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Bipartite oriented graphs with exactly two skew eigenvalues outside [−i, i]

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Abstract

This study begins with a classification of finite simple undirected bipartite graphs whose second largest eigenvalue λ_2 is less than 1, together with a complete characterization of those for which $\lambda_2 = 1$, expressed in terms of their star complements. The star complement technique proves to be particularly effective in this setting, enabling a systematic description of the aforementioned graphs. As an application of the classification, we extend our analysis to two related settings. First, we identify all bipartite signed graphs whose spectrum shares the same spectral property. Second, as the main objective, we consider bipartite oriented graphs and determine all such graphs in which all but two eigenvalues of the skew adjacency matrix lie within the interval [−i, i].

1 Introduction

We consider finite simple undirected graphs, as well as the corresponding signed graphs and oriented graphs. For a graph $G = (V, E)$, a *signed graph* $\dot{G}$ is a pair $(G, \dot{\sigma})$, where $\dot{\sigma}$ is the edge *signature* satisfying $\dot{\sigma}(ij) \in \{1, -1\}$, for every $ij \in E$. The edge set of $\dot{G}$ consists of positive and negative edges. Similarly, an *oriented graph* G' is a pair (G, σ'), where σ' is the edge *orientation* satisfying $\sigma'(ij) \in \{i, j\}$,

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for every $ij \in E$. For G', if $\sigma'(ij) = j$, we say that the edge ij is oriented from i to j. In both cases, we say that G is the *underlying graph* (of G' or $\dot{G}$).

The *adjacency matrix* (of $\dot{G}$) $A_{\dot{G}} = (a_{ij})$ is the $n \times n$ matrix obtained from the standard adjacency matrix of the corresponding underlying graph by reversing the sign of every entry that corresponds to a negative edge. The *eigenvalues* and the *spectrum* of $\dot{G}$ are identified as the eigenvalues and the spectrum of $A_{\dot{G}}$, respectively. In this paper a graph is interpreted as a signed graph with all-positive signature. Accordingly, we will use the same notation for the eigenvalues of a graph and the eigenvalues of a signed graph. In particular, the second largest eigenvalue is denoted by λ_2.

The *skew adjacency matrix* (of G') $S_{G'} = (s_{ij})$ is the $n \times n$ matrix defined by

$$s_{ij} = \begin{cases} 0 & \text{if } ij \notin E, \\ 1 & \text{if } \sigma'(ij) = j, \\ -1 & \text{if } \sigma'(ij) = i. \end{cases}$$

The eigenvalues of $S_{G'}$ are called the *skew eigenvalues* of G' and, together with their multiplicity, form the *skew spectrum* of G'. Since $S_{G'}$ is skew symmetric, its spectrum consists of strictly imaginary numbers. If $\mu \neq 0$ is an eigenvalue of $S_{G'}$, its conjugate $\overline{\mu} = -\mu$ is also an eigenvalue of $S_{G'}$ (since $S_{G'}$ is real), and thus the non-zero eigenvalues come in pairs μ and $-\mu$ (with equal algebraic multiplicity).

Let U be a subset of the vertex set $V(G)$. A signed graph $\dot{G}^U$ (resp. an oriented graph G'^U) obtained by reversing the sign (orientation) of every edge with one endpoint in U and the other in $V(G) \setminus U$ is said to be *switching equivalent* to $\dot{G} = (G, \dot{\sigma})$ (to $G' = (G, \sigma')$). Switching equivalence preserves the spectrum of the associated matrices defined above. A signed graph is said to be *balanced* if it switches to its underlying (unsigned) graph.

The determination of bipartite graphs whose second largest eigenvalue does not exceed 1 was one of the pioneering results obtained by Petrović [5] in addressing the long-standing problem of classifying graphs with comparatively small second largest eigenvalue. According to this result, for a connected bipartite graph G, the inequality $\lambda_2 \leq 1$ holds if and only if G is an induced subgraph of one of the graphs illustrated in the forthcoming Figure 1. In this context, the graphs shown in the same figure are recognized as the maximal connected bipartite graphs satisfying $\lambda_2 = 1$. It is worth mentioning that, at present, graphs with $\lambda_2 \leq 1$ have not yet been fully characterised [3].

