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Symmetric harmoniousness of odd-order groups

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Abstract

We prove that every odd-order group is symmetric harmonious; that is, there exists a permutation $g_0, g_1, \ldots, g_{\ell-1}$ of elements of G such that the consecutive products $g_0g_1, g_1g_2, \ldots, g_{\ell-1}g_0$ also form a permutation of elements of G and $g_{\ell-i} = g_i^{-1}$ for all $1 \leqslant i \leqslant \ell - 1$, where $\ell = |G|$. We apply this result to obtain new examples of R*-sequenceable groups as well as new examples of non-abelian #-harmonious groups.

1 Introduction

A *harmonious* sequence in a group G is a permutation $g_0, g_1, \ldots, g_{\ell-1}$ of the elements of G such that the consecutive products $g_0g_1, g_1g_2, \ldots, g_{\ell-1}g_0$ also form a permutation of the elements of G (where $\ell = |G|$). Such sequences were introduced by Beals et al. [1] in 1991 in connection with the study of complete mappings. A *complete mapping* of a group G is a bijection $\phi : G \to G$ such that the map $x \mapsto x\phi(x)$ is also a bijection.

If $g_0, g_1, \ldots, g_{\ell-1}$ is a harmonious sequence in G, then the map defined by

$$\phi : g_i \mapsto g_{i+1} \quad (0 \leqslant i \leqslant \ell - 1, g_\ell = g_0)$$

is a complete mapping of G that consists of a single cycle of length $|G|$. Conversely, if ϕ is a complete mapping of G that is a single $|G|$-cycle, then for any $g \in G$ the sequence

$$g, \phi(g), \phi^2(g), \ldots, \phi^{\ell-1}(g)$$

is a harmonious sequence in G.

Hall and Paige [6] proved that a necessary condition for a group G to admit a complete mapping is that every Sylow 2-subgroup of G is either trivial or non-cyclic (the Hall–Paige condition). Their conjecture that this condition is also sufficient was affirmed in 2009 [2, 4, 15].

Beals et al. [1] showed that every abelian group is harmonious except for the elementary abelian 2-groups and the binary groups (groups with a unique element of order 2).

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Among non-abelian groups, the dihedral and dicyclic groups of order $8n$, with $n > 1$, are harmonious [1, 11, 12]. More recently, Müesser and Pokrovskiy [8] proved that all sufficiently large groups satisfying the Hall–Paige condition, except the elementary abelian 2-groups, are harmonious.

Wang and Leonard [13] introduced the notion of *symmetric harmonious groups* and used such groups to construct new examples of *R^*-sequenceable* groups.

Definition 1.1. Let G be a group of order ℓ.

i) G is *symmetric harmonious* if there exists a permutation $g_0, g_1, \ldots, g_{\ell-1}$ of the elements of G such that $g_0g_1, g_1g_2, \ldots, g_{\ell-1}g_0$ is also a permutation of the elements of G and
$$g_i g_{\ell-i} = 1_G \text{ for all } 1 \leqslant i \leqslant \ell - 1.$$

ii) G is *R-sequenceable* if there exists a permutation $g_1, \ldots, g_{\ell-1}$ of the non-identity elements of G such that the consecutive quotients
$$g_1^{-1}g_2, g_2^{-1}g_3, \ldots, g_{\ell-1}^{-1}g_1$$
also form a permutation of the non-identity elements of G. If, in addition, there exists some index i such that $g_{i-1}g_{i+1} = g_{i+1}g_{i-1} = g_i$, then G is *R^*-sequenceable.*

Wang and Leonard [13] proved that if $q = 6h + 1$ is prime, then the non-abelian group of order $3q$ is symmetric harmonious. They further showed in [14] that every nilpotent group of odd order is symmetric harmonious. The following theorem is the main result of this article.

Theorem 1.2. *A group is symmetric harmonious if and only if it has odd order.*

The Hall–Paige condition is also necessary for the existence of R-sequencings (which produce complete mappings and orthogonal Latin squares) [10]. Friedlander et al. [5] introduced R*-sequenceable groups and showed that if every Sylow 2-subgroup of G is R*-sequenceable, then G itself is R-sequenceable.

Non-binary abelian groups whose Sylow 2-subgroups have order different from 8 are R*-sequenceable [5, 7]. Certain direct products involving $(\mathbb{Z}_2)^3$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$, and odd-order abelian groups with restricted Sylow 3-structure are also R*-sequenceable [5, 9]. Non-abelian examples include groups of order pq (distinct odd primes) [12], dihedral groups of order $4n$ with $n \geqslant 4$, and certain dicyclic groups [13].

