times H_1) \simeq \overrightarrow{\mathrm{EG}}(G_2 \times H_2)$.

5 Cyclic groups

We illustrate these concepts by considering the cyclic group $\mathbb{Z}_n$ of order n. We denote the elements of this group by $\{0, 1, \ldots, n-1\}$, the group operation being addition modulo n.

An endomorphism f is uniquely determined by the image of 1; if $1^f = a$ then $x^f = xa \pmod{n}$ for all x. We denote this endomorphism by f_a. If $\gcd(a, n) = d$, then the image of f_a consists of all multiples of d. Moreover, since $f_a f_b = f_{ab}$ (subscript $(\mathrm{mod}\ n)$), we see that f_a is a unit if and only if $\gcd(a, n) = 1$; and more generally, f_a generates the same endomorphisms as f_d, where $\gcd(a, n) = d$. Note that there is an endomorphism mapping x to y if and only if $\gcd(x, n)$ divides $\gcd(y, n)$. Hence the following are equivalent:

- x and y lie in the same endomorphism class;
- $\gcd(x, n) = \gcd(y, n)$;
- there is an automorphism mapping x to y.

So the endomorphism and automorphism classes coincide. In particular, we see that any automorphism class contains a unique divisor of n; in what follows, we use this element to label the class. So the compressed endomorphism digraph has vertices indexed by divisors of n (except 1), with an edge $[x] \to [y]$ whenever x divides y. In particular, $\mathrm{EG}_-(\mathbb{Z}_n)$ is complete if and only if the divisors of n form a chain, that is, if and only if n is a prime power.

Moreover, the number of elements in a class $[a]$ is $\phi(|a|)$, which is $\phi(n/a)$ if we choose a to be a divisor of n. (Here ϕ is Euler's function.)

Observation 5.1. The **directed power graph** $\vec{P}(G)$ of a finite group G has vertex set G, with an arc $x \to y$ whenever y is a power of x. From the description above, we see that $\overrightarrow{\mathrm{EG}}(\mathbb{Z}_n)$ is the same as $\vec{P}(\mathbb{Z}_n)$. It is known [4, 5] that groups with isomorphic power graphs have isomorphic directed power graphs (where the power graph $P(G)$ is obtained from $\vec{P}(G)$ by ignoring directions and collapsing multiple edges); but this does not hold in general for the undirected and directed endomorphism graphs, as we will see.

Theorem 5.2. *Let G be a cyclic group of order n and $1 < d_1 \leq d_2 \leq \cdots \leq d_k < n$ be the divisors of n. Then the total number of edges in* $\mathrm{EG}(G)$ *is*

$$\binom{n}{2} - \sum_{\substack{1 \leq i < j \leq k \\ d_i \nmid d_j}} \phi(d_i)\phi(d_j). \tag{1}$$

Proof. Let a and b be divisors of n. Then vertices in $[a]$ and $[b]$ are joined if and only if one of a and b divides the other. So a pair $\{x, y\}$ is not an edge if and only if neither of the orders of x and y divides the other; there are $\phi(d_i)\phi(d_j)$ such pairs, if d_i and d_j are the orders of x and y. □

Theorem 5.3. *The number of maximal cliques in* $\mathrm{EG}(\mathbb{Z}_n)$*, where $n = {p_1}^{n_1}{p_2}^{n_2}\cdots{p_k}^{n_k}$ in which $p_1, p_2, \ldots, p_k$ are distinct primes and $n_i \in \mathbb{N}$, for every $i \in \{1, 2, \ldots, k\}$ is,*

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1! n_2! \cdots n_k!}.$$

Proof. As noted, if we label isomorphism classes by divisors of n, then two classes are joined if and only if the label of one divides the label of the other. So the maximal cliques come from maximal chains in the lattice of divisors of n. Any such maximal chain has the property that each element is obtained by multiplying its predecessor by a prime. So the length of the chain is $n_1 + \cdots + n_k$, and we can form all possible chains by choosing subsets of the positions in the sequence with cardinalities $n_1, n_2, \ldots, n_k$ and inserting p_i in all positions in the ith set; then the index of the ith automorphism class in the chain is $p_1 p_2 \cdots p_i$.

So the number of maximal cliques is the multinomial coefficient

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1! n_2! \cdots n_k!}.$$

□

Theorem 5.4. *Let $n = p_1^{n_1} \cdots p_k^{n_k}$, where $p_1, \dots, p_k$ are distinct primes and $n_1, \dots, n_k$ positive integers. Write the primes $p_1, \dots, p_k$ (with multiplicities $n_1, \dots, n_k$) in a sequence of length $n_1 + \cdots + n_k$ where the primes are in nonincreasing order, say $(q_1, q_2, \dots, q_m)$, where $m = n_1 + \cdots + n_k$, and let r_i be the product of the first i terms in this sequence (with $r_0 = 1$). Then the clique number and chromatic number of $\mathrm{EG}(\mathbb{Z}_n)$ are equal to $\sum_{i=0}^{m} \phi(r_i)$.*

Proof. The proof of the preceding theorem shows that the size of a maximal clique is $\frac{m!}{n_1! n_2! \cdots n_k!}$, for some ordering of the m primes (with their multiplicities) dividing n. Now, if p and q are primes with $p < q$, then $\phi(ph) < \phi(qh)$ for any positive integer h; so we can repeatedly swap adjacent primes in the sequence to put them in nonincreasing order, and the size of the clique will increase at each step.

The chromatic number is equal to the clique number since, as noted earlier, the graph $\mathrm{EG}(G)$ is perfect for any group G. $\square$

Example 5.5. Consider $G = \mathbb{Z}_{12}$ (with addition modulo 12).

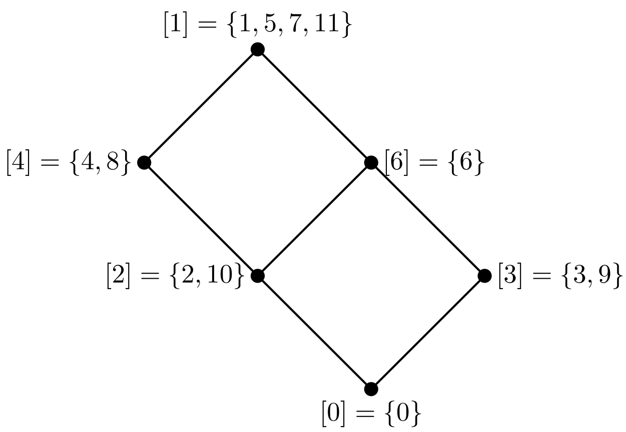

Figure 1: Lattice diagram of $\mathbb{Z}_{12}$

The maximal chains $\{0, 2, 10, 4, 8, 1, 5, 7, 11\}, \{0, 2, 10, 6, 1, 5, 7, 11\}, \{0, 3, 9, 6, 1, 5, 7, 11\}$ are maximal cliques in $\mathrm{EG}(\mathbb{Z}_{12})$, corresponding to the sequences $(3, 2, 2)$, $(2, 3, 2)$ and $(2, 2, 3)$ of primes in the proof above. The largest clique has size $\phi(1) + \phi(3) + \phi(6) + \phi(12) = 1 + 2 + 2 + 4 = 9$.

Remark 5.6. The clique number of the power graph of a cyclic group was determined by Alireza et al. [1, Theorem 7] and further discussed by Kumar et al. in their survey [19, Section 8.2].

We end this section with a general result.

