

Radio gracefulness of Moore graphs and beyond

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Abstract

The study of radio graceful labelings is motivated by modeling efficient frequency assignment to radio towers, cellular towers, and satellite networks. For a simple, connected graph $G = (V(G), E(G))$, a radio labeling is a mapping $f : V(G) \rightarrow \mathbb{Z}^+$ satisfying (for any distinct vertices u, v)

$$|f(u) - f(v)| + d(u, v) \geq \text{diam}(G) + 1,$$

where $d(u, v)$ is the distance between u and v in G and $\text{diam}(G)$ is the diameter of G . A graph is radio graceful if there is a radio labeling such that $f(V(G)) = \{1, \dots, |V(G)|\}$. In this paper, we determine the radio gracefulness of low-diameter graphs with connections to high-performance computing, including Moore graphs, bipartite Moore graphs, and approximate Moore graphs like (r, g) -cages, Erdős-Rényi polarity graph, and McKay-Miller-Širáň graphs. We prove a new necessary and sufficient condition for radio graceful bipartite graphs with diameter 3. We compute the radio number of (r, g) -cages arising from generalized n -gons. Additionally, we determine that Erdős-Rényi polarity graphs and McKay-Miller-Širáň graphs are radio graceful.

1 Introduction

Graph labeling is an integral topic of study in graph theory, with relations to problems in graph coloring, network design, and communication theory. A graph labeling is a function that assigns integers to the vertices and/or edges of a graph and that satisfies certain conditions. The labeling conditions often model real world problems, and the study of graph labeling reveals useful combinatorial structures or optimized assignments for practical applications.

One such application is the *frequency assignment problem* (or *channel assignment problem*). First introduced by Hale in 1980 [13], it addresses the challenge of assigning frequencies to transmitters such that interference is minimized. This foundational problem inspired the radio labeling problem formulated by Chartrand et al. in 2001 [4], as well as numerous variants [11, 3, 21], which has modern applications to satellite and wireless communication networks. In this setting, the graph represents a network of radio stations, with edges modeling geographic proximity, which can potentially lead to signal interference. The goal is to assign distinct frequencies to each station, such that stations that are closer in the network (i.e., have shorter distance in the graph) are assigned more widely separated frequencies. For a simple, connected graph $G = (V(G), E(G))$, a radio labeling can then be defined as a mapping $f : V(G) \rightarrow \mathbb{Z}^+$ satisfying

$$|f(u) - f(v)| + d(u, v) \geq \text{diam}(G) + 1,$$

where $d(u, v)$ is the distance between u and v in G and $\text{diam}(G)$ is the diameter of G .

A secondary objective of the frequency assignment problem is the minimization of bandwidth. This optimization corresponds to finding a labeling that minimizes the span, or maximum integer assigned as a label. We call a radio labeling *graceful* if the difference between the largest and smallest assigned labels is minimal, i.e. $|V(G)| - 1$. Previous research has investigated the radio gracefulness of some common families of graphs [20], their radio numbers [15, 19, 26], and more general characterizations of radio graceful graphs [4, 22].

Motivated by recent advances in high-performance computing network design, we are interested in studying efficiently structured networks, or low-latency, high-bandwidth, and scalable networks, and their applicability to the frequency assignment problem with the existence of a radio graceful labeling for such networks. In extremal graph theory, these networks relate to the degree-diameter problem and the degree-girth problem. The degree-diameter problem seeks to determine the largest possible number of vertices in a graph of given maximum degree Δ and diameter D . An upper bound on the order of such graphs is called the Moore bound $M(\Delta, D)$, and is given by the formula:

$$M(\Delta, D) = 1 + \Delta \sum_{i=0}^{D-1} (\Delta - 1)^i.$$

Graphs that attain this bound are known as Moore graphs. The known Moore graphs exhibit a highly symmetric structure, including being distance-regular and

vertex-transitive. Vertex-transitivity guarantees self-centeredness, a necessary condition for radio graceful graphs. Highly symmetric graphs are also often self-centered. All possible Moore graphs are summarized in Table 1 below [17].

Graph	Moore graph (Δ, D)	Cage (r, g)
Complete graphs K_n	$(n - 1, 1)$	$(n - 1, 3)$
Odd cycles C_{2n+1}	$(2, n)$	$(2, 2n + 1)$
Petersen graph	$(3, 2)$	$(3, 5)$
Hoffman-Singleton graph	$(7, 2)$	$(7, 5)$
Hypothetical graph of diameter 2, girth 5, degree 57	$(57, 2)$	$(57, 5)$

Table 1: List of all possible Moore graphs

A closely related problem is the degree-girth problem, which is to find the smallest possible regular graph of a given degree r and girth g . Such graphs are called (r, g) -cages. Notably, the Moore bound $M(\Delta, D)$ exactly provides the lower bound on the order of a $(\Delta, 2D + 1)$ -cage. As a result, every Moore graph is also a cage, though not every cage attains the Moore bound.

While all classical Moore graphs have odd girth, there is a generalized definition for even girth, that is, the bipartite Moore graphs, or generalized Moore graphs, which are the largest possible bipartite graphs of given maximum degree Δ and diameter D . The bipartite Moore bound $M_b(\Delta, D)$ given by

$$M_b(\Delta, D) = 2 \sum_{i=0}^{D-1} (\Delta - 1)^i$$

is the lower bound on the order of a cage with even girth, hence these graphs are the smallest possible graphs with the given maximum degree Δ and girth $g = 2D$, that is $(\Delta, 2D)$ -cages. As seen in Table 2, the bipartite Moore graphs with diameters 3, 4, 6 correspond to the incidence graphs of projective planes of order q , generalized quadrangles of order q , and generalized hexagons of order q , respectively. Such bipartite Moore graphs are only known to exist if q is a prime power.

Graph	bi-Moore graph (Δ, D)	Cage (r, g)
Complete bipartite graphs $K_{n,n}$	$(n, 2)$	$(n, 4)$
Even cycles C_{2n}	$(2, n)$	$(2, 2n)$
Incidence graph of projective planes of order q	$(q + 1, 3)$	$(q + 1, 6)$
Incidence graph of generalized quadrangles of order q	$(q + 1, 4)$	$(q + 1, 8)$
Incidence graph of generalized hexagons of order q	$(q + 1, 6)$	$(q + 1, 12)$

Table 2: List of all possible bipartite Moore graphs

Other interesting families of graphs related to the Moore bound, and specifically the Moore bound for diameter 2, are Erdős-Rényi polarity graphs [7] (or Brown

graphs [2]) and McKay-Miller-Širáň graphs (MMS graphs) [16]. For odd prime powers $q \geq 7$, the Erdős-Rényi polarity graphs give the current largest known order of graphs with diameter 2 and maximum degree $q + 1$. In particular, the graph with maximum degree $q + 1$ has $q^2 + q + 1$ vertices, which is only Δ less than the Moore bound. Modifications of these graphs have been constructed to show that the Moore bound can be approached asymptotically, even if it cannot be reached [1]. The MMS graphs are an infinite class of vertex-transitive graphs of order $\frac{8}{9}(\Delta + \frac{1}{2})^2$ with diameter 2 [16]. These constructions, whose sizes are “close” to the Moore bound, are called approximate Moore graphs [5]. In particular, Erdős-Rényi polarity graphs are constant additive approximations, while MMS graphs are constant multiplicative approximations.