Wang and Leonard [13, Prop. 1] proved that if G is even-order R*-sequenceable and H is symmetric harmonious, then $G \times H$ is R*-sequenceable. Combining this with Theorem 1.2 provides new examples of R*-sequenceable groups.

Theorem 1.3. *If G is even-order R^*-sequenceable and H is odd-order, then $G \times H$ is R^*-sequenceable.*

Beals et al. [1] also studied the variant of harmoniousness in which the identity is excluded.

Definition 1.4. A group G of order ℓ is *#-harmonious* if there exists a permutation $g_1, g_2, \ldots, g_{\ell-1}$ of the non-identity elements of G such that the consecutive products $g_1g_2, g_2g_3, \ldots, g_{\ell-1}g_1$ also form a permutation of the non-identity elements of G.

Like harmonious groups, #-harmonious groups must satisfy the Hall–Paige condition. However, unlike the harmonious case, all non-cyclic elementary abelian 2-groups are #-harmonious [1]. Additionally, every abelian group that is not binary is #-harmonious except for $\mathbb{Z}_3$ [1]. The following theorem presents the first examples of non-abelian #-harmonious groups.

Theorem 1.5. *Let K be an abelian group that is neither a binary group, nor an elementary abelian 2-group, nor isomorphic to $\mathbb{Z}_3$. Let $H_1, \ldots, H_k$ be odd-order groups. Then $G = (((K \ltimes H_1) \ltimes H_2) \ldots) \ltimes H_k$ is #-harmonious.*

The proof of Theorem 1.2 is given at the end of Section 2, and the proof of Theorem 1.5 is given at the end of Section 3.

2 Symmetric harmonious sequences

In this section, we prove that every odd-order group is symmetric harmonious. Since all odd-order abelian groups are symmetric harmonious [1] and odd-order groups are solvable, it is sufficient to describe how one can lift symmetric harmonious sequences via normal extensions (see the proof of Theorem 1.2 at the end of this section). We demonstrate this by an example. Let p, q be odd prime numbers such that $q \mid (p-1)$. Consider the semidirect product $G = \mathbb{Z}_p \rtimes \mathbb{Z}_q$:

$$G = \langle a, b \mid a^p = b^q = 1, bab^{-1} = a^r \rangle,$$

where r has order q modulo p (i.e., q is the least positive integer with $r^q = 1 \pmod p$). Then the sequence

$$\begin{array}{llllll}
a^0 & ab & b^{-1}ab^3 & b^{-3}ab^6 & \cdots & b^{-(q-2)(q-1)/2}a \\
a^2 & a^3b & b^{-1}a^3b^3 & b^{-3}a^3b^6 & \cdots & b^{-(q-2)(q-1)/2}a^3 \\
a^4 & a^5b & b^{-1}a^5b^3 & b^{-3}a^5b^6 & \cdots & b^{-(q-2)(q-1)/2}a^5 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a^{2(p-1)} & a^{2p-1}b & b^{-1}a^{2p-1}b^3 & b^{-3}a^{2p-1}b^6 & \cdots & b^{-(q-2)(q-1)/2}a^{2p-1}.
\end{array}$$

is a symmetric harmonious sequence in G. Here the jth term on the ith row, $1 < j \leqslant q$, is given by $b^{-t_j}a^{2i-1}b^{t_{j+1}}$, where $t_j = (j-1)(j-2)/2$.

In the case of $p = 7$ and $q = 3$ (where we let $r = 2$), this sequence is:

$$1, ab, a^4b^2, a^2, a^3b, a^5b^2, a^4, a^5b, a^6b^2, a^6, b, b^2, a, a^2b, ab^2, a^3, a^4b, a^2b^2, a^5, a^6b, a^3b^2,$$

with consecutive products:

$$ab, a^2, a^5b^2, a^5b, a^6, b^2, a^2b, a^3, a^2b^2, a^6b, 1, a^4b^2, a^3b, a^4, a^6b^2, b, a, ab^2, a^4b, a^5, a^3b^2.$$

In the next lemma, we generalize this construction to arbitrary normal extensions of odd-order groups.

