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# Chorded bipancyclicity for degree-sum conditions of two vertices with distance two in balanced bipartite graphs

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**Abstract**

Let $G$ be a connected balanced bipartite graph of order $2n$ with $n \geq 3$. Define $\mu_2(G) = \min\{d(x) + d(y) : x, y \in V(G), \mathrm{dist}(x, y) = 2\}$. A chord of a cycle is an edge joining two non-consecutive vertices of the cycle, and a chorded cycle is defined as a cycle containing at least one chord. A balanced bipartite graph $G$ of order $2n$ is chorded vertex (respectively, edge) bipancyclic if every vertex (respectively, edge) of $G$ lies on chorded cycles of every length from 6 to $2n$. In 2023, Guo et al. proved that if $\mu_2(G) \geq n + 2$, then $G$ is vertex bipancyclic. In this paper, we show that if $G$ is 2-connected and $\mu_2(G) \geq n + 1$, then $G$ is chorded vertex bipancyclic, with the exception two special bipartite graphs. We also show that if $\mu_2(G) \geq n + 2$, then $G$ is chorded edge bipancyclic.

## 1 Terminology and introduction

In this paper, we consider finite, simple and connected graphs. For the terminology not defined in this paper, the reader is referred to [3]. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For $u \in V(G)$, we denote by $d_G(u)$ or $d(u)$ the *degree* of $u$, and $N_G(u)$ or $N(u)$ the set of *neighbors* of $u$ in $G$; for a subgraph $H$ in $G$, let $N_H(u) = N_G(u) \cap V(H)$ and $d_H(u) = |N_H(u)|$. For any two vertices $x$ and $y$, let $e(x, y)$ be the binary indicator of the edge $xy$, where $e(x, y) = 1$ if $xy$ is an edge of $G$, and $e(x, y) = 0$ otherwise. For an integer $n$, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. For $u, v \in V(G)$, we denote by $\mathrm{dist}(u, v)$ the *distance* between $u$ and $v$, that is, the length of the shortest path connecting $u$ and $v$.

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We consider each cycle $C$ to be equipped with a clockwise direction. Let $C = u_1 \dots u_m u_1$ be a cycle in a graph $G$. For every vertex $u_i$ of $C$, $u_{i+1}$, also denoted by $u_i^+$, is the *successor* of $u_i$, and $u_{i-1}$, also denoted by $u_i^-$, is the *predecessor* of $u_i$. Moreover, we let $u_i^{+t} = u_{i+t}$ and $u_i^{-t} = u_{i-t}$. For $i \neq j$, the path $u_i u_{i+1} \dots u_j$ is denoted by $u_i \overrightarrow{C} u_j$; while the path $u_i u_{i-1} \dots u_j$ is denoted by $u_i \overleftarrow{C} u_j$. Here, the subscripts are taken modulo $m$.

Let $C$ be a cycle in a bipartite graph $G$, and let $y_1, y_2, y_3$ be three distinct vertices lying in the same partite set of $V(C)$, appearing in the cyclic order $y_1, y_2, y_3$ along $C$. We say that $y_1$ and $y_2$ *cross* at $y_3$ if $y_1 y_3^- \in E(G)$ and $y_2 y_3^+ \in E(G)$.

For a non-complete graph $G$, we define

$$\sigma_2(G) = \min\{d(x) + d(y) : x, y \in V(G), xy \notin E(G)\};$$

$$\mu_2(G) = \min\{d(x) + d(y) : x, y \in V(G),\ \mathrm{dist}(x, y) = 2\};$$

otherwise, let $\sigma_2(G) = \mu_2(G) = +\infty$.

For a non-complete bipartite graph $G = G[X, Y]$, we define

$$\sigma_{1,1}(G) = \min\{d(x) + d(y) : x \in X, y \in Y, xy \notin E(G)\},$$

otherwise, let $\sigma_{1,1}(G) = +\infty$.

The study of cycles in graphs is a rich and important area. A graph $G$ is hamiltonian if it has a cycle that includes every vertex of $G$. One of the earliest sufficient conditions for a graph to be hamiltonian is one due to Ore [10].

**Theorem 1.1.** *(Ore, [10]) Let $G$ be a graph of order $n$ with $n \geq 3$. If $\sigma_2(G) \geq n$, then $G$ is hamiltonian.*

In 1963, Moon and Moser considered a bipartite version of Ore's theorem (Theorem 1.1).

**Theorem 1.2.** *(Moon and Moser, [9]) Let $G$ be a balanced bipartite graph of order $2n$ with $n \geq 2$. If $\sigma_{1,1}(G) \geq n + 1$, then $G$ is hamiltonian.*

In 2023, the first author of this paper and Zhou [12] proposed a $\mu_2$-type degree condition for a balanced bipartite graph to be hamiltonian.

**Theorem 1.3.** *(Wang and Zhou, [12]) Let $G$ be a 2-connected balanced bipartite graph of order $2n$ with $n \geq 3$. If $\mu_2(G) \geq n + 1$, then $G$ is hamiltonian.*

In [12], the authors have shown that for the existence of hamiltonian cycles, the condition $\sigma_{1,1}(G) \geq n+1$ in Theorem 1.2 and the condition $\mu_2(G) \geq n+1$ in Theorem 1.3 are independent. In Section 2, we shall show that the bipartite graph satisfying the conditions of Theorem 1.2 or 1.3 contains at least $\frac{n(n+1)}{2}$ edges.

A cycle of order $k$ is called a $k$-cycle, denoted by $C_k$. In [2], Bondy introduced the notion of pancyclicity in graphs. A graph $G$ of order $n$ with $n \geq 3$ is pancyclic if $G$ contains a $k$-cycle for every integer $k$ with $3 \leq k \leq n$.

**Theorem 1.4.** *(Bondy, [2]) If $G$ is a graph of order $n$ with $n \geq 3$ and $\sigma_2(G) \geq n$, then either $G$ is pancyclic or it is isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$. Consequently, if $\sigma_2(G) \geq n+1$, $G$ is pancyclic.*

A balanced bipartite graph $G$ of order $2n$ is bipancyclic if it contains a $2k$-cycle for every integer $k$ with $2 \leq k \leq n$. In 1999, Bagga and Varma obtained the following bipancyclic result about the $\sigma_{1,1}$-type degree condition in a bipartite graph.

**Theorem 1.5.** *(Bagga and Varma, [1]) If a balanced bipartite graph $G$ of order $2n$ with $n \geq 4$ satisfies $\sigma_{1,1}(G) \geq n+1$, then $G$ is bipancyclic.*

Recently, the first author of this paper et al. in [11] completely characterized the extremal graph of Theorem 1.5. Given integers $n$ and $t$ with $n \geq 2$ and $1 \leq t \leq n-1$, we define $H_{t,n-t}$ to be a balanced bipartite graph formed from $K_{t,t} \cup K_{n-t,n-t}$ by selecting one partite set of each graph and adding all possible edges between these sets.

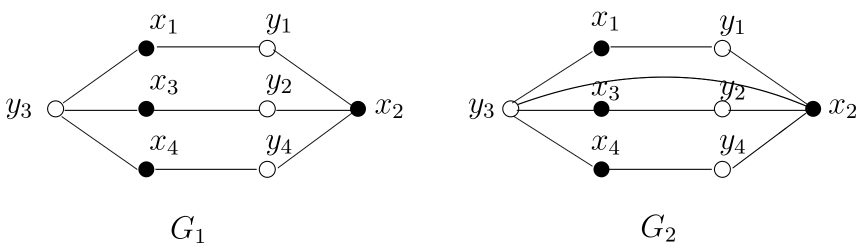


Figure 1: The balanced bipartite graphs $G_1$ and $G_2$.

**Theorem 1.6.** *(Wang et al., [11]) Let $G$ be a balanced bipartite graph of order $2n$ with $n \geq 3$. If $\sigma_{1,1}(G) \geq n$, then either $G$ is bipancyclic, or $G$ is isomorphic to one of $C_6$, $C_8$, $G_1$, $G_2$ (see Figure 1), or $K_{t,t} \cup K_{n-t,n-t} \subseteq G \subseteq H_{t,n-t}$ with $1 \leq t \leq n-1$.*

In 2023, the first author of this paper and Zhou proposed the $\mu_2$-type degree condition for a balanced bipartite graph to be bipancyclic.

**Theorem 1.7.** *(Wang and Zhou, [12]) Let $G$ be a 2-connected balanced bipartite graph of order $2n$ with $n \geq 4$. If $\mu_2(G) \geq n+1$, then $G$ is bipancyclic.*

In Section 2, we completely determine the extremal graph of Theorem 1.7. Specifically, we show that every 2-connected balanced bipartite graph $G$ of order $2n$ with $n \geq 3$ and $\mu_2(G) \geq n$ is either bipancyclic or isomorphic to one of $G_1$, $G_2$, $C_6$, $C_8$ or satisfies $K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}} \subseteq G \subseteq H_{\frac{n}{2},\frac{n}{2}}$.