In this paper, we consider the graphs outlined above. Section 2 introduces definitions and proof methods used throughout the paper. In Section 3, we discuss in more detail bipartite graphs, bipartite Moore graphs corresponding to (r, g) -cages, and approximate Moore graphs, including Erdős-Rényi polarity graphs and MMS graphs. Conditions for a bipartite graph to be radio graceful are given, as well as a necessary and sufficient condition for diameter 3 bipartite graphs to be radio graceful. In addition, the radio number of a bipartite Moore graph corresponding to a (r, g) -cage is determined, and the radio gracefulness of Erdős-Rényi polarity graphs and MMS graphs is determined. We conclude in Section 4 with a question on the radio gracefulness of all (r, g) -cages.

2 Preliminaries

Given a graph G , a *path* in G is a sequence of adjacent vertices without repeat. A sequence of adjacent vertices where only the initial and final vertex are the same is a *cycle*. A *Hamiltonian path (cycle)* in G is a path (cycle) that visits every vertex of G exactly once. If G contains a Hamiltonian path, then G is *traceable*. If G contains a Hamiltonian cycle, then G is *hamiltonian*. The *distance* between two vertices u and v is the length of the shortest path between them. We denote this as $d(u, v)$ (or $d_G(u, v)$ when specifying the graph G is necessary). When $d(u, v) = 1$, we write $u \sim v$ for convenience. The k^{th} *power* of a Hamiltonian path (cycle) H is the graph on the same vertex set where an edge occurs between two vertices u, v if and only if $d_H(u, v) \leq k$. The *diameter* of G is denoted and defined as $\text{diam}(G) = \{\max\{d(u, v)\} \mid u, v \in V(G)\}$. The *girth* of a graph is the length of the smallest cycle contained within it. Throughout this paper, we take all graphs G to be simple and finite. We also write r to denote the degree of a regular graph.

We now recall the definition of a radio labeling along with its natural refinement.

Definition 2.1. Given a simple, connected graph G , a labeling $f : V(G) \rightarrow \mathbb{Z}^+$ is a *radio labeling* if it satisfies the inequality

$$|f(u) - f(v)| + d(u, v) \geq \text{diam}(G) + 1 \quad (1)$$

for all distinct vertices u and v .

The *span* of a labeling f is the largest label assigned to a vertex. The *radio number* of G is $rn(G) = \min\{span(f) \mid f \text{ is a radio labeling of } G\}$. A radio labeling is *graceful* if $span(f) = |V(G)|$. A graph G is *radio graceful* if a radio graceful labeling exists, or equivalently, if $rn(G) = |V(G)|$.

The *antipodal graph* of a graph G , denoted as $A(G)$, is a graph on the same set of vertices as G where each pair of vertices u and v is connected in $A(G)$ if and only if $d_G(u, v) = \text{diam}(G)$. For $\text{diam}(G) = 2$, we have $A(G) = \overline{G}$. The following result proves useful for quickly determining if a graph is not radio graceful.

Theorem 2.2 (Saha and Basunia [22]). *If a graph G is radio graceful, then $A(G)$ is traceable.*

When restricting to graphs with diameter 2, Chartrand, Erwin, Harary, and Zhang were able to give a necessary and sufficient condition for radio gracefulness that proves very helpful below.

Theorem 2.3 (Chartrand, Erwin, Harary, and Zhang [4]). *Let G be a graph with diameter 2. Then G is radio graceful if and only if $A(G)$ is traceable.*

The requirement of traceability naturally leads to the use of the following results.

Theorem 2.4 (Dirac [6]). *If the minimum degree of a graph G on n vertices is at least $\frac{n-1}{2}$, then G is traceable.*

Theorem 2.5 (Corollary of Moon and Moser [18]). *If G is an r -regular bipartite graph $G = (U, V, E)$, where $|U| = |V| = n < 2r$, then G is traceable.*

2.1 Generalized polygons

Many of the graphs considered in this paper arise from *generalized polygons* (or *generalized n -gons*).

Definition 2.6. Let $\mathcal{I}$ be an ordered triple (P, L, I) where P is the nonempty set of points p of $\mathcal{I}$, L is the nonempty set of lines l of $\mathcal{I}$, and $I \subseteq P \times L$ is the incidence relation. Let G_I be the associated bipartite incidence graph on $P \cup L$ with edges joining the points of P to their incident lines in L . Then $\mathcal{I}$ is a *generalized n -gon* if the following four conditions are satisfied:

1. There exist $s \geq 1$ and $t \geq 1$ such that every line is incident to exactly $s + 1$ points and every point is incident to exactly $t + 1$ lines.
2. Any two distinct lines intersect in at most one point and there is at most one line through any two distinct points.
3. The diameter of the incidence graph G_I is n .
4. The girth of G_I is $2n$.

The generalized n -gon $\mathcal{I}$ has order (s, t) , or order s in the case $s = t$. If a generalized n -gon exists for (s, t) , then a dual exists for (t, s) . When $n = 3, 4, 6$, generalized

n -gons correspond to finite projective planes, generalized quadrangles, and generalized hexagons, respectively [24]. We now discuss each in detail, introducing the properties that are unique to each, beginning with $n = 3$. In this case, it is known that $s = t$ with the existence of the finite projective plane known when s is a prime power q .

Definition 2.7. A *projective plane* $\mathbb{P}$ of order q is an ordered triple (P, L, I) where P is the set of points p of $\mathbb{P}$, L is the set of lines l of $\mathbb{P}$, and $I \subseteq P \times L$ is the *incidence relation* satisfying the following four properties:

1. Any two points determine a line.
2. Any two lines determine a point.
3. Every point is incident with $q + 1$ lines.
4. Every line is incident with $q + 1$ points.

From this, a combinatorial argument can be made to prove the following statement.

Lemma 2.8. The incidence graph G_I of a finite projective plane of order q is a $(q + 1)$ -regular bipartite graph with diameter 3, girth 6, and parts corresponding to the set of points P and the set of lines L where $|P| = |L| = q^2 + q + 1$.

The smallest projective plane, the Fano plane, is seen in Figure 1. The Fano plane, arising from a field of order 2, has seven points and seven lines, with three points on every line and three lines through every point.

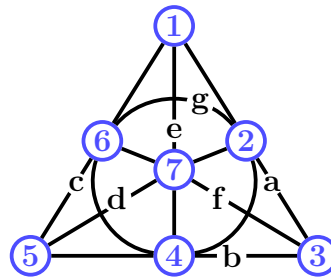
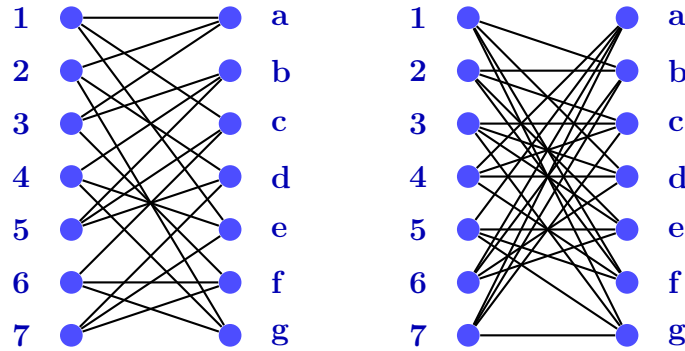


Figure 1: Fano plane

When $q = 2$ the incidence graph G_I of the Fano plane, as in Figure 2, is the unique $(3, 6)$ -cage (Heawood graph) [9]. The graph $A(G_I)$ has edges between points and lines that are not incident in $\mathbb{P} \cong PG(2, 2)$.