Theorem 5.7. *The directed endomorphism graphs (and hence the endomorphism graphs) of isomorphic groups are isomorphic. The converse is not true for endomorphism graphs.*

Proof. Consider two isomorphic groups, G_1 and G_2. Let $\phi : G_1 \to G_2$ be an isomorphism with $\phi(a_1) = a_2$ and $\phi(b_1) = b_2$ for some $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$. Consider the bijection $\psi : V(\overrightarrow{\mathrm{EG}}(G_1)) \to V(\overrightarrow{\mathrm{EG}}(G_2))$ defined as $\psi(a) = \phi(a)$, for every $a \in V(\overrightarrow{\mathrm{EG}}(G_1))$. There is a directed arc from a_1 to b_1 in $\overrightarrow{\mathrm{EG}}(G_1)$ if and only if there is an endomorphism f defined on G_1, such that $f(a_1) = b_1$. Consider the endomorphism $\phi f \phi^{-1}$ defined on G_2.

$$\phi f \phi^{-1}(a_2) = \phi f(a_1) = \phi(b_1) = b_2.$$

Hence there is a directed arc from a_2 to b_2 in $\overrightarrow{\mathrm{EG}}(G_2)$. The reverse implication can be proved by considering an endomorphism 'g' on G_2 such that $g(a_2) = b_2$.

Now, consider the non-isomorphic groups of order 4: the cyclic group $\mathbb{Z}_4$ under addition $+_4$ and the Klein four group. The endomorphism graph of both these groups is the complete graph K_4. Therefore, the converse is not true for endomorphism graphs. □

6 Abelian Groups

Theorem 6.1. *If G is an abelian group, then $\overrightarrow{\mathrm{EG}}(G)$ has a single point basis.*

Proof. Let $G \simeq \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$, $n_i \in \mathbb{N}$ for $1 \leq i \leq k$. Consider the element $(1, 1, \ldots, 1)$ and the endomorphism f on G

$$f(x_1, x_2, \ldots, x_k) = (a_1 x_1, a_2 x_2, \ldots, a_k x_k)$$

where $a_i \in \mathbb{Z}$ for $1 \leq i \leq k$. We can see that $(1, 1, \ldots, 1)$ can be mapped to any other element using these maps. □

Remark 6.2. The converse of the above theorem is not true. Consider the quaternion group, $Q_8 = \langle i, j \mid i^4 = 1, j^2 = i^2, i^{-1} j i = j^{-1} \rangle$, the element i can be mapped to all other elements in Q_8 via an endomorphism. Therefore, the converse is not true.

Theorem 6.3. *The endomorphism graph of an abelian group G is complete if and only if $G = (\mathbb{Z}_{p^a})^m \times (\mathbb{Z}_{p^{a+1}})^n$ for some $m, n \geq 0, a \geq 1$.*

Proof. Suppose that $G = (\mathbb{Z}_{p^a})^m \times (\mathbb{Z}_{p^{a+1}})^n$ for some $m, n \geq 0, a \geq 1$. Let

$$g_1 = (a_1 p^{b_1}, \ldots, a_m p^{b_m}, c_1 p^{d_1}, \ldots, c_n p^{d_n})$$

and

$$g_2 = (a_{m+1} p^{b_{m+1}}, \ldots, a_{2m} p^{b_{2m}}, c_{n+1} p^{d_{n+1}}, \ldots, c_{2n} p^{d_{2n}}) \in G\ ,$$

where $a_i, c_j \not\equiv 0 \pmod{p}, 0 \le b_i \le a, 0 \le d_j \le a+1$ for all i, j.

We will show that there exists an endomorphism Φ on G such that $\Phi(g_1) = g_2$ or $\Phi(g_2) = g_1$. Choose

$$b = \min\{b_i\},\ 1 \le i \le 2m$$
$$d = \min\{d_j - 1\},\ 1 \le j \le 2n.$$

<u>Case 1 : $d < b$</u>
Let the minimum be achieved at d_k. Then

$$d_j - d_k \ge 0 \text{ for all } j \text{ and } b_i - d_k \ge 0 \text{ for all } i.$$

Define $\phi_j : \mathbb{Z}_{p^{a+1}} \to \mathbb{Z}_{p^{a+1}}$ and $\psi_i : \mathbb{Z}_{p^{a+1}} \to \mathbb{Z}_{p^a}$ as

$$\phi_j(x) = {c_k}^{-1} c_j p^{d_j - d_k} x \ \text{ for all } j \text{ and for all } x \in \mathbb{Z}_{p^{a+1}}$$
$$\psi_i(x) = {c_k}^{-1} a_i p^{b_i - d_k} x \ \text{ for all } i \text{ and for all } x \in \mathbb{Z}_{p^{a+1}}.$$

Note that $\phi_j(c_k p^{d_k}) = c_j p^{d_j}$ and $\psi_i(c_k p^{d_k}) = a_i p^{b_i}$ for all i, j.

Without loss of generality assume that $k > n$. Let $\pi_1 : G \to \mathbb{Z}_{p^{a+1}}$ be the projection map such that $\pi_1(x_1, \ldots, x_m, y_1, \ldots, y_n) = y_{k-n}$. Then $\Phi = (\psi_1 \circ \pi_1, \ldots, \psi_m \circ \pi_1, \phi_1 \circ \pi_1, \ldots, \phi_n \circ \pi_1) : G \to G$ maps g_2 to g_1.

<u>Case 2 : $b \le d$</u>
Let the minimum be achieved at b_k. Then

$$b_i - b_k \ge 0 \text{ for all } i \text{ and } d_j - b_k \ge 1 \text{ for all } j.$$

Define $\phi_i : \mathbb{Z}_{p^a} \to \mathbb{Z}_{p^a}$ and $\psi_j : \mathbb{Z}_{p^a} \to \mathbb{Z}_{p^{a+1}}$ as

$$\phi_i(x) = {a_k}^{-1} a_i p^{b_i - b_k} x \text{ for all } i \ \text{ and for all } x \in \mathbb{Z}_{p^a}$$
$$\psi_j(x) = {a_k}^{-1} c_j p^{d_j - b_k} x \text{ for all } j \ \text{ and for all } x \in \mathbb{Z}_{p^a}.$$

Note that $\phi_i(a_k p^{b_k}) = a_i p^{b_i}$ and $\psi_j(a_k p^{b_k}) = c_j^{d_j}$ for all i, j. Without loss of generality assume that $k > m$. Let $\pi_2 : G \to \mathbb{Z}_{p^a}$ be the projection map such that $\pi_2(x_1, \ldots, x_m, y_1, \ldots, y_n) = x_{k-m}$. Then $\Phi = (\phi_1 \circ \pi_2, \ldots, \phi_m \circ \pi_2, \psi_1 \circ \pi_2, \ldots, \psi_n \circ \pi_2) : G \to G$ maps g_2 to g_1.

For the converse part, if $|G|$ contain at least two prime divisors, say p and q, then there exist elements of order p, q in G. But $\mathrm{EG}(G)$ cannot be complete since no element of order p can be mapped to an element of order q. Also, if $G \simeq \mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b} \times G_1$ where $a > b+1, b \ge 1$ and G_1 is any abelian p-group, we will prove that $\mathrm{EG}(G)$ is never a complete graph.

Take $g_1 = (p, 0, 0)$ and $g_2 = (0, 1, 0)$. Then $|g_1| = p^{a-1}$ and $|g_2| = p^b$. Since $a - 1 > b, g_2$ cannot be mapped to g_1.