A balanced bipartite graph $G$ of order $2n$ is vertex (respectively, edge) bipancyclic if every vertex (respectively, edge) is contained on a cycle of each length from 4 to $2n$ in $G$. Clearly, every vertex (respectively, edge) bipancyclic bipartite graph is necessarily bipancyclic. However, establishing vertex (respectively, edge) bipancyclicity for bipartite graphs presents significantly greater challenges in general. In 2023, Guo et al. obtained the following result concerning vertex bipancyclicity under the $\mu_2$-type degree condition.

**Theorem 1.8.** *(Guo et al., [7]) Let $G$ be a connected balanced bipartite graph of order $2n$ with $n \geq 2$. If $\mu_2(G) \geq n+2$, then $G$ is vertex bipancyclic.*

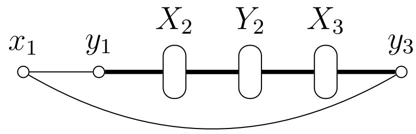


Figure 2: The balanced bipartite graph $G_3$. Thick lines represent all possible edges.

Given an integer $n \geq 3$, let $G_3 = G_3[X, Y]$ be a balanced bipartite graph of order $2n$ such that $X = \{x_1\} \cup X_2 \cup X_3$ and $Y = \{y_1\} \cup Y_2 \cup \{y_3\}$. Here $|X_2| + |X_3| = n-1$ and $|Y_2| = n-2$. The edge set $E(G_3)$ consists of the edges $x_i y_i$, $y_i x_{i+1}$ for all $i \in [3]$, where $x_2 \in X_2$, $x_3 \in X_3$ and $y_2 \in Y_2$ (noting $x_4 = x_1$). See Figure 2. It is important to note that $\mu_2(G_3) = n+1$, $x_1$ does not lie on any 4-cycle, and every vertex in $G_3$ lies on cycles of each length from 6 to $2n$.

In Section 3, we prove that every 2-connected balanced bipartite graph $G$ of order $2n$ with $n \geq 3$ and $\mu_2(G) \geq n+1$ is vertex bipancyclic, unless $G$ is isomorphic to the balanced bipartite graph $G_3$. Our approach differs from that of Guo et al. [7] and yields a significantly shorter proof.

Quite recently, the concept of chorded pancyclicity has been introduced by Cream et al. [5]. A chord of a cycle is an edge joining two non-consecutive vertices of the cycle and a chorded cycle is a cycle containing at least one chord. Note that a 4-cycle in a bipartite graph has no chord. A balanced bipartite graph $G$ of order $2n$ with $n \geq 3$ is called chorded bipancyclic if it contains a chorded $2k$-cycle for every integer $k$ with $3 \leq k \leq n$. A balanced bipartite graph of order $2n$ with $n \geq 3$ is called chorded vertex (respectively, edge) $2\ell$-bipancyclic if every vertex (respectively, edge) of $G$ lies on a chorded $2k$-cycle for every integer $k$ satisfying $3 \leq \ell \leq k \leq n$. In particular, when $\ell = 3$, we simply say that $G$ is chorded vertex (respectively, edge) bipancyclic. A survey of results and problems on chorded cycles can be found in [6].

In Section 3, we prove that every 2-connected balanced bipartite graph $G$ of order $2n \geq 6$ with $\mu_2(G) \geq n+1$ is chorded vertex bipancyclic, unless it is isomorphic to one of $G_3$ and $G_4$ (see Figures 2 and 3). As an immediate consequence, we show that every connected balanced bipartite graph $G$ of order $2n \geq 6$ with $\mu_2(G) \geq n+2$ is chorded vertex bipancyclic.

In Section 4, we prove that every connected balanced bipartite graph $G$ of order $2n \geq 6$ with $\mu_2(G) \geq n+2$ is chorded edge bipancyclic. Also, we give an example to show that the lower bound is sharp.

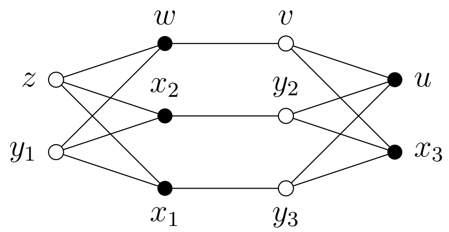


Figure 3: The balanced bipartite graph $G_4$. Every vertex in $G_4$ lies on chorded cycles of lengths 8 and 10, but does not lie on any chorded 6-cycle.

## 2 Edge number and the extremal graph of Theorem 1.7

In this section, we first prove that the balanced bipartite graph $G$ satisfying $\sigma_{1,1}(G) \geq n+1$ or $\mu_2(G) \geq n+1$ contains at least $\frac{n(n+1)}{2}$ edges.

**Theorem 2.1.** *Let $G$ be a balanced bipartite graph of order $2n$ with $n \geq 3$. If $\sigma_{1,1}(G) \geq n+1$, then $|E(G)| \geq \frac{n(n+1)}{2}$.*

*Proof.* Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two partite sets of $G$ and let $I = \{(i,j) : x_i y_j \notin E(G)\}$. Then $d(x_i) + d(y_j) \geq n+1$ for each $(i,j) \in I$. Denote $|E(G)| = m$. Hence,

$$\begin{aligned}
&\sum_{(i,j)\in I} (d(x_i) + d(y_j)) \geq (n+1)(n^2 - m) \\
\Leftrightarrow &\sum_{i=1}^{n} d(x_i)(n - d(x_i)) + \sum_{j=1}^{n} d(y_j)(n - d(y_j)) \geq (n+1)(n^2 - m) \\
\Leftrightarrow &\sum_{i=1}^{n} d^2(x_i) + \sum_{j=1}^{n} d^2(y_j) \leq 2mn - (n+1)(n^2 - m)
\end{aligned}$$

By the Cauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^{n} d(x_i)\right)^2 \leq n \sum_{i=1}^{n} d^2(x_i) \text{ and } \left(\sum_{i=1}^{n} d(y_i)\right)^2 \leq n \sum_{i=1}^{n} d^2(y_i).$$

This observation and the fact that $\sum\limits_{i=1}^{n} d(x_i) = \sum\limits_{i=1}^{n} d(y_i) = m$, imply

$$\sum_{i=1}^{n} d^2(x_i) \geq \frac{m^2}{n} \text{ and } \sum_{i=1}^{n} d^2(y_i) \geq \frac{m^2}{n}.$$

Then

$$\begin{aligned}
&\frac{2m^2}{n} \leq 2mn - (n+1)(n^2 - m) \\
\Leftrightarrow\ & 2m^2 \leq 2mn^2 - (n^2 + n)(n^2 - m) \\
\Leftrightarrow\ & 2m^2 - 2mn^2 + (n^2 + n)(n^2 - m) \leq 0 \\
\Leftrightarrow\ & 2m(m - n^2) - (n^2 + n)(m - n^2) \leq 0 \\
\Leftrightarrow\ & (2m - n^2 - n)(m - n^2) \leq 0.
\end{aligned}$$

This together with $m \leq n^2$ implies $m \geq \frac{n(n+1)}{2}$. $\square$

**Lemma 2.2.** *Given integers $n, t$ with $n \geq 3$ and $t \geq 4$, let $G$ be a hamiltonian balanced bipartite graph of order $2n$. If $\mu_2(G) \geq t$, then $|E(G)| \geq \frac{nt}{2}$.*

*Proof.* Let $X$ and $Y$ be two partite sets of $G$ and let $C = x_1y_1x_2y_2 \ldots x_ny_nx_1$, where $x_i \in X$ and $y_i \in Y$ with $i \in [n]$, be a hamiltonian cycle. From now on, the subscripts are taken modulo $n$. Note that for any $i \in [n]$, $\text{dist}(x_i, x_{i+1}) = 2$. By the hypothesis of this lemma, $d(x_i) + d(x_{i+1}) \geq \mu_2(G) \geq t$.

If $n$ is even, then let $n = 2r$. Then

$$|E(G)| = \sum_{i=1}^{2r} d(x_i) = \sum_{j=1}^{r} (d(x_{2j-1}) + d(x_{2j})) \geq rt = \frac{nt}{2}.$$

If $n$ is odd, then let $n = 2r + 1$. Since $d(x_{2r}) + d(x_{2r+1}) \geq t$, we may, without loss of generality, assume that $d(x_{2r+1}) \geq \frac{t}{2}$. Then

$$\begin{aligned} |E(G)| &= \sum_{i=1}^{2r+1} d(x_i) = \sum_{j=1}^{r} (d(x_{2j-1}) + d(x_{2j})) + d(x_{2r+1}) \\ &\geq rt + \frac{t}{2} = \frac{(n-1)t}{2} + \frac{t}{2} = \frac{nt}{2}. \end{aligned}$$

This establishes the lemma. $\square$

From Lemma 2.2 and Theorem 1.3, we can obtain the following result.