Lemma 2.1. *Suppose H is a normal subgroup of G and both H and G/H are odd-order symmetric harmonious groups. Then G is a symmetric harmonious group.*

Proof. Let $K_0, K_1, \ldots, K_{m-1}$ be a symmetric harmonious sequence in G/H, where each K_i is a coset of H in G and $K_0 = H$. Choose $k_i \in K_i \subseteq G$ so that $k_0 = 1_G$. Since $K_i K_{m-i} = H$ for $1 \leqslant i \leqslant m-1$, without loss of generality, we can choose $k_1, \ldots, k_{m-1}$ so that

$$k_i k_{m-i} = 1_G \quad \text{for all } 1 \leqslant i \leqslant m-1. \tag{2.1}$$

For $0 \leqslant i \leqslant m-1$, we let μ_i be the product of consecutive terms $k_{i+1}, \ldots, k_{m-1}$:

$$\mu_i = k_{i+1} \cdots k_{m-1}, \text{ for } 0 \leqslant i \leqslant m-2; \; \mu_{m-1} = 1_G.$$

Let $t = (m-1)/2$. It follows from (2.1) with $i = t$ that $k_t k_{t+1} = 1_G$ and so

$$\mu_0 = k_1 k_2 \cdots k_t k_{t+1} \cdots k_{m-1} = k_1 k_2 \cdots k_{t-1} k_{t+2} \cdots k_{m-1} = \cdots = 1_G. \tag{2.2}$$

Additionally, for $1 \leqslant r \leqslant m-1$, by (2.2), one has

$$\begin{aligned} k_{m-r}\mu_{m-r} &= k_{m-r} k_{m-r+1} \cdots k_{m-1} \\ &= k_r^{-1} k_{r-1}^{-1} \cdots k_1^{-1} \\ &= (k_1 \cdots k_r)^{-1} \\ &= k_{r+1} \cdots k_{m-1} = \mu_r. \end{aligned} \tag{2.3}$$

Next, let $h_0, h_1, \ldots, h_{n-1}$ be a symmetric harmonious sequence in H. For $0 \leqslant r \leqslant m-1$ and $0 \leqslant p \leqslant n-1$, we define

$$g_{pm+r} = \begin{cases} h_{2p} & \text{if } r = 0, \\ k_r \mu_r h_{2p+1} \mu_r^{-1} & \text{if } 0 < r \leqslant m-1, \end{cases} \tag{2.4}$$

where the indices in h_{2p} and h_{2p+1} are computed modulo n. Since H is a normal subgroup of G, we have $g_{pm+r} \in K_r$. We claim that $g_0, \ldots, g_{mn-1}$ is a symmetric harmonious sequence in G.

Step 1. We first show that $g_0, \ldots, g_{mn-1}$ is a permutation of elements of G. It suffices to prove that if $i, j \in \{0, \ldots, mn-1\}$ and $g_i = g_j$, then $i = j$. Write $i = pm + r$ and $j = qm + s$ with $p, q \in \{0, \ldots, n-1\}$ and $r, s \in \{0, \ldots, m-1\}$.

Since $g_{pm+r} \in K_r$ and $g_{qm+s} \in K_s$, the assumption $g_i = g_j$ implies that $r = s$. Suppose $r = 0$. Then (2.4) gives $h_{2p} = h_{2q}$, so $2p = 2q \pmod{n}$. As n is odd, $p = q$ and $i = j$. The case of $r > 0$ is similar.

Step 2. Next, we need to show that if $i, j \in \{0, \ldots, mn-1\}$ and $g_i g_{i+1} = g_j g_{j+1}$ (where $g_{mn} = g_0$), then $i = j$. Write $i = pm + r$ and $j = qm + s$ with $p, q \in \{0, \ldots, n-1\}$ and $r, s \in \{0, \ldots, m-1\}$. Since $g_i \in K_r$, we have $g_i g_{i+1} \in K_r K_{r+1}$.

Thus $g_i g_{i+1} = g_j g_{j+1}$ implies $K_r K_{r+1} = K_s K_{s+1}$ (where $K_m = K_0$). Since $K_0, \ldots, K_{m-1}$ is a harmonious sequence, it follows that $r = s$. To prove $p = q$, we consider three cases:

Case 1: $r \neq 0, m-1$. Then

$$\begin{aligned} g_i g_{i+1} &= k_r \mu_r h_{2p+1} \mu_r^{-1} \cdot k_{r+1} \mu_{r+1} h_{2p+1} \mu_{r+1}^{-1} \\ &= \mu_{r-1} (h_{2p+1})^2 \mu_{r+1}^{-1}. \end{aligned}$$

Therefore, it follows from $g_i g_{i+1} = g_j g_{j+1}$ that $h_{2p+1}^2 = h_{2q+1}^2$. In an odd-order group the mapping $h \mapsto h^2$ is a bijection, hence $2p+1 = 2q+1 \pmod{n}$, and so $p = q$.