Suppose there exists a map Φ that sends g_1 to g_2. Let $\Pi : G \to \mathbb{Z}_{p^b}$ be the projection map. Define $\Psi = \Pi \circ \Phi : G \to G$ as

$$\Psi(x, y, z) = cx + dy + \eta(z)$$

where c and d are some constants chosen depending on where 1 goes and $\eta : G_1 \to \mathbb{Z}_{p^b}$.

$$\Psi(p,0,0) = c \cdot p + d \cdot 0 + \eta(0) \equiv 1 \pmod{p^b}.$$

Therefore, p^b divides $cp-1$, which is a contradiction. So there does not exist any endomorphism on G mapping g_1 to g_2 or g_2 to g_1. □

Theorem 6.4. *Let G be a finite abelian group. For $a, b \in G$, if there is an endomorphism mapping a to b, then $|b|$ divides $|a|$. The converse holds if and only if $G \simeq \prod_{i=1}^{k} (\mathbb{Z}_{{p_i}^{n_i}})^{m_i}$ for distinct primes p_i and $n_i, m_i \in \mathbb{N}$.*

Proof. The order of the image of an endomorphism always divides the order of the preimage [11]. Suppose that $G \simeq \prod_{i=1}^{k} (\mathbb{Z}_{{p_i}^{n_i}})^{m_i}$. Let $a = (a_1, a_2, \ldots, a_k)$ and $b = (b_1, b_2, \ldots, b_k) \in G$, where $a_i, b_i \in (\mathbb{Z}_{{p_i}^{n_i}})^{m_i}$ for $1 \le i \le k$ and $|b|$ divides $|a|$. Assume $|a_i| = {p_i}^{c_i}$ and $|b_i| = {p_i}^{d_i}$ for each i between 1 and k. Then, we have

$$|a| = {p_1}^{c_1}{p_2}^{c_2} \cdots {p_k}^{c_k}, \qquad |b| = {p_1}^{d_1}{p_2}^{d_2} \cdots {p_k}^{d_k},$$

where $d_i \le c_i \le n_i$ for $1 \le i \le k$. Consider the proof of Theorem 6.3. We can see that for $a_i, b_i \in (\mathbb{Z}_{{p_i}^{n_i}})^{m_i}$, whenever $|b_i|$ divides $|a_i|$, there exists an endomorphism mapping a_i to b_i. Therefore, if $|b|$ divides $|a|$, there exists an endomorphism mapping each a_i to b_i for $1 \le i \le k$ (since ${p_i}^{d_i}$ divides ${p_i}^{c_i}$ for each i) and hence there is an endomorphism mapping a to b.

For the converse part assume that $G = \mathbb{Z}_{p^{u+v}} \times \mathbb{Z}_{p^u} \times G_1$ where p is any prime, $u, v > 0$ and G_1 is any abelian group. Consider the elements $a = (p^v, 0, 0)$ and $b = (0, 1, 0)$. Here $|a| = |b|$, but there is no endomorphism mapping a to b. (For further reading, refer to the section 'Degeneracy and ideals' in [10].) □

Theorem 6.5. *Let G be a group and suppose that there exists an $a \in G$ such that $|G : C(a)| > 3$, where $C(a)$ denotes the centralizer of a. Then* EG(G) *is non-planar.*

Proof. By assumption there are elements e, x, y, z which lie in distinct cosets of $C(a)$. This means that the elements $a, x^{-1}ax, y^{-1}ay, z^{-1}az$ are all distinct. Moreover, these elements carry a complete graph in EG(G). For example, conjugation by $x^{-1}y$ is an automorphism of G which carries $x^{-1}ax$ to $y^{-1}ay$. Moreover, e is joined to all four of these elements. So EG(G) has an induced subgraph K_5, and is not planar. □

Remark 6.6. The converse of the theorem does not hold. A counterexample is the cyclic group $\mathbb{Z}_p$, where $p > 3$ is a prime. The endomorphism graph EG($\mathbb{Z}_p$) is the complete graph K_p, which is non-planar for $p \ge 5$. However, since $\mathbb{Z}_p$ is abelian, the centralizer of every element is the group itself, and hence $|G : C(a)| = 1 \le 3$ for all $a \in \mathbb{Z}_p$.

Theorem 6.7. *Let G be an abelian group. Then* EG(G) *is planar if and only if $|G| \le 4$.*

Proof. The endomorphism graphs of of $\mathbb{Z}_2, (\mathbb{Z}_2)^2, \mathbb{Z}_3$, and $\mathbb{Z}_4$ are K_2, K_4, K_3 and K_4, respectively and hence they are planar.
If there exists an $a \in G$, with $|a| > 4$, then $\mathrm{EG}(G)$ is non-planar. Since $e - a^{-1} - a - a^2 - (a^2)^{-1} - e$ is a directed cycle in $\overrightarrow{\mathrm{EG}}(G)$ and composition of a homomorphisms is again a homomorphisms, there is an induced K_5 in $\mathrm{EG}(G)$. Therefore, $\mathrm{EG}(G)$ is non-planar.
Suppose $\mathrm{EG}(G)$ is planar. Then the order of each element of G will be at most 4. So G will be isomorphic to $\mathbb{Z}_2^a \times \mathbb{Z}_3^b \times \mathbb{Z}_4^c$ for some $a, b, c \in \mathbb{N}$.

Case 1 : $b > 0$
If $a \neq 0$ or $c \neq 0$, then we get a $g \in G$ with $|g| \geq 4$. So $G \simeq \mathbb{Z}_3^b$. From Theorem 6.3 $\mathrm{EG}(G)$ will be K_{3^b}. However, since $\mathrm{EG}(G)$ is planar, $b = 1$.

Case 2 : $b = 0$
Here $G \simeq \mathbb{Z}_2^a \times \mathbb{Z}_4^c$. Again from Theorem 6.3 $\mathrm{EG}(G)$ will be $K_{2^a 4^c}$. Therefore, $\mathrm{EG}(G)$ is planar in this case if and only if $a = 0,\ c = 1$ or $a = 1$ or $2,\ c = 0$. □

Proposition 6.8. *Let G be a non-trivial group. If $G \neq \mathbb{Z}_2$, then the girth $g(\mathrm{EG}(G))$ $= 3$. Moreover, $\mathrm{EG}(G)$ is bipartite if and only if $G = \mathbb{Z}_2$.*

Proof. A finite group G of order greater than 2 has a non-trivial automorphism f:

- if G is non-abelian, take f to be conjugation by a non-central element;
- if G is abelian with exponent greater than 2, take f to be the inverse map;
- if G is abelian of exponent 2, it is a vector space over the 2-element field: let f swap the first two vectors in a basis and fix the remaining vectors.

Therefore there are two distinct elements a and b different from e such that $f(a) = b$, so that $e - a - b - e$ induces a cycle of length 3 in $\mathrm{EG}(G)$.

Since there always exists the trivial homomorphism that maps each element of the group to identity, the identity element alone forms a partite set. The existence of an edge between any two non-identity elements would imply that the graph is not bipartite. □

Corollary 6.9. *The endomorphism graph of a group G is a tree if and only if $G = \mathbb{Z}_2$.*

6.1 Abelian p-groups

For a prime number p, a p-group is a group in which order of every element is a power of p. Consider the abelian p-group $G = (\mathbb{Z}_{p^{m_1}})^{l_1} \times (\mathbb{Z}_{p^{m_2}})^{l_2} \times \cdots \times (\mathbb{Z}_{p^{m_k}})^{l_k}$ where $n_i, m_i \in \mathbb{N}, m_1 < m_2 < \cdots < m_k$. Let $|G| = n$.

6.1.1 $(\mathbb{Z}_{p^m})^l$ under addition $(+_{p^m}, +_{p^m}, \ldots, +_{p^m})$

From Theorem 6.3, we have $\mathrm{EG}((\mathbb{Z}_{p^m})^k)$ is complete. However, let us determine the automorphism classes of this group. [12] presents a necessary and sufficient condition for an $n \times n$ matrix over integers to be associated with an automorphism.

Claim 1: $(p^i, 0, \dots, 0)$ can be assigned to any element of order p^{m-i}.
Let $a = (a_1, a_2, \dots, a_l) \in (\mathbb{Z}_{p^m})^l$ with $|a| = p^{m-i}$. We can represent a as $(c_1 p^{j_1}, c_2 p^{j_2}, \dots, c_l p^{j_l})$ where $(c_r, p) = 1, i \leq j_r$ for $1 \leq r \leq l$ and $j_r = i$ for at least one r. Without loss of generality, assume that $j_1 = i$. Since $i \leq j_r$ for all r, there exists an h_r such that $p^{j_r} = p^{i+h_r}$ for $2 \leq r \leq l$. Therefore we can rewrite a as

$$a = (c_1 p^i, c_2 p^{i+h_2}, \dots, c_l p^{i+h_l}.) \tag{2}$$

Consider the matrix

$$A = \begin{bmatrix} c_1 & 0 & 0 & \cdots & 0 \\ c_2 p^{h_2} & 1 & 0 & \cdots & 0 \\ c_3 p^{h_3} & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ c_l p^{h_l} & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

If we denote a in equation (2) as a column vector, then

$$A[p^i \; 0 \; 0 \cdots 0]^T = [c_1 p^i, c_2 p^{i+h_2}, \dots, c_l p^{i+h_l}]^T.$$

Note that even if $j_m = i$, for some $m \neq 1$, we can transform it to $j_1 = i$ using a permutation matrix.