**Theorem 2.3.** *Let $G$ be a 2-connected balanced bipartite graph of order $2n$ with $n \geq 3$. If $\mu_2(G) \geq n + 1$, then $|E(G)| \geq \frac{n(n+1)}{2}$.*

Next, we shall use the following two theorems to prove Theorem 2.6.

**Theorem 2.4.** *(Wang and Zhou, [12]) If $G$ is a 2-connected non-hamiltonian balanced bipartite graph of order $2n$ with $n \geq 3$ and $\mu_2(G) = n$, then either $G$ is one of $G_1$ and $G_2$, or $K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}} \subseteq G \subseteq H_{\frac{n}{2},\frac{n}{2}}$.*

**Theorem 2.5.** *(Hu, [8]) Let $G$ be a hamiltonian balanced bipartite graph of order $2n$ with $n \geq 4$. If $|E(G)| > \frac{n(n-1)}{2} + 2$, then $G$ is bipancyclic.*

**Theorem 2.6.** *Let $G$ be a 2-connected balanced bipartite graph of order $2n$ with $n \geq 3$. If $\mu_2(G) \geq n$, then either $G$ is bipancyclic or it is isomorphic to one of $G_1$, $G_2$, $C_6$, $C_8$, or $K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}} \subseteq G \subseteq H_{\frac{n}{2},\frac{n}{2}}$.*

*Proof.* Let $X$ and $Y$ be two partite sets of $G$. If $G$ is non-hamiltonian, then by Theorems 1.3 and 2.4, $G$ is one of $G_1$ and $G_2$, or $K_{\frac{n}{2},\frac{n}{2}} \cup K_{\frac{n}{2},\frac{n}{2}} \subseteq G \subseteq H_{\frac{n}{2},\frac{n}{2}}$ and so we are done. Below assume that $G$ is hamiltonian. By Lemma 2.2, $|E(G)| \geq \frac{n^2}{2}$. If $n \geq 5$, then $\frac{n^2}{2} > \frac{n(n-1)}{2} + 2$. By Theorem 2.5, $G$ is bipancyclic. If $n = 3$ or 4, then it is easy to see that either $G$ is bipancyclic or it is isomorphic to $C_6$ or $C_8$. $\square$

## 3 Vertex bipancyclicity

**Theorem 3.1.** *Let $G$ be a 2-connected balanced bipartite graph of order $2n$ with $n \geq 3$. If $\mu_2(G) \geq n+1$, then either $G$ is vertex bipancyclic or $G$ is isomorphic to $G_3$ (see Figure 2).*

*Proof.* Let $X$ and $Y$ be two partite sets of $G$. Let $x$ be an arbitrary vertex of $G$. By Theorem 1.3, $G$ is hamiltonian. Therefore, $x$ lies on a cycle of length $2n$. Now select a $2k$-cycle $C = x_1y_1x_2y_2\dots x_ky_kx_1$, where $x_i \in X$ and $y_i \in Y$ for each $i \in [k]$, such that $k$ is minimal with respect to the property that $G$ contains cycles of every even length from $2k$ to $2n$ that include the vertex $x$. From now on, the subscripts are taken modulo $k$. The case $k = 2$ is trivial, so we may assume $3 \leq k \leq n$. The minimality of $k$ implies that $G$ contains no cycle of length $2k-2$ through $x$. Without loss of generality, assume that $x = x_1$. We now show this assumption leads to a contradiction. Observe that for each $i \in [k]$, $\mathrm{dist}(x_i, x_{i+1}) = 2$ and $\mathrm{dist}(y_i, y_{i+1}) = 2$. By the hypothesis of this theorem,

$$d(x_i) + d(x_{i+1}) \geq n+1 \text{ and } d(y_i) + d(y_{i+1}) \geq n+1. \tag{3.1}$$

Let $H = G - V(C)$ and define $H_X = V(H) \cap X$ and $H_Y = V(H) \cap Y$. Now we divide two cases to consider.

**Case 1.** $k = 3$.

The presence of any chord in $C$ would generate a 4-cycle that includes $x_1$. Therefore, $C$ is an induced 6-cycle. Since $x_1$ is not contained in any 4-cycle, it follows that $x_1$ and $x_2$, (respectively, $x_1$ and $x_3$, $y_1$ and $y_3$) have no common neighbors outside $C$, that is, $N_H(x_1) \cap N_H(x_2) = \emptyset$, $N_H(x_1) \cap N_H(x_3) = \emptyset$ and $N_H(y_1) \cap N_H(y_3) = \emptyset$. Hence, $n+1 \leq d(x_1) + d(x_2) \leq 4 + n - 3 = n+1$, which implies that $d(x_1) + d(x_2) = n+1$. Analogously, $d(x_1) + d(x_3) = n+1$ and $d(y_1) + d(y_3) = n+1$. These also imply that $N_H(x_2) = N_H(x_3)$, $N_H(x_1) \cup N_H(x_2) = H_Y$ and $N_H(y_1) \cup N_H(y_3) = H_X$.

We now show that $d(x_1) = 2$, that is, $N_H(x_1) = \emptyset$. Assume, for contrary, that $N_H(x_1) \neq \emptyset$. For any $v \in N_H(x_1)$, $v$ has at least one neighbor in $H_X$ as $G$ is 2-connected and $N_H(x_1) \cap N_H(x_j) = \emptyset$ for $j \in \{2, 3\}$. Recall that $N_H(y_1) \cup N_H(y_3) = H_X$. This configuration necessarily creates a 4-cycle containing $x_1$ (through either $y_1$ or $y_3$), which contradicts our initial assumption. Therefore, $N_H(x_1)$ must indeed be empty and so $d(x_1) = 2$. Combining this with $N_H(x_2) = N_H(x_3)$ and $N_H(x_1) \cup N_H(x_2) = H_Y$, we have that $N_H(x_2) = N_H(x_3) = H_Y$.

For any $u \in N_H(y_1)$, we have that $\mathrm{dist}(x_1, u) = 2$, which implies that $d(x_1) + d(u) \geq n+1$. Combining this with $d(x_1) = 2$, we have that $d(u) \geq n-1$. Since $N_H(y_1) \cap N_H(y_3) = \emptyset$, we have that $uy_3 \notin E(G)$. Thus $u$ and every vertex $Y \setminus \{y_3\}$ are adjacent. By the arbitrariness of $u$, every vertex in $N_H(y_1)$ and every vertex of $Y \setminus \{y_3\}$ are adjacent. Analogously, we can obtain that every vertex in $N_H(y_3)$ and every vertex of $Y \setminus \{y_1\}$ are adjacent. Let $X_2 = \{x_2\} \cup N_H(y_1)$, $Y_2 = \{y_2\} \cup H_Y$ and $X_3 = \{x_3\} \cup N_H(y_3)$. It is not difficult to see that $G$ is isomorphic to $G_3$.

**Case 2.** $k \geq 4$.

First we give the following two observations.

**Observation 1.** *For any $z \in V(C) \setminus \{x_k, y_k\}$, $zz^{+3} \notin E(G)$.*

In fact, if $zz^{+3} \in E(G)$, then $zz^{+3}\overrightarrow{C}z$ is a cycle of length $2k-2$ containing $x_1$, a contradiction.

**Observation 2.** *For any $u \in V(C) \setminus \{y_{k-1}, x_k, y_k\}$, $u$ and $u^{+4}$ have no common neighbors outside $C$.*

In fact, if $u$ and $u^{+4}$ have common neighbors outside of $C$, say $v$, then $uvu^{+4}\overrightarrow{C}u$ is a cycle of length $2k-2$ containing $x_1$, a contradiction.