Case 2: $r = 0$. Then

$$\begin{aligned} g_i g_{i+1} &= h_{2p} \cdot k_1 \mu_1 h_{2p+1} \mu_1^{-1}, \\ &= (h_{2p} h_{2p+1}) \mu_1^{-1}. \end{aligned}$$

It follows from $g_i g_{i+1} = g_j g_{j+1}$ that $h_{2p} h_{2p+1} = h_{2q} h_{2q+1}$, which implies that $2p = 2q \pmod{n}$ (since $h_0, h_1, \ldots, h_{n-1}$ is harmonious), hence $p = q$.

Case 3: $r = m-1$. Then $i = pm + (m-1)$ and $i+1 = (p+1)m + 0$. Therefore,

$$g_i g_{i+1} = h_{2p+1} h_{2p+2},$$

and as in Case 2, we must have $p = q$.

Step 3. It is left to show that $g_i g_{mn-i} = 1_G$ for all $1 \leqslant i \leqslant mn$. Let $i = pm + r$ with $0 \leqslant p \leqslant n-1$ and $0 \leqslant r \leqslant m-1$. One has $mn - i = m(n-1-p) + m - r$. If $r > 0$, then by (2.3), one has

$$\begin{aligned} g_i g_{mn-i} &= k_r \mu_r h_{2p+1} \mu_r^{-1} \cdot k_{m-r} \mu_{m-r} h_{2(n-1-p)+1} \mu_{m-r}^{-1} \\ &= k_r \mu_r h_{2p+1} h_{2(n-1-p)+1} \mu_{m-r}^{-1} \\ &= k_r \mu_r h_{2p+1} h_{n-(2p+1)} \mu_{m-r}^{-1} \\ &= k_r \mu_r \mu_{m-r}^{-1} \\ &= k_r k_{m-r} \mu_{m-r} \mu_{m-r}^{-1} \\ &= 1_G. \end{aligned}$$

If $r = 0$, then $i = pm$ and $mn - i = (n-p)m$, and so

$$g_i g_{mn-i} = h_{2p} h_{2(n-p)} = h_{2p} h_{n-2p} = 1_G.$$

Thus G is symmetric harmonious. □

We are now ready to prove Theorem 1.2 which states that a group is symmetric harmonious if and only if it has odd order.

Proof of Theorem 1.2. We first show that if G is symmetric harmonious, then it has odd order. Let $g_0, \ldots, g_{\ell-1}$ be a symmetric harmonious sequence in G. There exists $0 \leqslant i \leqslant \ell - 1$ such that $g_{i-1} g_i = 1_G$ (indices modulo ℓ), so $g_{i-1} = g_i^{-1} = g_{\ell-i}$. Thus $i - 1 = \ell - i \pmod{\ell}$, which implies that ℓ is odd.

For the converse, we argue by contradiction. Suppose that G is a group of odd order that is not symmetric harmonious, and assume that $|G|$ is minimal among all groups with this property. Since every odd-order abelian group admits a symmetric harmonious sequence [1, 12], the group G must be non-abelian. By the

Feit–Thompson theorem, every finite group of odd order is solvable; in particular, the derived subgroup G' is a proper subgroup of G, so $|G'| < |G|$.

By minimality of $|G|$, both G' and G/G' are symmetric harmonious. Applying Lemma 2.1, it follows that G itself is symmetric harmonious, contradicting our assumption. Hence every odd-order group is symmetric harmonious. □

3 #-harmonious groups

In this section, we construct examples of #-harmonious groups through taking semidirect products of odd-order groups with *harmoniously matched* groups.

Definition 3.1. A finite group G is *harmoniously matched* if there exist a harmonious sequence $g_0, g_1, \ldots, g_{n-1}$ and a #-harmonious sequence $g'_1, g'_2, \ldots, g'_{n-1}$ in G such that $g_0 = g'_1$, $g_{n-1} = g'_{n-1}$, and

$$g_0 g_1 \cdots g_{n-1} = g'_1 g'_2 \cdots g'_{n-1} = 1_G. \tag{3.1}$$

Beals et al. showed that every finite abelian group except $\mathbb{Z}_3$, binary groups, and elementary 2-groups is harmoniously matched (although condition (3.1) does not appear explicitly in their definition, it is automatically satisfied in the setting of abelian groups).