Claim 2: Any element of order p^{m-i_1} can be mapped to an element of order p^{m-i_2} whenever $i_1 \leq i_2$.
Since $i_1 \leq i_2$, there exists an i_3 such that $p^{i_2} = p^{i_1+i_3}$. Consider the matrix

$$A_{12} = \begin{bmatrix} p^{i_3} & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

A_{12} is the matrix associated with the homomorphism that maps $(p^{i_1}, 0, \dots, 0)$ to $(p^{i_2}, 0, \dots, 0)$. Let $[p^i] = \{a \in \mathbb{Z}^{\mathbf{l}}_{\mathbf{p^m}} \mid |a| = p^{m-i}\}$. Then the automorphism classes of $\mathbb{Z}^{\mathbf{l}}_{\mathbf{p^m}}$ are

$$\Big\{[1],\ [p],\ [p^2], \dots, [p^{m-1}]\Big\}.$$

6.1.2 $\mathbb{Z}^{l_1}_{p^{m_1}} \times \mathbb{Z}^{l_2}_{p^{m_2}} \times \cdots \times \mathbb{Z}^{l_k}_{p^{m_k}}$

Consider the groups $G = \mathbb{Z}^{l_1}_{p^{m_1}} \times \mathbb{Z}^{l_2}_{p^{m_2}} \times \cdots \times \mathbb{Z}^{l_k}_{p^{m_k}}$ and $G' = \mathbb{Z}_{p^{m_1}} \times \mathbb{Z}_{p^{m_2}} \times \cdots \times \mathbb{Z}_{p^{m_k}}$ where $n_i, m_i \in \mathbb{N}, m_1 < m_2 < \cdots < m_k$. Let $(a_1, a_2, \dots, a_k) \in G'$. The diagonal function maps $(a_1, a_2, \dots, a_k)$ to $(\underbrace{a_1, a_1, \dots, a_1}_{l_1}, \underbrace{a_2, a_2, \dots, a_2}_{l_2}, \dots, \underbrace{a_k, a_k, \dots, a_k}_{l_k})$ in G.

We have already shown that $\mathbb{Z}^l_{p^m}$ is complete. So, for examining $\overrightarrow{\mathrm{EG}}(G)$, it is enough to study $\overrightarrow{\mathrm{EG}}(G')$.

Let $b = (p^{b_1}, p^{b_2}, \ldots, p^{b_k})$ and $c = (p^{c_1}, p^{c_2}, \ldots, p^{c_k}) \in G'$. Assume $|c|$ divides $|b|$. Claim: There exists a homomorphism mapping b to c if and only if for all i, there exists j such that $b_j + \max\{0, m_i - m_j\} \leq c_i$.
Let E be the matrix associated with the homomorphism that maps b to c and $(e_1, e_2, \ldots, e_k)$ be the i^{th} row of E. Now from [12]

$$e_j \equiv 0 \pmod{p^{m_i - m_j}}, \text{ if } j \leq i,$$
$$e_j \in \mathbb{Z}_{p^{m_j}}, \text{ if } j > i.$$

$$e_1 p^{b_1} + e_2 p^{b_2} + \cdots + e_k p^{b_k} = p^{c_i}.$$

Define f_j for $1 \leq j \leq i$ with $(f_j, p) = 1$. Then

$$f_1 p^{m_i - m_1} p^{b_1} + f_2 p^{m_i - m_2} p^{b_2} + \cdots + f_i p^{b_i} + e_{i+1} p^{b_i + 1} + \cdots + e_k p^{b_k} = p^{c_i},$$

or we can say

$$p^{c_i} = \sum_{j=1}^{n} d_j p^{\max\{0, m_i - m_j\} + b_j};$$
$$d_j = \begin{cases} f_j, & 1 \leq j \leq i; \\ e_j, & j > i. \end{cases}$$

Hence there exists a homomorphism that maps b to c if and only if there exists a j such that $\max\{0, m_i - m_j\} + b_j \leq c_i$ for all i.

7 Identity element deleted subgraphs

In endomorphism graphs the identity element e serves as a universal vertex and in the case of directed endomorphism graphs there is a directed arc to e from all other vertices. A similar nature (either serves as a universal vertex or serves as an isolated vertex) is observed for the vertex corresponding to identity element in various types of graphs defined from groups. So, at various situations, the presence of vertex corresponding to identity element is not interesting and hence it is a common practice to consider the graph induced by group elements other than the identity element. In this section, $\overrightarrow{\mathrm{EG}^*}(G)$ and $\mathrm{EG}^*(G)$, denotes the induced subgraph obtained from $\overrightarrow{\mathrm{EG}}(G)$ and $\mathrm{EG}(G)$, respectively, by deleting the vertex corresponding to the identity element. $\overrightarrow{\mathrm{EG}}(G)$ is not diconnected, since there is no endomorphism mapping the identity element to any other element.

Proposition 7.1. *For $\overrightarrow{\mathrm{EG}^*}(G)$ the following are equivalent:*

(i) $\overrightarrow{\mathrm{EG}^*}(G)$ *is diconnected;*

(ii) $\overrightarrow{\mathrm{EG}^*}(G)$ *is a complete digraph;*

(iii) $\overrightarrow{\mathrm{EG}^*}(G)$ *is a Hamiltonian digraph.*

Proof. The equivalence follows immediately from the fact that composition of two homomorphisms is again a homomorphism. □

Theorem 7.2. *Let G be an abelian group. Then $\overrightarrow{\mathrm{EG}^*}(G)$ is diconnected if and only if $G \simeq (\mathbb{Z}_p)^k$ under addition $\underbrace{(+_p, +_p, \dots, +_p)}_{k}$, for some prime p and $k \in \mathbb{N}$.*

Proof. Suppose that $\overrightarrow{\mathrm{EG}^*}(G)$ is diconnected. Then order of any vertex divides the order of any other. So the orders of all vertices will be same, say m. Let m be composite and p be a prime dividing m, Then there exists an element, say x with order m. But then x^p has order m/p, which is a contradiction. Therefore m must be a prime number. Since G is an abelian group, $G \simeq (\mathbb{Z}_p)^k$, for some prime p and $k \in \mathbb{N}$.

The converse holds by the proof of Theorem 6.4. □

Theorem 7.3. *Given a group G, $\mathrm{EG}^*(G)$ is a tree if and only if $G = \mathbb{Z}_2$ or $\mathbb{Z}_3$ under $+_2$ and $+_3$, respectively.*

Proof. If $G = \mathbb{Z}_2$ or $\mathbb{Z}_3$, then $\mathrm{EG}^*(G)$ is obviously a tree. Now assume that $\mathrm{EG}^*(G)$ is a tree. By the argument at the start of Proposition 6.8, there is a non-trivial automorphism f of G. Then, if $y = x^f \neq x$, the pair $\{x, y\}$ is a bi-directional edge. Since the graph is connected, there is a vertex z joined to one of x and y. Then, applying f or its inverse, we see that z is joined to both x and y, and so there is a triangle, a contradiction. See Figure 2. □

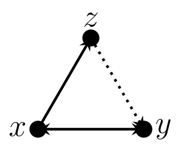

Figure 2: Formation of a triangle in $\mathrm{EG}^*(G)$

8 Compressed endomorphism digraph of dihedral, dicyclic, symmetric and metacyclic groups

8.1 Dihedral Group

$$D_{2n} = \langle r, s \mid r^n = 1, s^2 = 1, srs^{-1} = r^{-1} \rangle$$

First let us determine the automorphism classes. The automorphisms of D_{2n} are of the form $f : r \mapsto r^k,\ s \mapsto sr^j$, where $\gcd(k, n) = 1$ and $0 \leq j \leq n - 1$. For any two reflections sr^a and sr^b, the automorphism f with $j = b - a$ maps sr^a to sr^b, hence all reflections form a single automorphism class. Let $[s]$ denote the automorphism class of reflections.
For any divisor d of n, $[r^d] = \{r^e \mid (d, n) = (e, n)\}$, forms the remaining automorphism

classes. Note that the collection of all automorphism classes of D_{2n}, excluding $[s]$ is similar to the automorphism classes of $\mathbb{Z}_n$.