**Case 2.1.** $k = 4$**.**

By Observation 1, $C$ can have at most two chords. The only possible chords are $x_4y_1$ and $y_4x_2$. Therefore, $d_C(x_1) + d_C(x_2) \le 5$ and $d_C(x_3) + d_C(x_4) \le 5$. By Observation 2,

$$N_H(x_1) \cap N_H(x_3) = \emptyset \text{ and } N_H(x_2) \cap N_H(x_4) = \emptyset, \tag{3.2}$$

which follow that $d_H(x_1) + d_H(x_3) \le n-4$ and $d_H(x_2) + d_H(x_4) \le n-4$. Thus,

$$\begin{aligned} 2(n+1) &\le d(x_1) + d(x_2) + d(x_3) + d(x_4) \\ &= d_C(x_1) + d_C(x_2) + d_C(x_3) + d_C(x_4) \\ &\quad + d_H(x_1) + d_H(x_3) + d_H(x_2) + d_H(x_4) \\ &\le 10 + 2(n-4) = 2(n+1). \end{aligned}$$

Hence equality hold everywhere, in particular,

$$d(x_1) + d(x_2) + d(x_3) + d(x_4) = 2(n+1), \tag{3.3}$$

$$d_H(x_1) + d_H(x_3) = d_H(x_2) + d_H(x_4) = n-4, \tag{3.4}$$

$$d_C(x_1) + d_C(x_2) = d_C(x_3) + d_C(x_4) = 5. \tag{3.5}$$

By (3.5), $x_4y_1 \in E(G)$ and $y_4x_2 \in E(G)$, that is, $d_C(x_1) = d_C(x_3) = 2$ and $d_C(x_2) = d_C(x_4) = 3$. By (3.1) and (3.3), we have that $d(x_i) + d(x_{i+1}) = n+1$ for every $i \in [4]$, which implies $d(x_1) = d(x_3)$. This together with $d_C(x_1) = d_C(x_3)$ implies $d_H(x_1) = d_H(x_3)$. From this with (3.4) we have that $d_H(x_1) = \frac{n-4}{2}$. If $n = 4$, then $d(y_2) + d(y_3) = d_C(y_2) + d_C(y_3) = 4$, a contradiction to (3.1). Thus, $n \ge 5$ and moreover $d_H(x_1) > 0$.

Now we show that $N_H(x_1) \cap N_H(x_2) = \emptyset$ and $N_H(x_1) \cap N_H(x_4) = \emptyset$. In fact, if $x_1$ and $x_2$ have common neighbors outside $C$, say $z$, then $x_1zx_2y_4x_4y_1x_1$ is a cycle of length $2k-2$ containing $x_1$, a contradiction. Thus $N_H(x_1) \cap N_H(x_2) = \emptyset$. Analogously, we can show that $N_H(x_1) \cap N_H(x_4) = \emptyset$. By (3.2) and (3.4), we have that $N_H(x_2) \cup N_H(x_4) = H_Y$. Thus $d_H(x_1) = 0$, a contradiction to $d_H(x_1) > 0$.

**Case 2.2.** $k \ge 5$**.**

First we show that $d_C(y_1) + d_C(y_2) \le k+1$ and $d_C(y_3) + d_C(y_4) \le k$. Note that for any $i \in [k-1]$ and $j \in [k] \setminus \{i, i+1\}$, $y_i$ and $y_{i+1}$ does not cross at $y_j$, that is, $e(y_i, x_j) + e(y_{i+1}, x_{j+1}) \le 1$. Indeed, if $y_i$ and $y_{i+1}$ cross at $y_j$, then $y_ix_j \in E(G)$

and $y_{i+1}x_{j+1} \in E(G)$. It follows that $y_i x_j \overleftarrow{C} y_{i+1} x_{j+1} \overrightarrow{C} y_i$ is a cycle of length $2k-2$ containing $x_1$, a contradiction. Thus,

$$\begin{aligned} &\text{for any } j \in [k] \setminus \{1,2,3\},\ e(y_1,x_j) + e(y_2,x_{j+1}) \leq 1; \\ &\text{for any } r \in [k] \setminus \{2,3,4,5\},\ e(y_3,x_r) + e(y_4,x_{r+1}) \leq 1. \end{aligned}$$

Note that

$$\begin{aligned} &e(y_1,x_1) + e(y_2,x_2) = e(y_1,x_2) + e(y_2,x_3) = 2; \\ &e(y_1,x_3) + e(y_2,x_4) = 0; \\ &e(y_3,x_2) + e(y_4,x_3) = e(y_3,x_5) + e(y_4,x_6) = 0; \\ &e(y_3,x_3) + e(y_4,x_4) = e(y_3,x_4) + e(y_4,x_5) = 2. \end{aligned}$$

Therefore,

$$\begin{aligned} d_C(y_1) + d_C(y_2) &= \sum_{j=1}^{k} (e(y_1,x_j) + e(y_2,x_{j+1})) \\ &= \sum_{j=1}^{3} (e(y_1,x_j) + e(y_2,x_{j+1})) + \sum_{j=4}^{k} (e(y_1,x_j) + e(y_2,x_{j+1})) \\ &\leq 4 + (k-3) = k+1 \end{aligned}$$

and

$$d_C(y_3) + d_C(y_4) = \sum_{j=1}^{k} (e(y_3,x_j) + e(y_4,x_{j+1})) \leq 4 + (k-4) = k.$$

Thus we have shown that $d_C(y_1) + d_C(y_2) \leq k+1$ and $d_C(y_3) + d_C(y_4) \leq k$.

By Observation 2, $N_H(y_1) \cap N_H(y_3) = \emptyset$ and $N_H(y_2) \cap N_H(y_4) = \emptyset$, which imply that $d_H(y_1) + d_H(y_3) \leq n-k$ and $d_H(y_2) + d_C(y_4) \leq n-k$. Thus,

$$\begin{aligned} 2(n+1) &\leq d(y_1) + d(y_2) + d(y_3) + d(y_4) \\ &= d_C(y_1) + d_C(y_2) + d_C(y_3) + d_C(y_4) \\ &\quad + d_H(y_1) + d_H(y_3) + d_H(y_2) + d_H(y_4) \\ &\leq 2k + 1 + 2(n-k) = 2n+1, \end{aligned}$$

a contradiction. □

By Theorem 3.1, we can obtain the following corollary.

**Corollary 3.2.** *Let $G$ be a 2-connected balanced bipartite graph of order $2n$ with $n \geq 3$. If $\mu_2(G) \geq n+1$, then $G$ is vertex 6-bipancyclic.*

**Theorem 3.3.** *Let $G$ be a 2-connected balanced bipartite graph of order $2n$ with $n \geq 3$. If $\mu_2(G) \geq n+1$, then either $G$ is chorded vertex bipancyclic or $G$ is isomorphic to one of $G_3$ and $G_4$ (see Figures 2 and 3).*

*Proof.* Let $X$ and $Y$ be two partite sets of $G$ and let $b$ be an arbitrary vertex of $G$. By Corollary 3.2, $G$ is vertex 6-bipancyclic. This means that $b$ lies on cycles of every length from 6 to $2n$. In particular, $b$ lies on a hamiltonian cycle $C_{2n}$. If $n = 3$, then either $G$ is chorded vertex bipancyclic or $G$ is isomorphic to $G_3$. Now assume that $n \geq 4$. Since $\mu_2(G) \geq n + 1 \geq 5$, there exists a vertex on $C_{2n}$ whose degree is at least three. Thus, $C_{2n}$ contains at least one chord and the vertex $b$ is in a chorded $2n$-cycle.

Suppose that there exists an integer $k \in \{3, \ldots, n-1\}$ such that $b$ does not lie on any chorded $2k$-cycle. We will show that under this assumption, $G$ must be isomorphic to one of $G_3$ and $G_4$. Let $C = x_1y_1x_2y_2 \ldots x_ky_kx_1$ be a $2k$-cycle containing $b$, where $x_i \in X$ and $y_i \in Y$ for each $i \in [k]$. Since $b$ belongs to no any chorded $2k$-cycle, $C$ must be an induced cycle in $G$. Without loss of generality, we fix $b = x_1$. Define $H = G - V(C)$, $H_X = V(H) \cap X$ and $H_Y = V(H) \cap Y$. From now on, the subscripts are taken modulo $k$.

Assume that $k \geq 4$. Let $L = \{x_1, x_2, x_3, x_4\}$. If there exists a vertex $z \in H_Y$ such that $|N_L(z)| \geq 3$, then $x_1$ lies on a chorded $2k$-cycle, a contradiction. Now suppose that $|N_L(z)| \leq 2$ for each $z \in H_Y$. Note that $\text{dist}(x_1, x_2) = 2$ and $\text{dist}(x_3, x_4) = 2$. By the hypothesis of this theorem, $d(x_1) + d(x_2) \geq n + 1$ and $d(x_3) + d(x_4) \geq n + 1$. Thus,

$$2(n+1) \leq d(x_1) + d(x_2) + d(x_3) + d(x_4) \leq 8 + 2(n-k),$$

which yields that $k \leq 3$, a contradiction to $k \geq 4$.

Now assume that $k = 3$. To complete the proof, first we give the following several claims.