In the case that $n = 4$, it is known that $t^{1/2} \leq s \leq t^2$ and $s + t$ must divide $st(s + 1)(t + 1)$ with $s, t > 1$. The existence of generalized quadrangles is known when (s, t) is (q, q) , (q, q^2) , $(q - 1, q + 1)$, or (q^2, q^3) for a prime power of q .

Definition 2.9. A *generalized quadrangle* of order (s, t) has $(s + 1)(st + 1)$ points and $(t + 1)(st + 1)$ lines, and satisfies the following five properties:

Figure 2: G_I and $A(G_I)$

1. Any two points lie on at most one line.
2. Any two lines intersect in at most one point.
3. Every line is incident with $s + 1$ points.
4. Every point is incident with $t + 1$ lines.
5. For any point $p \in P$ and line $l \in L$ where $(p, l) \notin I$, there is exactly one line incident with p and intersecting l .

We focus on the case where $s = t$ for a prime power q , that is, a generalized quadrangle of order q .

Lemma 2.10. *The incidence graph G_I of a generalized quadrangle of order q is a $(q + 1)$ -regular bipartite graph with diameter 4, girth 8, and parts corresponding to the set of points P and the set of lines L where $|P| = |L| = (q + 1)(q^2 + 1)$.*

In the case that $n = 6$, it is known that $t^{1/3} \leq s \leq t^3$ and st is a perfect square with $s, t > 1$. The existence of generalized hexagons is known when (s, t) is (q, q) or (q^3, q) for a prime power of q .

Definition 2.11. A *generalized hexagon* of order (s, t) has $(s + 1)(s^2t^2 + st + 1)$ points and $(t + 1)(s^2t^2 + st + 1)$ lines, and satisfies the following five properties:

1. Any two points lie on at most one line.
2. Any two lines intersect in at most one point.
3. Every line is incident with $s + 1$ points.
4. Every point is incident with $t + 1$ lines.
5. For any point $p \in P$ and line $l \in L$ where $(p, l) \notin I$, there is a unique shortest path from p to l of length 3 or 5.

We focus on the case where $s = t$ for a prime power q , that is, a generalized hexagon of order q .

Lemma 2.12. *The incidence graph G_I of a generalized hexagon of order q is a $(q + 1)$ -regular bipartite graph with diameter 6, girth 12, and parts corresponding to the set of points P and the set of lines L where $|P| = |L| = (q^3 + 1)(q^2 + q + 1)$.*

3 Main Results

In this section, we present our main results. We begin with results on bipartite graphs in Section 3.1, giving conditions for these graphs to have a radio graceful labeling. In Section 3.2, we apply these results to explore when bipartite Moore graphs, or cages arising from generalized polygons, are radio graceful. We then address the approximate Moore Graphs, Erdős-Rényi polarity graphs and MMS graphs, and their complements in Section 3.3. Throughout this section, let q be a prime power.

3.1 Bipartite graphs

The necessary condition for the radio gracefulness of a graph G outlined in Theorem 2.2 extends to bipartite graphs. Two results follow when $\text{diam}(G)$ is even or $\text{diam}(G) = 3$.

Proposition 3.1. *If a bipartite graph G has even diameter, then G is not radio graceful.*

Proof. For two adjacent vertices u, v in $A(G)$, we have $d_G(u, v)$ even. Thus, only vertices in the same part are adjacent in $A(G)$, leaving $A(G)$ disconnected. It follows from Theorem 2.2 that G is not radio graceful. $\square$

Theorem 3.2. *Let G be a bipartite graph with diameter 3. Then G is radio graceful if and only if $A(G)$ is traceable.*

Proof. In the antipodal graph $A(G)$, $u \sim_{A(G)} v$ if and only if $d_G(u, v) = 3$. Thus, $A(G)$ is bipartite with the same parts as G . Assume $A(G)$ is traceable. Define a labeling $f : V(G) \rightarrow \{1, \dots, n\}$ on the vertices of G sequentially according to a Hamiltonian path H in $A(G)$. The distance along the path is given by $d_H(u, v) = |f(u) - f(v)|$. If $|f(u) - f(v)| = d_H(u, v) = 1$, then $d_G(u, v) = 3$. If $|f(u) - f(v)| = d_H(u, v) = 2$, then u and v are in the same part of both $A(G)$ and G , so $d_G(u, v) = 2$. If $|f(u) - f(v)| = d_H(u, v) \geq 3$, then $d_G(u, v) \geq 1$ is trivially satisfied. In all cases the radio condition is satisfied, implying that G is radio graceful. From Theorem 2.2, the other direction holds. $\square$

From the proof of Theorem 3.2, a useful observation can be made.

Observation 3.3. Given a bipartite graph with an odd diameter with parts U and V , the antipodal graph $A(G)$ is a bipartite graph with the same parts U and V . Furthermore, if $\text{diam}(G) = 3$, then for $u \in U$ and $v \in V$ we have $u \sim_G v$ or $u \sim_{A(G)} v$, but not both.

A natural question to ask is when $A(G)$ is traceable. Introducing the additional property of regularity to G yields sufficient conditions, from which the radio gracefulness of the graph follows.

Lemma 3.4. *For $1 < r < n/2$, let G be an r -regular bipartite graph on $2n$ vertices with diameter 3. Then G is radio graceful.*

Proof. By Observation 3.3, the antipodal graph $A(G)$ is a regular bipartite graph with the same parts as G where an edge between parts lies in G or $A(G)$, but not both. For a vertex $v \in V(A(G))$, we have $\deg(v) = n - r > \frac{n}{2}$. By Theorem 2.5, the graph $A(G)$ is traceable. From Theorem 3.2 it follows that G is radio graceful. $\square$

Employing $\frac{n}{2}$ as a lower bound on r guarantees $\text{diam}(G) = 3$, but not the traceability of $A(G)$. Necessary and sufficient conditions for highly connected regular bipartite graphs to be radio graceful are provided.

Lemma 3.5. *For $\frac{n}{2} < r < n$, let G be a r -regular bipartite graph on $2n$ vertices. Then G has diameter 3.*

Proof. Suppose G has parts U and V . Since G is regular, we have $|U| = |V| = n$. For any $u, u' \in U$, since $r > n/2$, u and u' share a neighbor in V and $d(u, u') = 2$. Since $r < n$, there exists $v \in V$ such that $u \approx v$. From this, G has diameter 3. $\square$

Proposition 3.6. *Let G be a bipartite graph on $2n$ vertices. For $n \geq 3$, if G is $(n-1)$ -regular, then G is not radio graceful. For $n \geq 5$, if G is $(n-2)$ -regular, then G is radio graceful if and only if $A(G) = C_{2n}$.*

Proof. Let G be a regular bipartite graph on $2n$ with parts U and V . Suppose G is $(n-1)$ -regular where $n \geq 3$. From Lemma 3.5, as $n/2 < n-1 < n$ when $n \geq 3$, G has diameter 3. As seen in Observation 3.3, for $u \in U$ and $v \in V$, we have $u \sim_{A(G)} v$ when $u \approx_G v$. Thus, $A(G) \cong nK_2$. It follows from Theorem 3.2 that G is not radio graceful. Now suppose G is $(n-2)$ -regular where $n \geq 5$. From Lemma 3.5, as $n/2 < n-2 < n$ when $n \geq 5$, G has diameter 3 and $A(G)$ is 2-regular. The antipodal graph $A(G)$ is disconnected if and only if $A(G) \neq C_{2n}$. Therefore, G is radio graceful if and only if $A(G) = C_{2n}$. $\square$

3.2 Cages

We now consider three families of cages that arise as incidence graphs of generalized polygons. These graphs can be recognized as bipartite Moore graphs, that is, graphs that attain the Moore bound for even girth. We show that the $(q+1, 6)$ -cage is radio graceful, while the $(q+1, 8)$ -cage and the $(q+1, 12)$ -cage are not radio graceful, requiring the minimum radio labeling span to expand by 1 beyond the graceful radio labeling span.