If $d \mid n$, then $\langle r^{n/d} \rangle$ is a normal subgroup of D_{2n}. Let $\langle r^{n/d} \rangle$ be the kernel of some endomorphism on D_{2n}. So $D_{2n}/\langle r^{\frac{n}{d}} \rangle$ has $\phi(m)$ cosets of order m, for each m dividing $\frac{n}{d}$ and $\frac{n}{d}$ cosets of order 2. Without further examinations, we get

$$D_{2n}/\langle r^{n/d} \rangle \simeq D_{2\frac{n}{d}}.$$

The map f given by $f(r) = r^d$ and $f(s) = s$ is a homomorphism with kernel $\langle r^{n/d} \rangle$ and image set isomorphic to $D_{2n/d}$. Hence there exists an edge from $[r]$ to $[r^d]$ for any d dividing n. Now for d dividing k, the homomorphism $f(r) = r^{k/d}$ and $f(s) = s$ maps $[r^d]$ to $[r^k]$. The compressed directed endomorphism graph of rotations in D_{2n} is same as the compressed directed endomorphism graph of $\mathbb{Z}_n$. Now let us determine the remaining edges.
Case 1 : n is odd.
The only normal subgroups of D_{2n} are $\langle r^d \rangle$ where d divides n and D_{2n} itself. No rotation can be mapped to a reflection or vice versa since their orders are co-prime.
Case 2 : n is even.
The normal subgroups of D_{2n} are $\langle r^d \rangle$ where d divides n, $\langle r^2, s \rangle$, $\langle r^2, rs \rangle$ and D_{2n} itself. Let f be an endomorphism on D_{2n}. We will find the image of the homomorphism f when the following normal subgroups are kernels:

(i) $\mathrm{Ker}(f) = \langle r \rangle$.
$D_{2n}/\langle r \rangle \simeq \mathbb{Z}_2$. The possible choices for f are

- $f(r) = e, f(s) = r^{\frac{n}{2}}$
- $f(r) = e, f(s) = r^a s,\ 0 \leq a \leq n-1$

(ii) $\mathrm{Ker}(f) = \langle r^2, s \rangle = \{e, r^2, r^4, \ldots, r^{n-2}, s, r^2 s, \ldots, r^{n-2} s\}$.
$D_{2n}/\langle r^2, s \rangle \simeq \mathbb{Z}_2$. The possible choices for f are

- $f(r) = r^a s, f(s) = e,\ 0 \leq a \leq n-1$

Note that any odd multiple of 'r' can be mapped to a reflection via this homomorphism.

(iii) $\mathrm{Ker}(f) = \langle r^2, rs \rangle = \{e, r^2, r^4, \ldots, r^{n-2}, rs, r^3 s, \ldots, r^{n-1} s\}$.
$D_{2n}/\langle r^2, rs \rangle \simeq \mathbb{Z}_2$. The possible choices for f are

- $f(r) = r^a s, f(s) = r^a s,\ 0 \leq a \leq n-1$

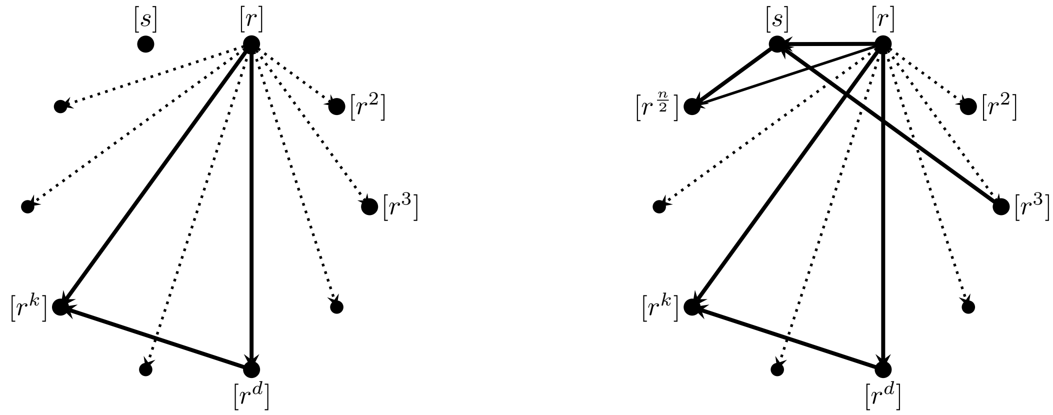

Figure 3: Compressed endomorphism digraph of dihedral group for odd and even cases, d dividing k

8.2 Dicyclic group

$$\mathrm{Dic}_n = \langle a, x | a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$$

For determining the automorphism classes, consider the function $f : \mathrm{Dic}_n \to \mathrm{Dic}_n$:

$$f(a) = a^k, f(x) = a^m x \text{ where } 0 \leq k, m \leq 2n-1, (k, 2n) = 1.$$

Clearly f is a homomorphism. Let $\mathrm{Im}(f)$ denote the image set of f. Note that $a \in \mathrm{Im}(f)$, since $a^k \in \mathrm{Im}(f)$. Consequently a^m and $a^{-m} \in \mathrm{Im}(f)$. Hence $a^{-m}a^m x = x \in \mathrm{Im}(f)$. Therefore, $\mathrm{Im}(f) = \mathrm{Dic}_n$. An onto homomorphism on a finite set is an isomorphism. Hence f is an automorphism. The automorphism classes are :

- $[a^d] = \{a^e \mid (e, 2n) = (d, 2n)\}$ for each divisor d of $2n$
- $[x] = \{a^m x \mid 0 \leq m \leq 2n-1\}$

Having established the automorphism classes, let us find the arcs between them.

(i) There exists an arc from $[a]$ to $[a^d]$ for any divisor d of $2n$ and from $[a^d]$ to $[a^k]$ for any divisor d of k. Consider the homomorphism f on Dic_n given by $f(a) = a^k$ and $f(x) = x$. $\mathrm{Ker}(f) = \langle a^{\frac{2n}{k}} \rangle$, which is a normal subgroup of Dic_n. The homomorphism f maps $[a]$ to $[a^d]$. Now the homomorphism given by $f(a) = a^{\frac{k}{d}}$ and $f(x) = x$ maps $[a^d]$ to $[a^k]$ for any divisor d of k.

(ii) Edges incident with x.

- $f(a) = 1$ and $f(x) = a^n$ is a homomorphism with kernel $\langle a \rangle$ and image isomorphic to $\mathbb{Z}_2$.
- Suppose $f(a) = x$ and $f(x) = a^k$. Since $|x| = 4$ divides $|a|$, n must be even for this case.

$$f(x^{-1}ax) = f(x^{-1})f(a)f(x) = a^{-k}xa^k = a^{-2k}x.$$

However,

$$f(a^{-1}) = x^{-1} = a^n x,$$

$$a^n x = a^{-2k} x \text{ if and only if } \quad k = \frac{n}{2}.$$

Also

$$f(x^2) = f(a^n) \text{ implies } a^n = x^n.$$

Since $x^2 = a^n$ and $x^4 = 1$, n should be even and not divisible by 4. Therefore, there exists an arc from $[x]$ to $[a^k]$ if and only if $n \equiv 2 \pmod 4$. Note that the homomorphism $f(a) = x, f(x) = a^{\frac{n}{2}}$ maps $[a^d]$ to $[x]$ for $d \equiv 1 \pmod 4$ and $f(a) = a^n x, f(x) = a^{\frac{n}{2}}$ maps $[a^d]$ to $[x]$ for $d \equiv 3 \pmod 4$.