**Claim 1.** *For any $z \in V(H)$, $d_C(z) \leq 2$.*

*Proof.* Without loss of generality, assume that $z \in H_X$ and $d_C(z) = 3$. Then $x_1y_1zy_2x_3y_3x_1$ is a 6-cycle containing $x_1$ with the chord $zy_3$, a contradiction to the choice of $x_1$. □

**Claim 2.** *Let $Q = u_1v_1u_2v_2u_1$, where $u_i \in X$ and $v_i \in Y$ for $i \in [2]$, be a 4-cycle containing $x_1$ and $u \in X, v \in Y$ be two vertices in $G - V(Q)$. If $u$ and $v$ are adjacent, then $N_Q(u) = \emptyset$ or $N_Q(v) = \emptyset$. Moreover, if $N_Q(u) \neq \emptyset$, say $uv_i \in E(G)$, then $N_{G-V(Q)}(u) \cap N_{G-V(Q)}(u_i) = N_{G-V(Q)}(u) \cap N_{G-V(Q)}(u_{i+1}) = \emptyset$.*

*Proof.* Assume that $u$ and $v$ are adjacent. Suppose, on the contrary, that $N_Q(u) \neq \emptyset$ and $N_Q(v) \neq \emptyset$, then by symmetry, assume that $uv_1 \in E(G)$ and $vu_1 \in E(G)$. Then $u_1vuv_1u_2v_2u_1$ is a 6-cycle containing $x_1$ with the chord $u_1v_1$, a contradiction. Thus, $N_Q(u) = \emptyset$ or $N_Q(v) = \emptyset$. By the above argument, if $uv_i \in E(G)$ for some $i \in [2]$, then $N_{G-V(Q)}(u) \cap N_{G-V(Q)}(u_i) = N_{G-V(Q)}(u) \cap N_{G-V(Q)}(u_{i+1}) = \emptyset$. □

Since $G$ is 2-connected, the degree of every vertex in $G$ is more than or equal to two.

**Claim 3.** *For any $w \in V(G)$, let $z_1, z_2 \in N(w)$ be two different vertices. Then $d_H(z_1) + d_H(z_2) \geq n - 3$. Moreover, if $N_H(z_1) \cap N_H(z_2) = \emptyset$, then $d_H(z_1) + d_H(z_2) = n - 3$ and $d_C(z_1) = d_C(z_2) = 2$.*

*Proof.* Since $\mathrm{dist}(z_1, z_2) = 2$, the hypothesis of this theorem implies $d(z_1) + d(z_2) \geq n + 1$. By Claim 1 and the fact that $C$ is an induced cycle, we have $d_C(z_1) \leq 2$ and $d_C(z_2) \leq 2$. Consequently, we deduce

$$\begin{aligned} d_H(z_1) + d_H(z_2) &= d(z_1) + d(z_2) - d_C(z_1) - d_C(z_2) \\ &\geq n + 1 - 4 = n - 3. \end{aligned}$$

If $N_H(z_1) \cap N_H(z_2) = \emptyset$, then $d_H(z_1) + d_H(z_2) \leq n - 3$. Therefore, we conclude the equality $d_H(z_1) + d_H(z_2) = n - 3$ and $d_C(z_1) = d_C(z_2) = 2$. □

**Claim 4.** *For any $w \in N_H(x_1) \cup N_H(y_1) \cup N_H(y_3)$, if $d_C(w) = 1$, denoted by $N_C(w) = \{u\}$, then the following hold:*

$$\begin{aligned} &N_H(w) \cap N_H(u^+) = N_H(w) \cap N_H(u^-), \\ &|N_H(w) \cap N_H(u^+)| = |N_H(w) \cap N_H(u^-)| = 1, \\ &|N_H(w)| + |N_H(u^+)| = |N_H(w)| + |N_H(u^-)| = n - 2. \end{aligned}$$

*Proof.* Clearly, $x_1 \in \{u, u^+\}$ or $x_1 \in \{u, u^-\}$. Note that $\{w, u^+, u^-\} \subseteq N(u)$. Then

$$\begin{aligned} d_H(w) + d_H(u^+) &= d(w) + d(u^+) - d_C(w) - d_C(u^+) \\ &\geq (n + 1) - 3 = n - 2. \end{aligned} \tag{3.6}$$

Similarly, $d_H(w) + d_H(u^-) \geq n - 2$. According to Claim 3, we can deduce that

$$N_H(w) \cap N_H(u^+) \neq \emptyset \text{ and } N_H(w) \cap N_H(u^-) \neq \emptyset.$$

Let $a \in N_H(w) \cap N_H(u^+)$ and $b \in N_H(w) \cap N_H(u^-)$. Then $a = b$, as otherwise $bwau^+uu^-b$ is a 6-cycle containing $x_1$ with the chord $wu$, a contradiction. This implies that $N_H(w) \cap N_H(u^+) = N_H(w) \cap N_H(u^-)$ and moreover $|N_H(w) \cap N_H(u^+)| = |N_H(w) \cap N_H(u^-)| = 1$. Then

$$\begin{aligned} |N_H(w)| + |N_H(u^+)| &= |N_H(w) \cup N_H(u^+)| + |N_H(w) \cap N_H(u^+)| \\ &\leq (n - 3) + 1 = n - 2. \end{aligned}$$

Combining this with (3.6), we have $|N_H(w)| + |N_H(u^+)| = n - 2$. Similarly, $|N_H(w)| + |N_H(u^-)| = n - 2$. □

By Claim 3, for each $i \in [3]$,

$$d_H(x_i) + d_H(x_{i+1}) \geq n - 3 \text{ and } d_H(y_i) + d_H(y_{i+1}) \geq n - 3. \tag{3.7}$$

In particular, $d_H(y_1) + d_H(y_3) \geq n - 3$. Thus $d_H(y_1) \geq \frac{n-3}{2}$ or $d_H(y_3) \geq \frac{n-3}{2}$. Without loss of generality, assume that $d_H(y_1) \geq \frac{n-3}{2}$.

Now we divide into two cases to consider.

**Case 1.** $N_H(x_1) \cap N_H(x_2) \neq \emptyset$**.**

Let $z \in N_H(x_1) \cap N_H(x_2)$ be arbitrary. This together with Claim 1 implies $zx_3 \notin E(G)$. Applying the cycle $x_1zx_2y_1x_1$ and the edges $x_2y_2$ and $x_1y_3$ to Claim 2, we have that

$$\begin{aligned} N_H(y_2) \cap N_H(z) = \emptyset,\ N_H(y_2) \cap N_H(y_1) = \emptyset, \\ N_H(y_3) \cap N_H(z) = \emptyset,\ N_H(y_3) \cap N_H(y_1) = \emptyset. \end{aligned} \tag{3.8}$$

Combining these with Claim 3, we have that

$$\begin{aligned} d_H(y_2) + d_H(z) = n-3,\ d_H(y_2) + d_H(y_1) = n-3, \\ d_H(y_3) + d_H(z) = n-3,\ d_H(y_3) + d_H(y_1) = n-3. \end{aligned}$$

These imply that

$$N_H(y_1) = N_H(z), N_H(y_2) = N_H(y_3) \ \text{ and } \ N_H(y_1) \cup N_H(y_2) = H_X. \tag{3.9}$$

In this case, up to now, see Figure 4. Recall that

$$d_H(y_1) \geq \frac{n-3}{2} \text{ and } d_H(y_2) + d_H(y_3) \geq n-3.$$

Then $n$ is odd, and

$$d_H(y_1) = d_H(z) = \frac{n-3}{2} \text{ and } d_H(y_2) = d_H(y_3) = \frac{n-3}{2}.$$

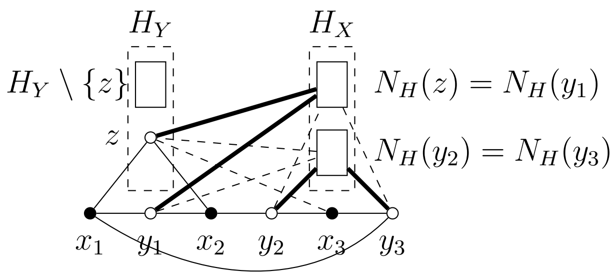


Figure 4: The thick lines represent all possible edges and the dashed lines represent no edge.

Now we shall show that $d_H(x_1) = 1$. Suppose, on the contrary, that $d_H(x_1) \geq 2$. Let $z' \in N_H(x_1) \setminus \{z\}$. By Claim 3, $d_H(z') + d_H(z) \geq n-3$. Thus $d_H(z') \geq \frac{n-3}{2} \geq 1$. Let $u \in N_H(z')$ be arbitrary. By (3.8) and (3.9), we know that $N_H(y_1) \cap N_H(y_2) = \emptyset$ and $N_H(y_1) \cup N_H(y_2) = H_X$. Thus, $u \in N_H(y_1)$ or $u \in N_H(y_2)$. If $u \in N_H(y_1)$, then $x_1zx_2y_1uz'x_1$ is a 6-cycle containing $x_1$ with the chord $x_1y_1$, a contradiction. If $u \in N_H(y_2)$, then by (3.9), $u \in N_H(y_3)$. Note that $x_1z'uy_2x_3y_3x_1$ is a 6-cycle containing $x_1$ with the chord $uy_3$, a contradiction. Therefore, $d_H(x_1) = 1$, that is, $N_H(x_1) = \{z\}$.