3.2.1 Cages with girth 6

The incidence graph G_I of a projective plane of order q is a $(q+1)$ -regular graph on $2(q^2 + q + 1)$ vertices with diameter 3 and girth 6, which meets the Moore bound for even girth. Hence G_I is a $(q+1, 6)$ -cage and, by [23], any $(q+1, 6)$ -cage can be associated with the incidence graph of a projective plane of order q .

Theorem 3.7. *Let G_I be the incidence graph of a projective plane of order q . Then G_I is radio graceful.*

Proof. From Lemma 2.8, the graph G_I is a $(q+1)$ -regular bipartite graph on $2(q^2 + q + 1)$ vertices with diameter 3. By Lemma 3.4, as $1 < q + 1 < q^2 + q + 1$ for all prime powers q , we have that G_I is radio graceful. $\square$

Remark 3.8. Any $(q+1, 6)$ -cage is in one-to-one correspondence with the incidence graph of a projective plane of order q [23], so a $(q+1, 6)$ -cage is radio graceful for any prime power q . A radio graceful labeling can be constructed by assigning odd labels $1, 3, \dots, 2n-1$ to vertices corresponding to points and even labels $2, 4, \dots, 2n$ to vertices corresponding to lines, such that vertex i is not in the neighborhood of vertex $i-1$. This labeling is guaranteed by Theorem 3.2 and the traceability of the antipodal graph of the $(q+1, 6)$ -cage, a bipartite graph whose parts correspond to the set of points and the set of lines.

3.2.2 Cages with girth 8

The incidence graph G_I of a generalized quadrangle of order q is the unique $(q+1, 8)$ -cage [24], a bipartite $(q+1)$ -regular graph on $2(q+1)(q^2+1)$ vertices with diameter 4 and girth 8. We have the following result on the radio gracefulness of $(q+1, 8)$ -cage.

Proposition 3.9. *Let G_I be the incidence graph of a generalized quadrangle of order q . Then G_I is not radio graceful.*

Proof. From Lemma 2.10, we have G_I is a bipartite graph with diameter 4. By Proposition 3.1, it follows that G_I is not radio graceful. $\square$

We use the following Lemma to show the existence of the square of a Hamiltonian cycle in each of the components of $A(G_I)$, a fact that we will find useful in determining the radio number of G_I .

Lemma 3.10. *(Fan and Häggkvist [10]) Let G be a graph on n vertices with minimum degree $\delta(G)$. If $\delta(G) \geq \frac{5}{7}n$, then G contains the square of a Hamiltonian cycle.*

Let G_P be the graph induced by vertices corresponding to the set of points P in $A(G_I)$ and let G_L be the graph induced by vertices corresponding to the set of points L in $A(G_I)$.

Lemma 3.11. *The antipodal graph $A(G_I)$ of the incidence graph G_I of a generalized quadrangle of order q is a q^3 -regular graph with two components G_P and G_L . The graphs G_P and G_L both contain the square of a Hamiltonian cycle.*

Proof. For $u, v \in V(G)$, we have $d(u, v) = 4 = \text{diam}(G)$ if either u and v are noncollinear points, or u and v are nonconcurrent lines. Each point p in a generalized quadrangle is collinear with $q(q+1)$ other points, since each point lies on $q+1$ lines and each of these lines is incident with q points other than p . Therefore, for each point p , there are $(q+1)(q^2+1) - 1 - q(q+1) = q^3$ points that are not collinear with p . The same argument applies to lines, so $A(G_I)$ is a q^3 -regular graph with two components corresponding to the set of points P and the set of lines L . Since

$q^3 \geq \frac{5}{7}(q+1)(q^2+1)$ for $q > 3$, Lemma 3.10 shows the existence of the square of a Hamiltonian cycle both in G_P and G_L .

For $q = 2$ and $q = 3$, we can find the square of a Hamiltonian cycle in each bipartite part using edge lists available at

<https://aeb.win.tue.nl/graphs/cages/cages.html>.

When $q = 2$, let the parts of G be labeled $(0, 1, \dots, 14)$ and $(15, 16, \dots, 29)$. The square of a Hamiltonian cycle in each part is $(0, 1, 2, 4, 5, 6, 7, 14, 13, 12, 3, 8, 11, 9, 10, 0)$ and $(15, 19, 23, 25, 16, 18, 29, 22, 20, 26, 21, 17, 24, 28, 27, 15)$. When $q = 3$, let the parts of G be labeled $(0, 1, \dots, 39)$ and $(40, 41, \dots, 79)$. The square of a Hamiltonian cycle in each part is $(0, 1, 2, 3, 5, 7, 8, 6, 9, 10, 12, 11, 13, 22, 4, 14, 16, 17, 15, 18, 19, 21, 20, 24, 26, 25, 23, 27, 28, 31, 29, 30, 33, 34, 35, 32, 36, 37, 39, 38, 0)$ and $(40, 45, 50, 43, 44, 49, 42, 47, 48, 41, 46, 51, 52, 57, 59, 55, 56, 61, 64, 54, 58, 60, 53, 62, 67, 69, 72, 65, 63, 70, 71, 79, 75, 66, 77, 76, 68, 78, 73, 74, 40)$. $\square$

We now compute the radio number of a $(q+1, 8)$ -cage.

Theorem 3.12. *Let G be the $(q+1, 8)$ -cage. Then $rn(G) = 2(q+1)(q^2+1) + 1$.*

Proof. Since G is the incidence graph of a generalized quadrangle of order q , we have the natural partition of its vertices to parts P and L , corresponding to the set of points and the set of lines, respectively. Let $n = |P| = |L| = (q+1)(q^2+1)$.

From Proposition 3.9 we have that G is not radio graceful, so $rn(G) \geq 2n + 1$. We give a radio labeling $f : V(G) \rightarrow \{1, 2, \dots, 2n + 1\}$, showing this lower bound can be achieved. We first show that a consecutive labeling exists for both P and L , then verify the existence of a gluing strategy, all of which satisfy the radio labeling condition $|f(u) - f(v)| + d_G(u, v) \geq 5$ for all $u, v \in V(G)$.