Example 3.2. The dihedral group $D_{12} = \langle r, s | r^6 = s^2 = (sr)^2 = 1\rangle$ is harmoniously matched:

$$\begin{aligned} \mathbf{g} &: r, 1, r^3, s, r^5, rs, r^2, r^4, r^4s, r^2s, r^3s, r^5s, \\ \mathbf{g}' &: r, r^2, r^3, s, r^4, r^3s, r^2s, r^4s, r^5, rs, r^5s. \end{aligned}$$

Similarly, the dihedral group $D_{16} = \langle r, s | r^8 = s^2 = (sr)^2 = 1\rangle$ is harmoniously matched:

$$\begin{aligned} \mathbf{g} &: r^2, r, r^6, s, r^3s, r^7, r^3, r^5, r^4s, r^6s, r^5s, rs, r^4, r^7s, 1, r^2s, \\ \mathbf{g}' &: r^2, r^5, r^6s, r^4, r^3s, rs, s, r^5s, r^7s, r^3, r, r^4s, r^6, r^7, r^2s. \end{aligned}$$

Lemma 3.3. *Let H be an odd-order normal subgroup of a group G with a harmoniously matched complement K so that $G = HK$ and $H \cap K = \{1_G\}$. Then G is harmoniously matched.*

Proof. Since K is harmoniously matched, there exist a harmonious sequence $u_0, u_1, \ldots, u_{n-1}$ and a #-harmonious sequence $u'_1, u'_2, \ldots, u'_{n-1}$ in K such that $u_0 = u'_1$, $u_{n-1} = u'_{n-1}$, and $u'_1 u'_2 \cdots u'_{n-1} = u_0 u_1 \cdots u_{n-1} = 1_G$. Since H is odd-order, by Theorem 1.2, it has a symmetric harmonious sequence $h_0, h_1, \ldots, h_{m-1}$ such that $h_i h_{m-i} = 1_G$ for all $1 \leqslant i \leqslant m-1$. Define

$$\sigma_i = u_0 u_1 \cdots u_i, \text{ for } 0 \leqslant i \leqslant n-1.$$

In particular, $\sigma_{n-1} = u_0 u_1 \cdots u_{n-1} = 1_G$. Then the sequence:

$$\begin{array}{llllll}
u'_1 & u'_2 & \dots & u'_{n-1} & & \\
h_1\sigma_0 & \sigma_0^{-1}h_1\sigma_1 & \sigma_1^{-1}h_1\sigma_2 & \sigma_2^{-1}h_1\sigma_3 & \dots & \sigma_{n-2}^{-1}h_1\sigma_{n-1} \\
h_2\sigma_0 & \sigma_0^{-1}h_2\sigma_1 & \sigma_1^{-1}h_2\sigma_2 & \sigma_2^{-1}h_2\sigma_3 & \dots & \sigma_{n-2}^{-1}h_2\sigma_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
h_{m-1}\sigma_0 & \sigma_0^{-1}h_{m-1}\sigma_1 & \sigma_1^{-1}h_{m-1}\sigma_2 & \sigma_2^{-1}h_{m-1}\sigma_3 & \dots & \sigma_{n-2}^{-1}h_{m-1}\sigma_{n-1}
\end{array}$$

is #-harmonious. The same sequence with the first row replaced by $u_0, \dots, u_{n-1}$ is harmonious. It is straightforward to check that the two sequences obtained in this manner begin and end at the same terms. To see that the product of all the terms equals the identity element, we note that

$$u'_1 u'_2 \cdots u'_{n-1}(h_1\sigma_0)(\sigma_0^{-1}h_1\sigma_1)\cdots(\sigma_{n-2}^{-1}h_{m-1}\sigma_{n-1}) = h_1^n h_2^n \cdots h_{m-1}^n,$$

which equals the identity, since $h_0, h_1, \dots, h_{m-1}$ is a symmetric harmonious sequence (the proof is similar to the proof of (2.2)). □

Proof of Theorem 1.5. Theorem 1.5 follows by repeatedly applying Lemma 3.3 and the result that every abelian group that is not binary, nor an elementary 2-group, nor isomorphic to $\mathbb{Z}_3$ is harmoniously matched [1]. □

Example 3.4. Let $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \ltimes \mathbb{Z}_9$ be the semidirect product induced by the map $\phi : \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_9) = \mathbb{Z}_9^\times$ with

$$\phi(1,0) = 4, \ \phi(0,1) = 1.$$

In terms of generators and relations, we have

$$G = \langle a, x, y | a^9 = x^3 = y^3 = 1, xy = yx, xax^{-1} = a^4, yay^{-1} = a \rangle.$$

Group G is harmoniously matched by Theorem 1.5, hence it is #-harmonious.

Example 3.5. If m is odd, then the dihedral group D_{16m} of order $16m$ is the semidirect product:

$$D_{16m} = D_{16} \ltimes \mathbb{Z}_m.$$

Thus by Theorem 1.5 and Example 3.2, D_{16m} is #-harmonious for odd m. Similarly, D_{12m} is #-harmonious if $\gcd(m, 6) = 1$.

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