Recall that $zx_3 \notin E(G)$ and so $N_H(x_1) \cap N_H(x_3) = \emptyset$. According to Claim 3, $d_H(x_1) + d_H(x_3) = n-3$. Thus, $N_H(x_3) = H_Y \setminus \{z\}$. Let $w \in N_H(z)$ be arbitrary. By (3.8) and (3.9), we know that $w \in N_H(y_1)$ and $w \notin N_H(y_2) = N_H(y_3)$. Thus

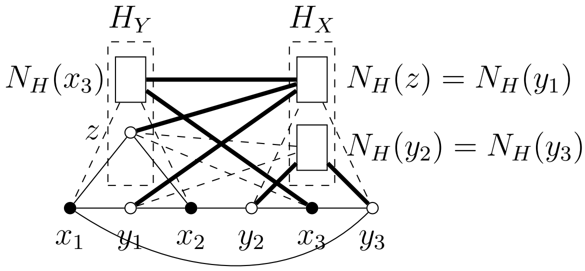


Figure 5: The thick lines represent all possible edges and the dashed lines represent no edge.

$d_C(w) = 1$. Combining this with Claim 4 and $wy_1 \in E(G)$, we have that $d_H(w) + d_H(x_1) = n - 2$. Since $d_H(x_1) = 1$, we have $d_H(w) = n - 3$. This means that $N_H(w) = H_Y$. Let $v \in H_Y \setminus \{z\}$ be arbitrary. Applying the cycle $x_1zx_2y_1x_1$ and the edge $wv$ to Claim 2, we have that $vx_2 \notin E(G)$. By the arbitrariness of $v$, $x_2$ and every vertex of $H_Y \setminus \{z\}$ are not adjacent. This means $N_H(x_2) = \{z\}$. In this case, up to now, see Figure 5. By (3.7), $d_H(x_1) + d_H(x_2) \geq n - 3$. It follows that $n \leq 5$. Since $n$ is odd and $n \geq 4$, we have that $n = 5$ and so $H_Y = \{v, z\}$.

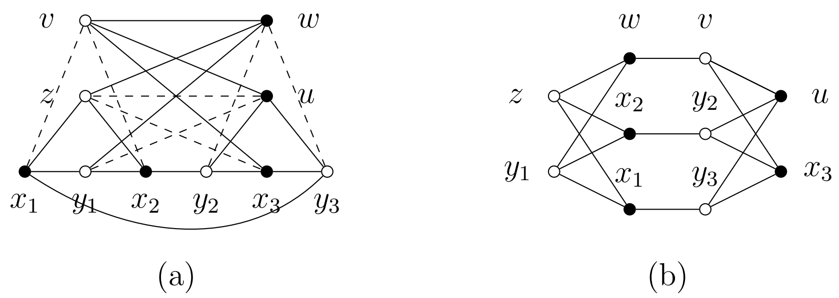


Figure 6: The dashed lines represent no edge.

Recalling that $N_H(x_1) = N_H(x_2) = \{z\}$ and $N_H(x_3) = H_Y \setminus \{z\} = \{v\}$, we have that $d_C(v) = 1$. By $\{v, y_2\} \subseteq N(x_3)$, we have that $d(v) + d(y_2) \geq n + 1$. Therefore,

$$\begin{aligned} d_H(v) + d_H(y_2) &= d(v) + d(y_2) - d_C(v) - d_C(y_2) \\ &\geq n + 1 - 3 = n - 2. \end{aligned}$$

Since $d_H(y_2) = \frac{n-3}{2} = 1$, we have $d_H(v) \geq n - 3$. This means that $N_H(v) = H_X$. Denote $N_H(y_2) = \{u\}$, that is, $N_H(y_3) = \{u\}$. Thus, $vu \in E(G)$. By $N_H(y_2) \cap N_H(z) = \emptyset$ in (3.8), we know that $w \neq u$ and $zu \notin E(G)$. We have shown that $y_1w, zw, zx_1, zx_2, wv, vu, vx_3, uy_2, uy_3 \in E(G)$ (see Figure 6(a)). Another drawing of the graph in Figure 6(a) is shown in Figure 6(b). It is not difficult to see that $G$ is isomorphic to the graph $G_4$ (see Figure 3).

**Case 2.** $N_H(x_1) \cap N_H(x_2) = \emptyset$.

If $N_H(x_1) \cap N_H(x_3) \neq \emptyset$, then similar to Case 1, we can show that $G$ is isomorphic

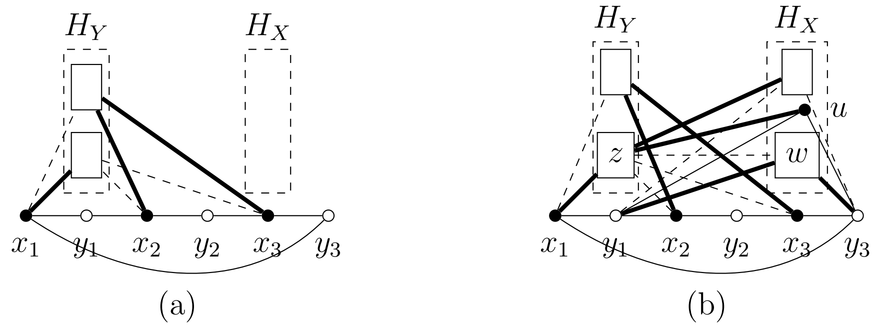


Figure 7: The thick lines represent all possible edges and the dashed lines represent no edge.

to the graph $G_4$. Now we assume that $N_H(x_1) \cap N_H(x_3) = \emptyset$. By Claim 3,

$$d_H(x_1) + d_H(x_2) = n - 3,\ d_H(x_1) + d_H(x_3) = n - 3,$$
$$d_H(x_2) + d_H(x_3) \geq n - 3,$$

which imply that

$$\begin{aligned} &N_H(x_2) = N_H(x_3),\ N_H(x_1) \cup N_H(x_2) = H_Y, \\ &d_H(x_2) = d_H(x_3) \geq \frac{n-3}{2},\ d_H(x_1) \leq \frac{n-3}{2}. \end{aligned} \tag{3.10}$$

In this case, up to now, see Figure 7(a). By (3.10) and $n \geq 4$, we have that

$$d_H(x_2) = d_H(x_3) \geq 1. \tag{3.11}$$

Assume that $N_H(x_1) \neq \emptyset$. By (3.11) and $N_H(x_1) \cap N_H(x_2) = \emptyset$, we have $n \geq 5$. Let $z \in N_H(x_1)$ be arbitrary. Since $N_H(x_1) \cap N_H(x_2) = \emptyset$ and $N_H(x_1) \cap N_H(x_3) = \emptyset$, we have that $zx_2, zx_3 \notin E(G)$. Thus $d_C(z) = 1$. By Claim 4,

$$N_H(z) \cap N_H(y_1) = N_H(z) \cap N_H(y_3),$$
$$|N_H(z) \cap N_H(y_1)| = |N_H(z) \cap N_H(y_3)| = 1,$$
$$|N_H(z)| + |N_H(y_1)| = |N_H(z)| + |N_H(y_3)| = n - 2.$$

Hence, $N_H(z) \cup N_H(y_1) = N_H(z) \cup N_H(y_3) = H_X$ and $N_H(y_1) = N_H(y_3)$. Denote $N_H(z) \cap N_H(y_1) = \{u\}$, that is, $N_H(z) \cap N_H(y_3) = \{u\}$. If $|N_H(y_1)| \geq 2$, then take $w \in N_H(y_1) \setminus \{u\}$. Then $N_H(y_1) = N_H(y_3)$ implies that $wy_3 \in E(G)$. Note that $x_1zuy_3wy_1x_1$ is a 6-cycle containing $x_1$ with the chord $x_1y_3$, a contradiction (see Figure 7(b)). Therefore, $|N_H(y_1)| = 1$. Consequently, we have that $N_H(z) = H_X$ and $|N_H(y_3)| = 1$. By Claim 3, $d_H(y_1) + d_H(y_3) \geq n - 3$. Then $n = 5$. Denote that $H_X \setminus \{u\} = \{a\}$ and $N_H(x_2) = \{v\}$. Applying the cycle $uzx_1y_1u$ and the edges $x_2v$ and $x_2y_2$ to Claim 2, we have that $uv, uy_2 \notin E(G)$. So $d(u) = 3$. Since $N_H(z) = H_X$, we have that $\{a, u\} \subseteq N(z)$. This implies that $d(a) + d(u) \geq n + 1 = 6$. So $d(a) \geq 3$. Then $av, ay_2 \in E(G)$. We have shown that $va, vx_2, vx_3, ay_2, az, zu, zx_1, uy_1, uy_3 \in$