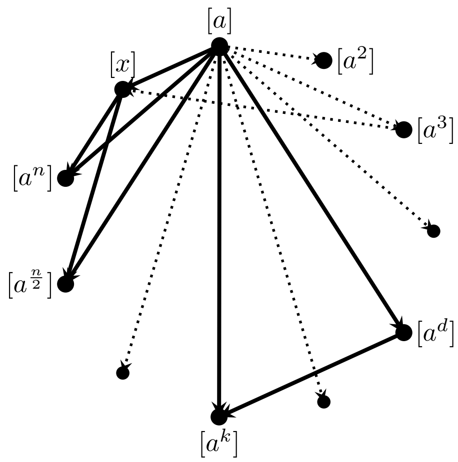

Figure 4: $\overrightarrow{\mathrm{EG}}_{-}(\mathrm{Dic}_n)$, when $n \equiv 2 \pmod 4$

8.3 Symmetric group

The symmetric group, S_n is the group of all permutations on n symbols. The automorphism group of S_n for different cases of n is listed below.

(i) $n \neq 6$
When $n \neq 6$, the automorphism group of S_n is simply the collection of inner automorphisms, $Inn(S_n)$. For $g \in S_n$, an inner automorphism $f_g : S_n \to S_n$ is defined as

$$f_g(x) = g^{-1} x g, \text{ for all } x \in S_n.$$

Therefore the automorphism classes are precisely the conjugacy classes of S_n. Note that elements with the same cyclic structure belong to the same conjugacy classes. We will represent the automorphism classes using the cyclic structure possessed by the permutation.

(ii) $n = 6$
For $n = 6$, apart from the inner automorphism, there are also outer automorphisms, $Out(S_n)$ [15]. With this outer automorphisms, elements with the following cyclic structures can be mapped together

- $(2) \longleftrightarrow (2)^3$
- $(3) \longleftrightarrow (3)^2$
- $(2)(3) \longleftrightarrow (6)$

Thus the automorphism classes of the group S_6 are:

$$[e],\ [(2)] = [(2)^3],\ [(2)^2],\ [(3)] = [(3)^2],\ [(2)(3)] = [(6)],\ [(4)],\ [(4)(2)],\ [(5)].$$

For identifying the endomorphisms, let us consider two cases.

* Case 1 : $n \neq 4$
The only trivial proper normal subgroup of S_n when $n \neq 4$, is the alternating group A_n, and $S_n/A_n \simeq \mathbb{Z}_2$. So the image set of the endomorphism with kernel A_n will be isomorphic to $\mathbb{Z}_2$. With this endomorphism, any odd permutation can be mapped to an element of order 2 in S_n. Note that an odd permutation is a permutation that can be expressed as the product of an odd number of transpositions.

* Case 2 : $n = 4$
If we consider the normal subgroup A_4 as the kernel of an endomorphism, automorphism class $[(4)]$ can mapped to $[(2)^2]$ and $[(2)]$. In addition to A_4, $V_4 = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of S_4. Even if we take V_4 as the kernel of an endomorphism, there are still no mappings among the elements since the order of the image should divide the order of the preimage of an endomorphism.

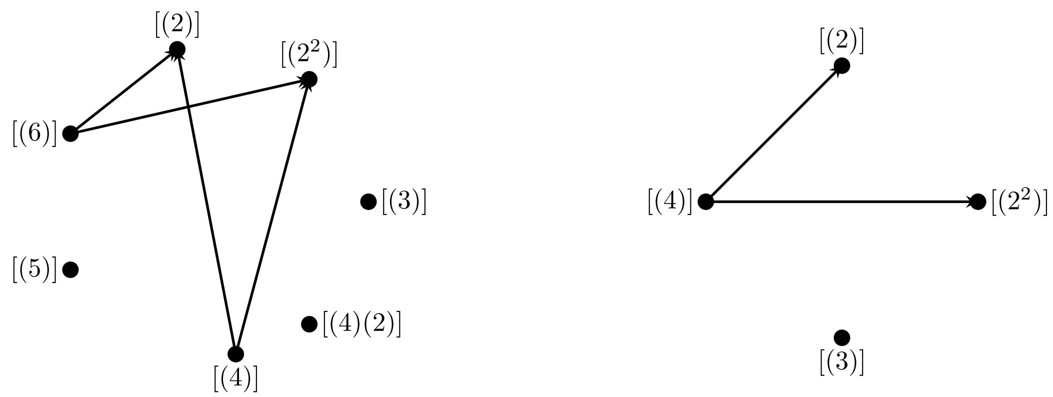

Figure 5: Compressed endomorphism digraph of S_6 and S_4

Remark 8.1. For the alternating group, A_n, the automorphism classes are the conjugacy classes of S_n that are present in A_n except for the case $n = 6$. This is because

$$\mathrm{Aut}(A_n) = \begin{cases} \mathbb{Z}_1, & \text{if } n = 1, 2; \\ \mathbb{Z}_2, & \text{if } n = 3; \\ S_n, & \text{if } n \geq 4, n \neq 6; \\ S_6 \rtimes \mathbb{Z}_2, & \text{if } n = 6. \end{cases}$$

For $n = 6$, apart from the conjugacy classes of S_6, that belong to A_6, a (3) cycle and (3)(3) cycle belong to the same automorphism class.

As for the endomorphisms, when $n \neq 4$, there are no non-trivial normal subgroups for A_n and hence the $\overrightarrow{\mathrm{EG}}_-(A_n)$ is a disconnected graph. For $n = 4$, the vertex set of $\overrightarrow{\mathrm{EG}}_-(A_4), V(A_4) = \{[(3)], [(2)(2)], e\}$. Even though $V_4 = \{(12)(34), (13)(24), (14)(23)\}$ is a non-trivial normal subgroup of A_4, there do not exist any edges between the vertices, since the order of the image should divide the order of the pre-image.

8.4 Some metacyclic groups

Let q be a fixed prime. For each divisor m of $q-1$ with $m > 1$, let G_m be the semidirect product of $\mathbb{Z}_q$ by $\mathbb{Z}_m$ where the latter acts faithfully on the former; that is,

$$G_m = \langle x, y \mid x^q = y^m = 1, y^{-1}xy = x^s \rangle,$$

where s is a primitive m-th root of 1 (mod q).

Theorem 8.2. *The compressed endomorphism graph of G_m has two connected components, an isolated vertex and the uncompressed endomorphism graph of $\mathbb{Z}_m$ with the identity removed.*

The endomorphism graph of a cyclic group is described in Section 5.

Proof. First, we establish the following claim:

$$\mathrm{Aut}(G_m) \cong G_{q-1}.$$

For this, we note that G_m is a subgroup of G_{q-1}. It is a normal subgroup by the Correspondence Theorem [11], since $\mathbb{Z}_m$ is a normal subgroup of $\mathbb{Z}_{q-1}$. So the inner automorphisms of G_{q-1} (which form a group isomorphic to G_{q-1}) act as automorphisms of G_m.

This action is transitive on pairs consisting of a non-identity power of x and a subgroup of order m; so, if there are any further automorphisms, we may assume that one of them, say σ, fixes x and maps y to y^j. Then we have

$$x^s = (x^s)^\sigma = y^{-\sigma}x^\sigma y^\sigma = y^{-j}xy^j = x^{s^j};$$

so s^j is congruent to s (mod q). By our assumption on m, it follows that $j \equiv 1$ (mod m), so $y^j = y$ and σ is the identity.