Consider the subgraph G_P of $A(G)$ induced by the vertex set $P = \{p_1, p_2, \dots, p_n\}$. By Lemma 3.11, G_P contains the square of a Hamiltonian cycle. Let $(p_1, p_2, \dots, p_n, p_1)$ be the square of a Hamiltonian cycle in G_P and let $f(p_i) = i$. Then, $d_{A(G)}(p_i, p_{i+1}) = 1$ for $i = 1, 2, \dots, n-1$ and $d_{A(G)}(p_i, p_{i+2}) = 1$ for $i = 1, 2, \dots, n-2$. In the graph G , we then have $d_G(p_i, p_{i+1}) = 4$ for $i = 1, 2, \dots, n-1$ and $d_G(p_i, p_{i+2}) = 4$ for $i = 1, 2, \dots, n-2$. Since G is bipartite, $d_G(p_i, p_j) \geq 2$ for all $i \neq j$. Thus, $|f(p_i) - f(p_j)| + d_G(p_i, p_j) = |i - j| + d_G(p_i, p_j) \geq 5$ for all $i \neq j$, so the consecutive labeling of P satisfies the radio labeling condition. Similarly, the subgraph G_L contains the square of a Hamiltonian cycle by Lemma 3.11, say $(l_1, l_2, \dots, l_n, l_1)$, so we can define a consecutive labeling of L that satisfies the radio labeling condition. We will first show that there exists an index $t \in \{1, 2, \dots, n-1\}$ such that l_t is not adjacent to p_{n-1} and p_n , and l_{t+1} is not adjacent to p_n , allowing the consecutive labeling of both P and L to be concatenated without violating the radio labeling condition.

There are $q+1$ possible indices t such that l_t is adjacent to p_{n-1} , $q+1$ possible indices t such that l_t is adjacent to p_n , and $q+1$ possible indices t such that l_{t+1} is adjacent to p_n . Hence there are at most $3(q+1) < (q+1)(q^2+1) - 1 = n-1$ such indices that violate these conditions and so there exists an index $t \in \{1, 2, \dots, n-1\}$ such that l_t is not adjacent to p_{n-1} and p_n , and l_{t+1} is not adjacent to p_n . Since G is bipartite, it follows that $d_G(p_{n-1}, l_t) \geq 3$, $d_G(p_n, l_t) \geq 3$, and $d_G(p_n, l_{t+1}) \geq 3$.

Let t be such an index. For $i = 1, 2, \dots, t-1$, let $f(l_i) = 2n - t + 2 + i$ and, for $i = t, t+1, \dots, n$, let $f(l_i) = n - t + 2 + i$. Then, since $(l_1, l_2, \dots, l_n, l_1)$ is the square of a Hamiltonian cycle in G_L and the biparteness of G guarantees that $d_G(l_i, l_j) \geq 2$ for all $i \neq j$, it is simple to verify that $|f(l_i) - f(l_j)| + d_G(l_i, l_j) \geq 5$ for all $i \neq j$. The existence of t guarantees that $|f(p_{n-1}) - f(l_t)| + d_G(p_{n-1}, l_t) \geq 3 + 3 \geq 5$, $|f(p_n) - f(l_t)| + d_G(p_n, l_t) \geq 3 + 2 \geq 5$, and $|f(p_n) - f(l_{t+1})| + d_G(p_n, l_{t+1}) \geq 3 + 3 \geq 5$. Lastly, we have $|f(p_i) - f(l_j)| + d_G(p_i, l_j) \geq 4 + 1 \geq 5$ for all other values of i and j . Thus, $f : V(G) \rightarrow \{1, 2, \dots, 2n+1\}$ satisfies the radio labeling condition.

We have defined the following radio labeling:

v	p_1	p_2	$\dots$	p_{n-1}	p_n	l_t	l_{t+1}	$\dots$	l_n	l_1	$\dots$	l_{t-1}
$f(v)$	1	2	$\dots$	$n-1$	n	$n+2$	$n+3$	$\dots$	$2n-t+2$	$2n-t+3$	$\dots$	$2n+1$

Since f is a radio labeling whose span is $2n+1$, we have $rn(G) \leq 2n+1$. Therefore, $rn(G) = 2n+1 = 2(q+1)(q^2+1)+1$. $\square$

3.2.3 Cages with girth 12

The incidence graph G_I of a generalized hexagon of order q is the unique $(q+1, 12)$ -cage [24], a bipartite $(q+1)$ -regular graph on $2(q^3+1)(q^2+q+1)$ vertices with diameter 6 and girth 12. We have the following result on the radio gracefulness of $(q+1, 12)$ -cage.

Proposition 3.13. *Let G_I be the incidence graph of a generalized hexagon of order q . Then G_I is not radio graceful.*

Proof. From Lemma 2.12, we have G_I is a bipartite graph with diameter 6. By Proposition 3.1, it follows that G_I is not radio graceful. $\square$

The following lemma is used to show the existence of the 4^{th} power of a Hamiltonian cycle in each of the components of $A(G_I)$, a fact that we will find useful in determining the radio number of G_I . While we only use it for the 4^{th} power of a Hamiltonian cycle, we state it for the general ℓ^{th} power of a Hamiltonian cycle.

Lemma 3.14. *Let G be a graph on n vertices with minimum degree $\delta(G)$. If $\delta(G) \geq \frac{4\ell-1}{4\ell}n$, then G contains the ℓ^{th} power of a Hamiltonian cycle.*

Proof. Let G be an n -vertex graph with minimum degree $d = \delta(G) \geq \frac{4\ell-1}{4\ell}n$. We say a cycle is *special* if the subgraph induced by the cycle contains the ℓ^{th} power of that cycle. Suppose the longest special cycle has k vertices. We will prove that $k = n$ by contradiction. Assume that $k < n$ and $C = (v_1, v_2, \dots, v_k, v_1)$ is a special cycle. Then any vertex u outside of this cycle is not adjacent to at least one of the 2ℓ vertices $v_{k-\ell+1}, v_{k-\ell+2}, \dots, v_k, v_1, v_2, \dots, v_\ell$, otherwise the cycle could be extended contradicting maximality. Each vertex in G has at most $n-1-d$ non-neighbors, so the number of vertices not adjacent to at least one of these 2ℓ vertices is at most $2\ell(n-1-d)$. Therefore, the number of vertices outside of the cycle C is

$$n - k \leq 2\ell(n - 1 - d). \quad (2)$$

Let u be a vertex of G that is not part of this cycle. Then for each index i , we have u is not adjacent to at least one of the vertices $(v_i, v_{i+1}, v_{i+2}, \dots, v_{i+2\ell-1})$ (let $v_{n+j} = v_j$ when $j > 0$), otherwise $(v_1, v_2, \dots, v_{i+2}, v_{i+\ell-1}, u, v_{i+\ell}, v_{i+\ell+1}, \dots, v_k, v_1)$ is a special cycle of length $k+1 > k$, a contradiction. The number of indices i such that u is not adjacent to v_i is at most $n-1-d$. Similarly for $v_{i+1}, v_{i+2}, \dots, v_{i+2\ell-1}$, we thus have the number of indices i such that u is not adjacent to at least one of the vertices $(v_i, v_{i+1}, v_{i+2}, \dots, v_{i+2\ell-1})$ is at most $2\ell(n-1-d)$. Since all indices $i = 1, 2, \dots, k$ have to satisfy this condition, we must have

$$k \leq 2\ell(n-1-d). \quad (3)$$

From (2) and (3), we have $n \leq 4\ell(n-1-d)$, which contradicts $d \geq \frac{4\ell-1}{4\ell}n$. $\square$

Let G_P be the graph induced by vertices corresponding to the set of points P in $A(G_I)$ and let G_L be the graph induced by vertices corresponding to the set of points L in $A(G_I)$.