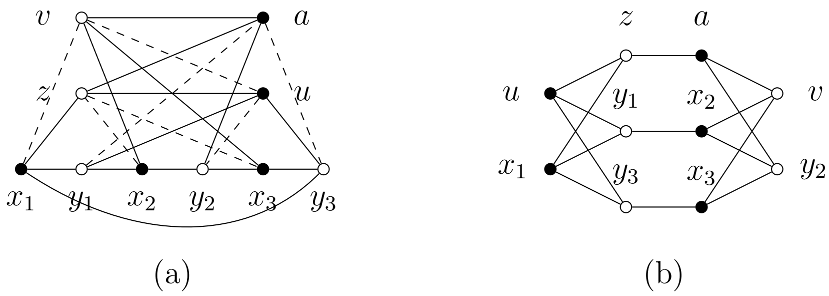


Figure 8: The dashed lines represent no edge.

$E(G)$ (see Figure 8(a)). Another drawing of the graph in Figure 8(a) is shown in Figure 8(b). It is easy to see that $G$ is isomorphic to the graph $G_4$ (see Figure 3).

Now assume that $N_H(x_1) = \emptyset$. By (3.10), $N_H(x_2) = N_H(x_3) = H_Y$. By (3.7) and $n \geq 4$, we have $d_H(y_1) + d_H(y_3) \geq n - 3 \geq 1$. For any $w \in N_H(y_1) \cup N_H(y_3)$, since $N_H(x_1) \cap N_H(w) = \emptyset$, and $\{x_1, w\} \subseteq N(y_1)$ or $\{x_1, w\} \subseteq N(y_3)$, by Claim 3, we have that $d_H(w) = n - 3$ and so $N_H(w) = H_Y$ and $d_C(w) = 2$. Note that $N_H(y_1) \cap N_H(y_3) = \emptyset$, as otherwise, let $u \in N_H(y_1) \cap N_H(y_3)$ and $v \in H_Y$, $x_1y_1uvx_3y_3x_1$ is a 6-cycle containing $x_1$ with the chord $uy_3$, a contradiction. This together with Claim 3 implies that $d_H(y_1) + d_H(y_3) = n - 3$ and so $N_H(y_1) \cup N_H(y_3) = H_X$. Moreover, $d_C(w) = 2$ implies that $wy_2 \in E(G)$. By the above argument, we have known that $N_H(x_2) = N_H(x_3) = H_Y$, $N_H(y_1) \cap N_H(y_3) = \emptyset$, $N_H(y_1) \cup N_H(y_3) = H_X$, $H$ is a complete bipartite graph and $y_2$ is adjacent to every vertex of $H_X$. Denote $X_2 = N_H(y_1) \cup \{x_2\}$, $Y_2 = H_Y \cup \{y_2\}$ and $X_3 = N_H(y_3) \cup \{x_3\}$. It is not difficult to see that $G$ is isomorphic to the graph $G_3$. □

By Theorem 3.3, we can obtain the following corollary.

**Corollary 3.4.** *Let $G$ be a 2-connected balanced bipartite graph of order $2n$ with $n \geq 4$. If $\mu_2(G) \geq n + 1$, then $G$ is chorded vertex 8-bipancyclic.*

**Corollary 3.5.** *Let $G$ be a connected balanced bipartite graph of order $2n$ with $n \geq 4$. If $\mu_2(G) \geq n + 2$, then $G$ is chorded vertex bipancyclic.*

*Proof.* By Theorem 1.8, we know that $G$ is 2-connected. Note that $\mu_2(G_3) = \mu_2(G_4) = n + 1$. By Theorem 3.3, $G$ is chorded vertex bipancyclic. □

## 4 Edge bipancyclicity

As a direct consequence of Theorem 11 in [4], we obtain the following theorem.

**Theorem 4.1.** *(Chiba et al., [4]) Let $G$ be a connected balanced bipartite graph of order $2n$ with $n \geq 3$. If $\mu_2(G) \geq n + 2$, then every edge of $G$ lies on a hamiltonian cycle.*

Now we extend Theorem 4.1 to edge bipancyclicity (Theorem 4.2) and provide an example demonstrating that the bound in Theorem 4.2 is tight.

**Theorem 4.2.** *Let $G$ be a connected balanced bipartite graph of order $2n$ with $n \geq 3$. If $\mu_2(G) \geq n+2$, then $G$ is edge bipancyclic.*

*Proof.* Let $X$ and $Y$ be two partite sets of $G$ and let $xy$ be an arbitrary edge of $G$, where $x \in X$ and $y \in Y$. By Theorem 4.1, the edge $xy$ lies on a hamiltonian cycle, which also implies that the degree of every vertex is greater or equal to two. If $d(x) = n$ or $d(y) = n$, then the edge $xy$ lies on cycles of every length from 4 to $2n$. Now assume that $d(x) \leq n-1$ and $d(y) \leq n-1$. This implies that for any $x' \in X \setminus \{x\}$ (respectively, $y' \in Y \setminus \{y\}$), if $\mathrm{dist}(x, x') = 2$ (respectively, $\mathrm{dist}(y, y') = 2$), then $d(x') \geq 3$ (respectively, $d(y') \geq 3$).

Observe that for any two vertices $u, v$ in $X$ or $Y$, if $\mathrm{dist}(u, v) = 2$, then $|N(u) \cap N(v)| \geq 2$. In fact, $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq (n+2) - n = 2$.

First we show that the edge $xy$ lies on a 4-cycle. Let $y' \in N(x) \setminus \{y\}$. Since $\mathrm{dist}(y, y') = 2$, we have that $|N(y) \cap N(y')| \geq 2$. Therefore, there exists a vertex $x' \in (N(y) \cap N(y')) \setminus \{x\}$. Note that $xyx'y'x$ is a 4-cycle containing the edge $xy$.

Now we show that the edge $xy$ lies on a 6-cycle. If $n = 3$, it is obvious. Assume that $n \geq 4$. According to the above argument, the edge $xy$ lies on a 4-cycle, denoted it by $xyx'y'x$. Since $\mathrm{dist}(x, x') = 2$ and $\mathrm{dist}(y, y') = 2$, we have that $d(x') \geq 3$ and $d(y') \geq 3$. Let $x'' \in N(y') \setminus \{x, x'\}$ and $y'' \in N(x') \setminus \{y, y'\}$. Since $\{x', x''\} \subseteq N(y')$ and $\{y', y''\} \subseteq N(x')$, we have $|N(x') \cap N(x'')| \geq 2$ and $|N(y') \cap N(y'')| \geq 2$. If $(N(x') \cap N(x'')) \setminus \{y, y'\} \neq \emptyset$, then let $y''' \in (N(x') \cap N(x'')) \setminus \{y, y'\}$. Note that $x''y'''x'yxy'x''$ is a 6-cycle containing the edge $xy$. Assume that $(N(x') \cap N(x'')) \setminus \{y, y'\} = \emptyset$. It follows that $x''y \in E(G)$. Analogously, we may assume that $y''x \in E(G)$. Then $x''y'x'y''xyx''$ is a 6-cycle containing the edge $xy$. Therefore, the edge $xy$ lies on a 6-cycle.

Assume, for contradiction, that there exists an integer $k$ such that the edge $xy$ lies on a $2k$-cycle $C = x_1y_1x_2y_2 \ldots x_ky_kx_1$, where $x_i \in X$ and $y_i \in Y$ for each $i \in [k]$, but does not lie on any $(2k-2)$-cycles. We now show this leads to a contradiction. Since the edge $xy$ is contained in both 4-cycles and 6-cycles, we conclude that $k \geq 5$. Without loss of generality, assume that $xy = x_1y_1$.

Observe that for each $i \in [k]$, $\mathrm{dist}(x_i, x_{i+1}) = 2$ and $\mathrm{dist}(y_i, y_{i+1}) = 2$. By the hypothesis of this theorem,

$$d(x_i) + d(x_{i+1}) \geq n+2 \text{ and } d(y_i) + d(y_{i+1}) \geq n+2.$$

Let $H = G - V(C)$ and define $H_X = V(H) \cap X$ and $H_Y = V(H) \cap Y$.