So the claim is proved.

Now we can read off the isomorphism classes of G_m; they are

$[x] = \{x^i : 1 \leq i \leq q-1\}$, and

$[y^j] = \{x^i y^j : 0 \leq i \leq q-1\}$ for $1 \leq j < m$.

No endomorphism can map an element of order q to a non-identity element of order dividing m, or *vice versa*; so $[x]$ is an isolated vertex in the endomorphism graph.

For the other isomorphism classes, there is a natural bijection to the non-identity elements of $\langle y \rangle \cong \mathbb{Z}_m$; and any endomorphism of G_m induces an endomorphism of $\mathbb{Z}_m$. Conversely, if τ is an endomorphism of $\mathbb{Z}_m$, then we may assume that the domain of τ is G_m (with $\langle x \rangle$ in the kernel) and the range is $\langle y \rangle$; then τ is an endomorphism of G_m. Thus the induced subgraph of the endomorphism graph on the set of automorphism classes of the second type is isomorphic to the uncompressed endomorphism graph of $\mathbb{Z}_m$. This concludes the proof. □

Corollary 8.3. *If q is prime and m is a prime power with $m \mid q-1$, then the endomorphism graph of G_m is the disjoint union of K_1 and K_{m-1}.*

9 Negative answers to Question 3.1

In this section we present examples giving negative answers to the remaining parts of Question 3.1: non-isomorphic groups with isomorphic endomorphism digraphs; groups with isomorphic endomorphism graphs but non-isomorphic endomorphism digraphs; and groups in which the endomorphism and automorphism classes are different.

Proposition 9.1. *Let p be an odd prime. Then the following three groups of order p^3 all have isomorphic endomorphism digraphs: the cyclic group $\mathbb{Z}_{p^3}$; the group $\mathbb{Z}_{p^2} \times \mathbb{Z}_p$; and the non-abelian group of order p^3 and exponent p^2.*

Proof. The endomorphism digraph of a group is a preorder; we will show that the preorders are the same in all three cases. (The convention is that $a > b$ if there is an endomorphism mapping a to b.)

We let $O_n(G)$ denote the set of elements of order n in G.

In the cyclic group of order p^3, the preorder is

$$O_{p^3}(G) > O_{p^2}(G) > O_p(G) > O_1(G) = \{e\},$$

where the sets have cardinalities $p^2(p-1)$, $p(p-1)$, $p-1$ and 1 respectively. For an element of order p^3 is a generator and can be mapped to any element; thus its pth power can be mapped to any element of order p^2 (since any such element has a pth root); and so on. There are no further arcs since endomorphisms cannot increase orders of elements.

Now consider $G = \mathbb{Z}_{p^2} \times \mathbb{Z}_p = \langle a, b \rangle$, where a and b have orders p^2 and p. Now a can be chosen to be any element of order p^2, and b any element not in $\langle a \rangle$. From this it is easy to see that there are endomorphisms mapping elements of order p^2 to all elements of the group. (Map b to e and a to the required element.) Now elements in $\langle a^p \rangle$ can only be mapped into this set since they are the only elements of the group which are pth powers; but other elements of order p can be mapped to arbitrary elements of order p or 1. So the preorder is

$$O(p^2) > O(p) \setminus \langle a^p \rangle > \langle a^p \rangle \setminus \{e\} > \{e\},$$

and the sets have cardinalities $p^3 - p^2$, $p^2 - p$, $p - 1$ and 1. So the graphs for these two groups are isomorphic.

Note that this argument also holds for $p = 2$, so $\mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$ have isomorphic endomorphism digraphs.

Finally, consider $G = \langle a, b : a^{p^2} = b^p = e, b^{-1}ab = a^{1+p} \rangle$. The elements of order p in this group form a subgroup $H = \langle a^p, b \rangle$ of order p^2. Any element outside this subgroup can play the role of a. So as in the previous case there are arcs from elements of order p^2 to all elements of the group. Also,

$$\begin{array}{rcl} (a^i b^j)^p & = & a^i \cdot b^j a^i b^{-j} \cdot b^{2j} a^i b^{-2j} \cdots b^{(p-1)j} a^i b^{-(p-1)j} \\ & = & a^{pi} \cdot a^{-i(p+2p+\cdots+(p-1)p)} \\ & = & a^{pi} \cdot a^{-p(p-1)i/2} \end{array}$$

which is a pth power (for p is odd, so p divides $p(p-1)/2$). So again, only the powers of a^p are pth powers, and the argument proceeds as before. □

Now Proposition 4.2 gives many more examples of pairs of groups with isomorphic endomorphism digraphs.

For the next question, we use the remaining two groups of order p^3.

Proposition 9.2. *Let p be an odd prime. Let H_p be the nonabelian group of order p^3 and exponent p. Then $\mathrm{EG}(H_p)$ is isomorphic to $\mathrm{EG}(\mathbb{Z}_p^3)$ but $\overrightarrow{\mathrm{EG}}(H_p)$ is not isomorphic to $\overrightarrow{\mathrm{EG}}(\mathbb{Z}_p^3)$.*

Proof. The automorphism group of $\mathbb{Z}_p^3$ is the general linear group $\mathrm{GL}(3, p)$, and acts transitively on the non-identity elements. So $\overrightarrow{\mathrm{EG}}(\mathbb{Z}_p^3)$ with the identity deleted is the complete directed graph, and $\mathrm{EG}(\mathbb{Z}_p^3)$ is the complete graph.

Now let Z be the center of H_p. Then $|Z| = p$. No automorphism can map an element of Z to an element outside Z. Also, Z is the Frattini subgroup of G, and so is contained in any nontrivial maximal subgroup; there are $p + 1$ such subgroups, corresponding to the subgroups of order p in H_p/Z. So all proper endomorphisms map Z to the identity. Thus $\overrightarrow{\mathrm{EG}}(H_p)$ is not isomorphic to $\overrightarrow{\mathrm{EG}}(\mathbb{Z}_p^3)$. However, there is an endomorphism of H_p whose image is $H_p/Z \cong \mathbb{Z}_p^2$, so any element a outside Z can be mapped to a non-identity element of $\mathbb{Z}_p^2$. Since H_p is the union of subgroups isomorphic to $\mathbb{Z}_p^2$, we see that a is joined to every element of H_p in $\mathrm{EG}(H_p)$. Thus $\mathrm{EG}(H_p)$ is complete, and is isomorphic to $\mathrm{EG}(\mathbb{Z}_p^3)$. □

Finally, here is a family of examples of groups where automorphism and endomorphism equivalence are not the same.

Proposition 9.3. *Let q and r be odd primes such that r divides $q-1$, and let G denote the nonabelian group of order qr. Then all elements of order r in G lie in the same endomorphism class, but these elements fall into $r-1$ different automorphism classes.*

Proof. This group contains a normal subgroup N of order q; so G/N is cyclic of order r. Every element outside N has order r; so every element outside N can be mapped to a generator of a cyclic group of order r, and so to any other element of G of order r, by an endomorphism. So $G \setminus N$ is a single endomorphism class. But, if h has order r, then there is a number $m = m(h)$, an rth root of unity mod q, such that $h^{-1}xh = x^m$ for all elements x of order p. If an automorphism maps h to h', then $m(h) = m(h')$. So there are $r-1$ automorphism classes of elements of order r, each of size q. $\square$

10 Miscellaneous topics and questions

In the final section, we discuss a few additional topics and pose some questions for further research.

10.1 Groups with $\overrightarrow{\mathrm{AG}}(G) = \overrightarrow{\mathrm{EG}}(G)$

Recall that $\overrightarrow{\mathrm{AG}}(G)$ is the directed automorphism graph of G. Which groups G satisfy $\overrightarrow{\mathrm{AG}}(G) = \overrightarrow{\mathrm{EG}}(G)$? These groups can be divided into two types:

(a) groups in which every non-trivial endomorphism is an automorphism;
(b) groups with non-trivial proper endomorphisms, but satisfying the condition that, if there is an endomorphism mapping x to y, then there is an automorphism doing the same.