Lemma 3.15. *The antipodal graph $A(G_I)$ of the incidence graph G_I of a generalized hexagon of order q is a q^5 -regular bipartite graph with two components G_P and G_L . For $q > 15$, the graphs G_P and G_L both contain the 4^{th} power of a Hamiltonian cycle.*

Proof. Let $\Gamma_i(v)$ be the set of vertices that are at distance i from v in G_I . Each point v has exactly $q+1$ neighbors, so $|\Gamma_1(v)| = q+1$. Each neighbor of v has q neighbors other than v , and no two neighbors u_1, u_2 of v have the same neighbor $w \neq v$, otherwise resulting in two irreducible paths of length 2 from v to w in G_I . Therefore, $|\Gamma_2(v)| = (q+1)q$. Similarly, we must have $|\Gamma_3(v)| = (q+1)q^2$, $|\Gamma_4(v)| = (q+1)q^3$, and $|\Gamma_5(v)| = (q+1)q^4$. Then,

$$|\Gamma_6(v)| = 2(q^3+1)(q^2+q+1) - (1 + |\Gamma_1(v)| + |\Gamma_2(v)| + |\Gamma_3(v)| + |\Gamma_4(v)| + |\Gamma_5(v)|) = q^5.$$

Hence, each vertex v in G has q^5 vertices at distance 6 from it, so $A(G_I)$ is a q^5 -regular graph with two components corresponding to the set of points P and the set of lines L . Since $q^5 \geq \frac{15}{16}(q^3+1)(q^2+q+1)$ for $q > 15$, Lemma 3.14 for $\ell = 4$ shows the existence of the 4^{th} power of a Hamiltonian cycle in both G_P and G_L . $\square$

Theorem 3.16. *Let G be the $(q+1, 12)$ -cage where $q > 15$. Then*

$$rn(G) = 2(q^3+1)(q^2+q+1) + 1.$$

Proof. Since G is the incidence graph of a generalized hexagon of order q , we can divide its vertex set into 2 parts P and L , corresponding to the set of points and the set of lines, respectively. Let $n = |P| = |L| = (q^3+1)(q^2+q+1)$

Since G is not radio graceful by Proposition 3.13, we have $rn(G) \geq |V(G)| + 1 = 2(q^3+1)(q^2+q+1) + 1 = 2n + 1$. We will prove that this lower bound is attained. Following the same argument as in the proof of Theorem 3.12, we construct a radio labeling $f : V(G) \rightarrow \{1, 2, \dots, 2n+1\}$ by establishing the existence of consecutive labelings for both P and L , together with a gluing strategy, all of which satisfy the radio labeling condition $|f(u) - f(v)| + d_G(u, v) \geq 7$ for all $u, v \in V(G)$.

Consider the subgraph G_P of $A(G)$ induced by the vertex set $P = \{p_1, p_2, \dots, p_n\}$. By Lemma 3.15, G_P contains the 4^{th} power of a Hamiltonian cycle. Let $(p_1, p_2, \dots, p_n, p_1)$ be the 4^{th} power of a Hamiltonian cycle in G_P and let $f(p_i) = i$. Then, $d_{A(G)}(p_i, p_{i+k}) = 1$ for all $i = 1, 2, \dots, n-k$ and $k = 1, 2, 3, 4$. In the graph G , we then have $d_G(p_i, p_{i+k}) = 6$ for all $i = 1, 2, \dots, n-k$ and $k = 1, 2, 3, 4$. Since G is bipartite, $d_G(p_i, p_j) \geq 2$ for all $i \neq j$. Thus, $|f(p_i) - f(p_j)| + d_G(p_i, p_j) = |i - j| + d_G(p_i, p_j) \geq 7$ for all $i \neq j$, so the consecutive labeling of P satisfies the radio labeling condition. Similarly, the subgraph G_L contains the 4^{th} power of a Hamiltonian cycle by Lemma 3.15, say $(l_1, l_2, \dots, l_n, l_1)$, so we can define a consecutive labeling of L that satisfies the radio labeling condition. We will first show that there exists an index $t \in \{1, 2, \dots, n-3\}$, allowing the consecutive labeling of both P and L to be concatenated without violating the radio labeling condition.

Since G is bipartite and $(q+1)$ -regular, the distance between a vertex in P and a vertex in L must be odd. Then there are at most $2q(q+1)^2 + 2(q+1)$ possible indices t such that $d_G(p_{n-k}, l_t) \leq 4-k$ for $k = 0, 1, 2, 3$, at most $q(q+1)^2 + 2(q+1)$ possible indices t such that $d_G(p_{n-k}, l_{t+1}) \leq 3-k$ for $k = 0, 1, 2$, at most $2(q+1)$ possible indices t such that $d_G(p_{n-k}, l_{t+2}) \leq 2-k$ for $k = 0, 1$, and at most $q+1$ possible indices t such that $d_G(p_n, l_{t+3}) = 1$. Since $3q(q+1)^2 + 7(q+1) < (q^3+1)(q^2+q+1) - 1 = n-1$ (as $q \geq 3$), there exists an index $t \in \{1, 2, \dots, n-3\}$ such that $d_G(p_{n-k}, l_t) > 4-k$ for $k = 0, 1, 2, 3$, $d_G(p_{n-k}, l_{t+1}) > 3-k$ for $k = 0, 1, 2$, $d_G(p_{n-k}, l_{t+2}) > 2-k$ for $k = 0, 1$, and $d_G(p_n, l_{t+3}) > 1$. Since G is bipartite, it follows that $d_G(p_{n-k}, l_t) \geq 5-k$ for $k = 0, 1, 2, 3$, $d_G(p_{n-k}, l_{t+1}) \geq 4-k$ for $k = 0, 1, 2$, $d_G(p_{n-k}, l_{t+2}) \geq 3$ for $k = 0, 1$, and $d_G(p_n, l_{t+3}) \geq 3$.

Let t be such an index. For $i = 1, 2, \dots, t-1$, let $f(l_i) = 2n - t + 2 + i$ and, for $i = t, t+1, \dots, n$, let $f(l_i) = n - t + 2 + i$. Then, since $(l_1, l_2, \dots, l_n, l_1)$ is the 4^{th} power of a Hamiltonian cycle in G_L and the bipartiteness of G guarantees that $d_G(l_i, l_j) \geq 2$ for all $i \neq j$, it is simple to verify that $|f(l_i) - f(l_j)| + d_G(l_i, l_j) \geq 7$ for all $i \neq j$. The existence of t guarantees that $|f(p_{n-k}) - f(l_t)| + d_G(p_{n-k}, l_t) \geq (2+k) + (5-k) \geq 7$ for $k = 0, 1, 2, 3$, $|f(p_{n-k}) - f(l_{t+1})| + d_G(p_{n-k}, l_{t+1}) \geq (3+k) + (4-k) \geq 7$ for $k = 0, 1, 2$, $|f(p_{n-k}) - f(l_{t+2})| + d_G(p_{n-k}, l_{t+2}) \geq (4+k) + 3 \geq 7$ for $k = 0, 1$, and $|f(p_n) - f(l_{t+3})| + d_G(p_n, l_{t+3}) \geq 5+3 \geq 7$. Lastly, we have $|f(p_i) - f(l_j)| + d_G(p_i, l_j) \geq 6+1 \geq 7$ for all other values of i and j . Thus, $f : V(G) \rightarrow \{1, 2, \dots, 2n+1\}$ satisfies the radio labeling condition.