Since $G$ contains no cycle of length $2k-2$ through the edge $x_1y_1$, we have that for any $z \in V(C) \setminus \{x_k, y_k, x_1\}$, $zz^{+3} \notin E(G)$. In fact, if $zz^{+3} \in E(G)$, then $zz^{+3}\overrightarrow{C}z$ is a cycle of length $2k-2$ containing the edge $x_1y_1$, a contradiction. From this, we know that

$$\begin{aligned} &e(y_1, x_3) + e(y_2, x_4) = 0; \\ &e(y_3, x_2) + e(y_4, x_3) = e(y_3, x_5) + e(y_4, x_6) = 0. \end{aligned} \tag{4.1}$$

Note that for any $i \in [k-1]$ and $j \in [k] \setminus \{1, i, i+1\}$, $y_i$ and $y_{i+1}$ does not cross at $y_j$, that is, $e(y_i, x_j) + e(y_{i+1}, x_{j+1}) \leq 1$. Indeed, if $y_i$ and $y_{i+1}$ cross at $y_j$, then

$y_ix_j \in E(G)$ and $y_{i+1}x_{j+1} \in E(G)$. This follows that $y_ix_j\overleftarrow{C}y_{i+1}x_{j+1}\overrightarrow{C}y_i$ is a cycle of length $2k-2$ containing the edge $x_1y_1$, a contradiction. This follows that

$$\begin{aligned} &\text{for any } j \in [k] \setminus \{1,2,3\},\ e(y_1,x_j) + e(y_2,x_{j+1}) \le 1,\\ &\text{for any } j \in [k] \setminus \{1,2,3,4,5\},\ e(y_3,x_j) + e(y_4,x_{j+1}) \le 1. \end{aligned} \tag{4.2}$$

Note that

$$\begin{aligned} &e(y_1,x_1) + e(y_2,x_2) = e(y_1,x_2) + e(y_2,x_3) = 2;\\ &e(y_3,x_3) + e(y_4,x_4) = e(y_3,x_4) + e(y_4,x_5) = 2;\\ &e(y_3,x_1) + e(y_4,x_2) \le 2. \end{aligned} \tag{4.3}$$

Therefore, by (4.1)-(4.3),

$$\begin{aligned} d_C(y_1) + d_C(y_2) &= \sum_{j=1}^{k}(e(y_1,x_j) + e(y_2,x_{j+1}))\\ &= \sum_{j=1}^{3}(e(y_1,x_j) + e(y_2,x_{j+1})) + \sum_{j=4}^{k}(e(y_1,x_j) + e(y_2,x_{j+1}))\\ &\le 2+2+0+(k-3) = k+1 \end{aligned}$$

and

$$\begin{aligned} d_C(y_3) + d_C(y_4) &= \sum_{j=1}^{k}(e(y_3,x_j) + e(y_4,x_{j+1}))\\ &= \sum_{j=1}^{5}(e(y_3,x_j) + e(y_4,x_{j+1})) + \sum_{j=6}^{k}(e(y_3,x_j) + e(y_4,x_{j+1}))\\ &\le 2+0+2+2+0+(k-5) = k+1. \end{aligned}$$

Note that $y_1$ and $y_3$ (respectively, $y_2$ and $y_4$) have no common neighbors outside $C$, that is, $N_H(y_1) \cap N_H(y_3) = \emptyset$ (respectively, $N_H(y_2) \cap N_H(y_4) = \emptyset$). Indeed, if $N_H(y_1) \cap N_H(y_3) \ne \emptyset$, then, let $u \in N_H(y_1) \cap N_H(y_3)$, $x_1y_1uy_3\overrightarrow{C}x_1$ is a cycle of length $2k-2$ containing the edge $x_1y_1$, a contradiction. Thus, $N_H(y_1) \cap N_H(y_3) = \emptyset$. Analogously, $N_H(y_2) \cap N_H(y_4) = \emptyset$. Then

$$\begin{aligned} 2(n+2) &\le d(y_1) + d(y_2) + d(y_3) + d(y_4)\\ &= d_C(y_1) + d_C(y_2) + d_C(y_3) + d_C(y_4)\\ &+ d_H(y_1) + d_H(y_3) + d_H(y_2) + d_H(y_4)\\ &\le 2(k+1) + 2(n-k) = 2n+2, \end{aligned}$$

a contradiction. □

**Remark 4.3.** *The sharpness of the lower bound of the degree condition in Theorem 4.2 can be seen from the following graph.*

*Let $n$ be an integer with $n \ge 3$. We prepare a complete balanced bipartite graph $G_0[X_0, Y_0]$, where $|X_0| = |Y_0| = n-2$. Let $G_5 = G_5[X, Y]$ be the balanced bipartite* *graph with* $X = X_0 \cup \{x_1, x_2\}$ *and* $Y = Y_0 \cup \{y_1, y_2\}$ *such that* $E(G) = E(G_0) \cup \{x_1y, y_1x : x \in X_0, y \in Y_0\} \cup \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ *(see Figure* 9*). It is easy to see that* $\mu_2(G_5) = n + 1$ *and the edge* $x_1y_1$ *does not lie on any hamiltonian cycle.*

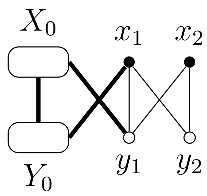


Figure 9: The balanced bipartite graph $G_5$. Thick lines represent all possible edges.

Now we prove that a balanced bipartite graph satisfying the conditions of Theorem 4.2 is in fact also chorded edge bipancyclic (Theorem 4.4). By the balanced bipartite graph $G_5$ in Figure 9, we can see that the lower bound in Theorem 4.4 is tight.

**Theorem 4.4.** *Let* $G$ *be a connected balanced bipartite graph of order* $2n$ *with* $n \geq 3$. *If* $\mu_2(G) \geq n + 2$, *then* $G$ *is chorded edge bipancyclic.*

*Proof.* Let $X$ and $Y$ be two partite sets of $G$ and let $xy$ be an arbitrary edge of $G$, where $x \in X$ and $y \in Y$. By Theorem 4.2, $G$ is edge bipancyclic. This means that the edge $xy$ lies on cycles of every length from 6 to $2n$. In particular, the edge $xy$ lies on a hamiltonian cycle $C_{2n}$. Since $\mu_2(G) \geq n + 2 \geq 5$, there exists a vertex on $C_{2n}$ whose degree is at least three. Thus, $C_{2n}$ contains at least one chord and the edge $xy$ is in a chorded $2n$-cycle.

Suppose that there exists an integer $k \in \{3, \ldots, n-1\}$ such that the edge $xy$ does not lie on any chorded $2k$-cycles. We now show that this leads to a contradiction. Let $C = x_1y_1x_2y_2 \ldots x_ky_kx_1$ be an $2k$-cycle containing the edge $xy$, where $x_i \in X$ and $y_i \in Y$ with $i \in [k]$. Since the edge $xy$ does not lie on any chorded $2k$-cycles, $C$ must be an induced cycle in $G$. Without loss of generality, we fix $xy = x_1y_1$. Define $H = G - V(C)$, $H_X = V(H) \cap X$ and $H_Y = V(H) \cap Y$. From now on, the subscripts are taken modulo $k$.

Assume that $k \geq 4$. Let $L = \{y_1, y_2, y_3, y_4\}$. If there exists a vertex $z \in H_X$ such that $|N_L(z)| \geq 3$, then the edge $x_1y_1$ is on a chorded $2k$-cycle, a contradiction. Now suppose that $|N_L(z)| \leq 2$ for each $z \in H_X$. Note that $\text{dist}(y_1, y_2) = 2$ and $\text{dist}(y_3, y_4) = 2$. By the hypothesis of this theorem, $d(y_1) + d(y_2) \geq n + 2$ and $d(y_3) + d(y_4) \geq n + 2$. Thus,

$$2(n + 2) \leq d(y_1) + d(y_2) + d(y_3) + d(y_4) \leq 8 + 2(n - k),$$

which yields $k \leq 2$, a contradiction to $k \geq 4$.

Now assume that $k = 3$. Observe that for any two vertices $u, v$ in $X$ or $Y$, if $\text{dist}(u, v) = 2$, then $|N(u) \cap N(v)| \geq 2$. In fact, $|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)| \geq (n+2) - n = 2$. Since $C$ is an induced cycle, $|N_C(x_1) \cap N_C(x_2)| = 1$.

Then $|N_H(x_1) \cap N_H(x_2)| \geq 1$. Similarly, $|N_H(y_1) \cap N_H(y_2)| \geq 1$. Let $z \in N_H(x_1) \cap N_H(x_2)$ and $u \in N_H(y_1) \cap N_H(y_2)$. Then $x_1 z x_2 y_2 u y_1 x_1$ is a 6-cycle with the chord $y_1 x_2$ containing the edge $x_1 y_1$, a contradiction. □

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