For the first type, we have the following. Recall that G is simple if its only normal subgroups are G and $\{e\}$, and G is perfect if its derived subgroup G' is equal to G.

Proposition 10.1. *Let z_G denote the trivial endomorphism of G mapping every element to e.*

(a) If G is simple, then $\mathrm{End}\, G = \mathrm{Aut}(G) \cup \{z_G\}$.
(b) If $\mathrm{End}\, G = \mathrm{Aut}(G) \cup \{z_G\}$, *then G is perfect.*
(c) Neither of these implications reverses.

Proof. (a) The kernel of an endomorphism f is a normal subgroup; so, if G is simple, then $\ker(f)$ is either $\{e\}$ (in which case G is an automorphism) or G (in which case $f = z_G$).

(b) Suppose that $G' \neq G$. Then G/G' is abelian, so we can choose a normal subgroup N containing G' such that G/N has prime order p. Then p divides $|G|$, so by Cauchy's theorem, G has a subgroup P of order p. Now an isomorphism from G/N to P lifts to an endomorphism f of G with $\ker(f) = N$.

(c) The group $G_1 = \mathrm{SL}(2, 5)$ of 2×2 matrices over the field of five elements has the property that its only proper non-trivial normal subgroup is $Z = \{\pm I\}$; and G_1/Z is isomorphic to $\mathrm{PSL}(2, 5)$, which is the simple group A_5. Also, if A_5 were a subgroup of G_1, then it would have index 2 and be normal, which is not so. So there is no endomorphism of G_1 with kernel Z, but G is not simple.

The group $G_2 = \mathrm{AGL}(3, 2)$, the three-dimensional affine group over the field of 2 elements (generated by translations and invertible linear maps of the 3-dimensional vector space) has a unique proper non-trivial subgroup T consisting of the translations; the quotient G_1/N is the simple group $\mathrm{GL}(3, 2)$, so G_2 is perfect; but $\mathrm{GL}(3, 2)$ is a subgroup of G_2 consisting of the linear maps, so there is an endomorphism with kernel T. □

For the second type, we have a classification of the abelian groups with this property.

Proposition 10.2. *Let G be a finite abelian group. Then G has the property that $\overrightarrow{\mathrm{EG}}(G) = \overrightarrow{\mathrm{AG}}(G)$ if and only if G is an elementary abelian p-group for some prime p.*

Proof. If G contains an element g of composite order rs where $r, s > 1$, then G has an endomorphism (the power map $x \mapsto x^s$) which maps g to g^s; but no automorphism can do this, since g and g^s have different orders. □

The classification for nonabelian groups is still open.

10.2 Endomorphism digraph and power digraph

We remarked at the start of Section 5 that, if G is a cyclic group, then the directed endomorphism graph $\overrightarrow{\mathrm{EG}}(G)$ and directed power graph $\overrightarrow{P}(G)$ coincide. What happens in general?

Theorem 10.3. *Let G be a finite group. Then $\overrightarrow{\mathrm{EG}}(G) = \vec{P}(G)$ if and only if G is cyclic.*

Proof. We have seen the reverse implication. Suppose that $\overrightarrow{\mathrm{EG}}(G) = \vec{P}(G)$. Then every endomorphism of G maps x to x^k for some k, possibly depending on x. In particular, this is true for inner automorphisms; this means that every subgroup of G is normal. So G is a *Dedekind group.*

From the classification of Dedekind groups [9], either G is abelian, or $G = Q_8 \times A$, where Q_8 is the quaternion group and A an abelian group with no elements of order 4. The second case is impossible, since Q_8 contains an automorphism permuting the three subgroups of order 4 cyclically. As we have seen, non-cyclic abelian groups have endomorphisms which do not map every element to a power. So G is cyclic. □

In general, neither the endomorphism digraph nor the power digraph contains the other. Our proof above shows that, for non-cyclic abelian groups, there are arcs in the endomorphism digraph which are not in the power digraph. In the other direction, take the group $G = \mathrm{PSL}(3,3)$. This group is simple, so all its non-trivial endomorphisms are automorphisms; its outer automorphism group is trivial, so all its automorphisms are inner. Now let x be an element of order 13. Then the only powers of x which are conjugate to x are x^3 and x^9; so the arc $x \to x^2$ belongs to the power digraph but not to the endomorphism digraph.

If G is abelian, we can define a "difference digraph" whose arcs are all the arcs of $\overrightarrow{\mathrm{EG}}(G)$ which are not arcs of $\overrightarrow{P}(G)$, and a corresponding undirected difference graph. The properties of these graphs await investigation.

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References

[1] D. Alireza, E. Ahmad and J. Abbas, Some results on the power graphs of finite groups, *Sci. Asia* 41 (2015), 73–78.

[2] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Universitext, Springer, New York, 2nd edition, 2009.

[3] R. Brauer and K. A. Fowler, On groups of even order, *Ann. Math.* 62(3) (1955), 565–583.

[4] D. Bubboloni and N. Pinzauti, Critical classes of power graphs and reconstruction of directed power graphs, *J. Group Theory* 28(3) (2025), 713–739.

[5] P. J. Cameron, The power graph of a finite group, II, *J. Group Theory* 13 (2010), 779–783.

[6] A. Cayley, On the theory of groups, *Proc. London Math. Soc.* 9 (1878), 126–133.

[7] I. Chakrabarty, S. Ghosh and M. K. Sen, Undirected power graphs of semigroups, *Semigroup Forum* 78(3) (2009), 410–426.

[8] M. Conder, On symmetries of Cayley graphs and the graphs underlying regular maps, *J. Algebra* 321(11) (2009), 3112–3127.

[9] R. Dedekind, Ueber Gruppen, deren sämmtliche Theiler Normaltheiler sind, *Math. Ann.* 48 (1897), 548–561.

[10] K. Dutta and A. Prasad, Degenerations and orbits in finite abelian groups, *J. Combin. Theory Ser. A* 118(6) (2011), 1685–1694.

[11] J. A. Gallian, *Contemporary Abstract Algebra*, 9th edition, 2019.

[12] M. Golasinski and D. L. Goncalves, On automorphisms of finite abelian p-groups, *Math. Slovaca* 58 (2008), 405–412.

[13] M. Gromov, Hyperbolic Groups, in: *Essays in Group Theory*, (Ed.: S.M. Gersten), MSRI Publ. 8, Springer, New York, 1987, pp. 75–263.

[14] F. Harary, *Graph Theory*, Narosa Publication House, 2001.

[15] G. Janusz and J. Rotman, Outer automorphisms of S_6, *Amer. Math. Monthly* 89(6) (1982), 407–410.

[16] J. B. Jensen and G. Gutin, *Classes of Directed Graphs*, Springer Monographs in Mathematics, Springer International Publishing, 2018.

[17] A. Kelarev, Cayley graphs as classifiers for data mining: The influence of asymmetries, *Discrete Math.* 309(17) (2009), 5360–5369.

[18] A. Kelarev and S. J. Quinn, A combinatorial property and power graphs of groups, *Contrib. General Algebra* 12(58) (2000), 3–6.

[19] A. Kumar, L. Selvaganesh, P. J. Cameron and T. Tamizh Chelvam, Recent developments on the power graph of finite groups—a survey, *AKCE Int. J. Graphs Combin.* 18 (2021), 65–94.

[20] M. W. Liebeck and A. Shalev, Simple groups, probabilistic methods, and a conjecture of Kantor and Lubotzky, *J. Algebra* 184(1) (1996), 31–57.

[21] L. Mirsky, A dual of Dilworth's decomposition theorem, *Amer. Math. Monthly* 78(8) (1971), 876–877.

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