We have defined the following radio labeling:

v	p_1	p_2	$\dots$	p_{n-1}	p_n	l_t	l_{t+1}	$\dots$	l_n	l_1	$\dots$	l_{t-1}
$f(v)$	1	2	$\dots$	$n-1$	n	$n+2$	$n+3$	$\dots$	$2n-t+2$	$2n-t+3$	$\dots$	$2n+1$

We have defined a radio labeling f whose span is $2n+1$, so $rn(G) \leq 2n+1$. Therefore, $rn(G) = 2n+1 = 2(q^3+1)(q^2+q+1) + 1$. $\square$

For $q \leq 13$, the radio number of the $(q+1, 12)$ -cage is not considered and is therefore unknown. For small values of q ($2 \leq q \leq 5$) edge lists available at <https://aeb.win.tue.nl/graphs/cages/cages.html> can be used to verify a 4^{th} power of a Hamiltonian cycle in each of the components of the antipodal graph of

the $(q+1, 12)$ -cage. For the remaining values of q , computational methods, such as Sage’s command “graphs.GeneralisedHexagonGraph(s,t)”, seem feasible for determining a 4^{th} power of a Hamiltonian cycle in each bipartite part of the $(q+1, 12)$ -cage. The existence of the necessary gluing index for $3 \leq q \leq 13$ would follow from the proof of Theorem 3.16 and can be verified separately for $q = 2$.

3.3 Approximate-Moore graphs and their complements

The radio gracefulness of well-known approximate Moore graphs is now considered. Let $PG(2, q)$ denote the projective plane over $\mathbb{F}_q$, whose points are equivalence classes of nonzero vectors $(x, y, z) \in \mathbb{F}_q^3 \setminus \{0\}$ under scalar multiplication. The *Erdős-Rényi polarity graph* (or Brown graph) ER_q is the graph whose vertex set is the set of all points of $PG(2, q)$ where an edge between two vertices (x_0, y_0, z_0) and (x_1, y_1, z_1) exists if and only if $x_0x_1 + y_0y_1 + z_0z_1 = 0$ in $\mathbb{F}_q$.

We now recall key structural properties of these graphs.

Lemma 3.17 (Bachratý and Širáň [1]). *Let ER_q be the Erdős-Rényi polarity graph of order q . Then:*

1. $|V(ER_q)| = q^2 + q + 1$;
2. *the vertex set can be partitioned into $q+1$ quadric vertices of degree q , and the remaining q^2 vertices of degree $q+1$;*
3. ER_q has diameter 2;
4. *for adjacent vertices $u \sim v$, there is at most one common neighbor.*

We use these facts to prove that Erdős-Rényi polarity graphs are radio graceful.

Lemma 3.18. *For all $q \geq 2$, the graph $\overline{ER_q}$ is traceable.*

Proof. Since ER_q has maximum degree $q+1$, every vertex in $\overline{ER_q}$ has a degree at least $q^2 + q + 1 - 1 - (q+1) = q^2 - 1$. The inequality $q^2 - 1 \geq \frac{q^2+q}{2}$ holds for all $q \geq 2$ so by Theorem 2.4, it follows that $\overline{ER_q}$ is traceable. $\square$

Proposition 3.19. *For all $q \geq 2$, the Erdős-Rényi polarity graph ER_q is radio graceful.*

Proof. By Lemma 3.17, $\text{diam}(ER_q) = 2$. By Lemma 3.18, $\overline{ER_q}$ is traceable. From Theorem 2.3, we have ER_q is radio graceful. $\square$

Proposition 3.19 is an example of the following result that holds more generally for diameter 2 graphs with bounded degree, which follows from Theorem 2.3 and Theorem 2.4.

Corollary 3.20. *Let G be a graph of diameter 2 on n vertices with maximum degree $\Delta(G)$. If $\Delta(G) \leq \frac{n-1}{2}$, then G is radio graceful.*

Let $q > 2$, and let ξ be a primitive element of the finite field $\mathbb{F}_q$. Let $X = \{1, \xi^2, \dots\}$ and $X' = \{\xi, \xi^3, \dots\}$ be the subsets of $\mathbb{F}_q$ as defined in [12]. The *MMS graph* H_q is the graph whose vertex set is $\mathbb{Z}_2 \times \mathbb{F}_q \times \mathbb{F}_q$ with edges between given by:

- $(0, x, y)$ is adjacent to $(0, x, y')$ if and only if $y - y' \in X$;
- $(1, m, c)$ is adjacent to $(1, m, c')$ if and only if $c - c' \in X'$;
- $(0, x, y)$ is adjacent to $(1, m, c)$ if and only if $y = mx + c$.

We can apply Corollary 3.20 to show that the MMS graph H_q is radio graceful.

Lemma 3.21 (Mckay, Miller, Širáň [16]). *Every MMS graph corresponds to an odd integer $d \equiv 1 \pmod{6}$ such that $\frac{2d+1}{3}$ is prime. The corresponding MMS graph is of degree d and order*

$$n = \frac{8}{9} \left(d + \frac{1}{2} \right)^2.$$

Corollary 3.22 (of Corollary 3.20). *Let G be a MMS graph. Then G is radio graceful.*

Proof. By Corollary 3.20, as MMS graphs have diameter 2, to show G is radio graceful it suffices to show that

$$d \leq \frac{n-1}{2}.$$

Equivalently,

$$d + \frac{1}{2} \leq \frac{n}{2} = \frac{4}{9} \left(d + \frac{1}{2} \right)^2,$$

where the equality follows from Lemma 3.21. This inequality holds for all $d \geq 7$, and so it holds when $d \equiv 1 \pmod{6}$ is an odd integer such that $\frac{2d+1}{3}$ is prime. $\square$

Corollary 3.20 can also be applied to show the largest known vertex-transitive graphs with diameter 2 of a given degree d (for sufficiently large d), a Cayley graph of order $d^2 - 6\sqrt{2}d^{3/2}$ whose existence is known for $d = 2^{2m+\delta} + (2+\delta)2^{m+1} - 6$ where m is any positive integer and $\delta \in \{0, 1\}$ [25], is radio graceful when $d \geq 76$. A sort of converse of Corollary 3.20 is also known:

Theorem 3.23 (Saha and Basunia [22]). *Let G be an n -vertex graph with $\Delta(G) < \frac{n}{2}$. If G is traceable, then the complement graph $\overline{G}$ is always radio graceful.*

Let $T_{m,n}$ be the tadpole graph, the graph constructed by adding an edge between an endpoint of P_n and a vertex in C_m . From Theorem 3.23, we can obtain three infinite families of radio graceful graphs from low-degree traceable graphs.

Example 3.24. The graphs $\overline{C_m}$, $\overline{P_n}$, and $\overline{T_{m,n}}$ are radio graceful for $m \geq 5$, $n \geq 5$, and $m + n \geq 7$, respectively.

One natural corollary of Theorem 3.23 is as follows:

Corollary 3.25. *For odd $n \geq 5$, let G be an $\frac{n-1}{2}$ -regular graph on n vertices. Then G and $\overline{G}$ are radio graceful.*

Proof. First, note that the complement of an $\frac{n-1}{2}$ -regular graph is also an $\frac{n-1}{2}$ -regular graph. By Theorem 2.4, both G and its complement are traceable. Hence, by Theorem 3.23, both G and its complement are radio graceful. $\square$

We wish to apply Theorem 3.23 to Erdős-Rényi polarity graphs. To confirm that Erdős-Rényi polarity graphs are traceable, we introduce an equivalent construction using Singer difference sets. A *Singer difference set* D is a set of $q + 1$ elements of $\mathbb{Z}_{q^2+q+1}$ such that the set $\{(d_i - d_j) \pmod{q^2 + q + 1} \mid d_i, d_j \in D \text{ and } d_i \neq d_j\}$ is the set of all integers from 1 to $q^2 + q$ without repetition.

Definition 3.26. [8] Let $D = \{d_0, d_1, \dots, d_q\}$ be the Singer difference set D over $\mathbb{Z}_{q^2+q+1}$. The Singer graph S_q has vertices $V = \{i \mid 0 \leq i < q^2 + q + 1\}$, and edges $E = \{(i, j) \mid (i + j) \pmod{q^2 + q + 1} \in D\}$.

Lemma 3.27 (Lakhotia, Isham, Monroe, Besta, Hoefler, Petrini [14]). *For all $q \geq 2$, the Singer graph S_q is Hamiltonian.*

Lemma 3.28 (Erskine, Fratrič, and Širáň [8]). *Let q be any prime power. Then the graphs S_q and ER_q are isomorphic.*

Proposition 3.29. $\overline{ER_q}$ is radio graceful.

Proof. From Lemma 3.27 and Lemma 3.28 it follows that ER_q is traceable. For $q \geq 2$, we have $\Delta(ER_q) = q + 1 < \frac{q^2+q}{2} = \frac{|V(ER_q)|-1}{2}$. By Theorem 3.23, it follows that $\overline{ER_q}$ is radio graceful. $\square$

3.3.1 Graceful Radio Labeling of ER_q and $\overline{ER_q}$

We now give a graceful radio labeling of ER_q and $\overline{ER_q}$ using Singer difference sets. First, we verify that the necessary parameters exist. A Singer difference set D is *reduced* if $\{0, 1\} \subset D$. Since a Singer difference set D contains two elements whose difference is congruent to 1 $\pmod{q^2 + q + 1}$, there exists a translation that maps these two elements to 0 and 1, and so every Singer difference set can be translated into a reduced Singer difference set.

Lemma 3.30. *For $d_0 = 0$ and $d_1 = 1$ in a reduced Singer difference set D , there exist natural numbers j_0 and j_1 such that $d_0 - j_0 \notin D$, $d_1 - j_1 \notin D$, and $\gcd((d_0 - j_0) - (d_1 - j_1), q^2 + q + 1) = 1$.*

Proof. Let D be a reduced Singer difference set with $d_0 = 0$ and $d_1 = 1$. For $q \geq 2$, we have $2(q - 1) + 1 < q^2 + q - 1$. Since $\{0, 1\} \in D$ and the elements of D generate the set of all integers from 1 to $q^2 + q$, there is always a pair of consecutive integers $2 \leq n_0, n_1 \leq q^2 + q$ such that $n_0, n_1 \notin D$ where $n_0 > n_1$. Let $j_0 \equiv d_0 - n_0 \pmod{q^2 + q + 1}$ and $j_1 \equiv d_1 - n_1 \pmod{q^2 + q + 1}$. Then $d_0 - j_0 \equiv n_0 \pmod{q^2 + q + 1} \notin D$ and $d_1 - j_1 \equiv n_1 \pmod{q^2 + q + 1} \notin D$. In addition, we have $(d_0 - j_0) - (d_1 - j_1) \equiv n_0 - n_1 \equiv 1 \pmod{q^2 + q + 1}$ and so $\gcd((d_0 - j_0) - (d_1 - j_1), q^2 + q + 1) = 1$. The existence of such parameters has thus been verified. $\square$

Construction 3.31. Let $d_0 = 0$ and $d_1 = 1$ from a reduced Singer difference set D and choose $j_0, j_1 \in \mathbb{N}$ such that $d_0 - j_0 \notin D$, $d_1 - j_1 \notin D$, and $\gcd((d_0 - j_0) - (d_1 - j_1), q^2 + q + 1) = 1$. A Hamiltonian path $v_1, v_2, \dots, v_{q^2+q+1}$ in $\overline{S_q} \cong \overline{ER_q}$ can

be constructed by setting $v_1 = \frac{q^2+q+2}{2}d_1 - 1$, setting $v_i = d_0 - j_0 - v_{i-1}$ if i is even, and setting $v_i = d_1 - j_1 - v_{i-1}$ if i is odd. To generate the radio graceful labeling of ER_q , give each vertex v_i the label i .

Note, by Definition 3.26, that $(v_{i-1}, v_i) \notin E(ER_q)$ as $v_{i-1} + v_i = d_0 - j_0$ if i is even where $d_0 - j_0 \notin D$ and $v_{i-1} + v_i = d_1 - j_1$ if i is odd where $d_1 - j_1 \notin D$.

Construction 3.32. Let $d_0 = 0$ and $d_1 = 1$ from a reduced Singer difference set D , then $\gcd(d_0 - d_1, q^2 + q + 1) = 1$. A Hamiltonian path $v_1, v_2, \dots, v_{q^2+q+1}$ in $S_q \cong ER_q$ can be constructed by setting $v_1 = \frac{q^2+q+2}{2}d_1$, setting $v_i = d_0 - v_{i-1}$ if i is even, and setting $v_i = d_1 - v_{i-1}$ if i is odd [14]. To generate the radio graceful labeling of $\overline{ER_q}$ give each vertex v_i the label i .

4 Conclusions and Open Problems

We have examined the radio gracefulness of different families of cages, including cages with girth 6, 8, or 12, which can be nicely represented as incidence graphs of generalized polygons. In addition, we considered many other known (r, g) -cages detailed in Figure 3. The $(r, 3)$ -cage, the complete graph K_{r+1} , is radio graceful as a direct consequence of the definition of radio labeling. It is known that the $(2, g)$ -cage is radio graceful only when $g \in \{3, 5\}$ [26]. For the remaining graphs, examination of the antipodal graphs for traceability shows almost all of these cages are not radio graceful by Theorem 2.2 or Theorem 2.3. The only exceptions are complete graphs, which are Moore graphs of girth 3, and Moore graphs of girth 5 and 6. Given these facts, we ask the following question.

Question 1. Are the only radio graceful (r, g) -cages Moore graphs of girth 3, 5, and 6?

		g									
		3	4	5	6	7	8	9	10	11	12
r	2	✓	✗	✓	✗	✗	✗	✗	✗	✗	✗
	3	✓	✗	✓	✓	✗	✗	✗	✗	✗	✗
	4	✓	✗	✗	✓		✗				✗
	5	✓	✗	✗	✓		✗				✗
	6	✓	✗	✗	✓		✗				✗
	7	✓	✗	✓	✗						
	8	✓	✗		✓		✗				✗
	9	✓	✗		✓		✗				✗
	10	✓	✗		✓		✗				✗
	11										

Figure 3: Radio gracefulness of various (r, g) -cages

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