In this paper, we revisit the result of [5] and provide a an alternative characterization of the same class of graphs using the concept of star complements. All necessary background on star complements is presented in the next section. Compared to the original result of [5], an advantage of our approach is that the corresponding star complements fall into only a few structural types, which leads to a significantly shorter proof. However, the result of [5] offers the benefit of an explicit enumeration of all maximal connected bipartite graphs satisfying the required spectral condition.

As an application of both the result of [5] and our characterization, we establish a classification of bipartite signed graphs with $\lambda_2 \leq 1$, as well as a classification of

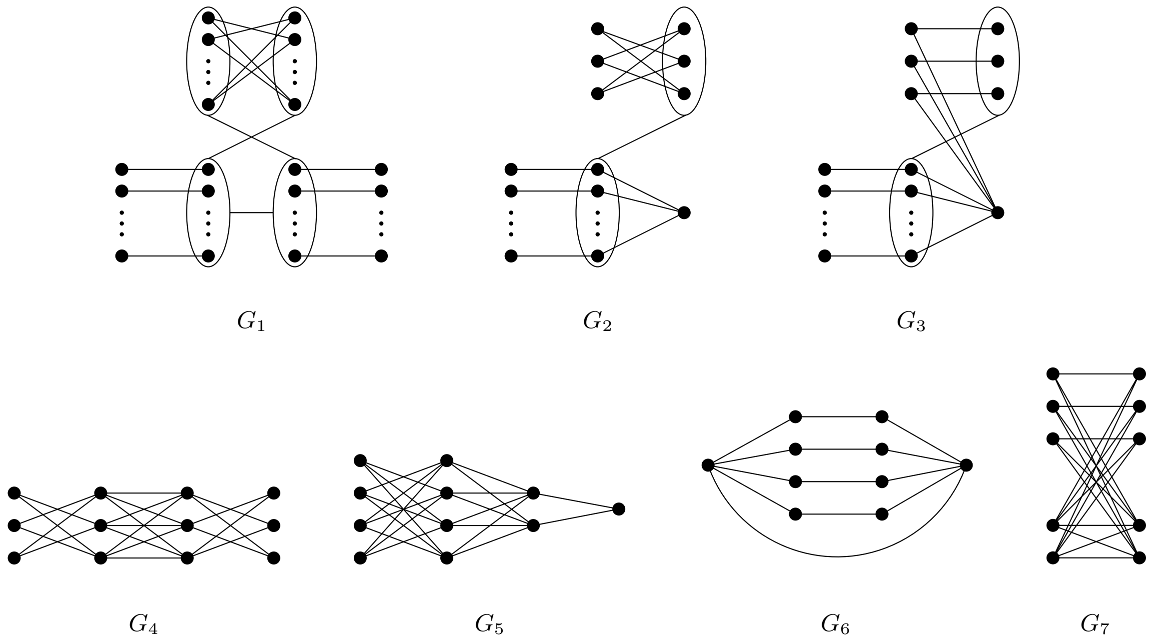

Figure 1: Maximal connected bipartite graphs with $\lambda_2 = 1$. A line between two ellipses indicates that every vertex in one ellipse is adjacent to every vertex in the other ellipse.

oriented graphs with exactly two skew eigenvalues outside the interval $[-\mathrm{i}, \mathrm{i}]$. The latter classification relies on recently established relationships between the eigenvalues of a signed graph and the skew eigenvalues of an associated oriented graph, with a precise meaning of the term 'associated' discussed in due course.

This paper contributes to the extensive body of research on graphs whose second largest eigenvalue is bounded by a fixed comparatively small constant, a topic of significant interest in spectral graph theory due to its connections with expansion properties, pseudo-randomness, and structural characterisations; see [7] for a comprehensive survey, and [3, 10] along with the references therein for recent developments. In particular, the study is extended here to encompass both signed and oriented graphs, broadening the scope of classical results. In addition, a strong connection is established with the work of [2], which provides a complete classification of signed graphs having at most two eigenvalues distinct from ± 1. The bipartite instances within that classification naturally emerge as a special case of the framework developed in this paper, and can be identified through a careful analysis of the solutions obtained herein.

Section 2 presents preliminary results, primarily focusing on star complements in (signed) graphs. In Section 3, we determine all bipartite signed graphs with $\lambda_2 < 1$, which in turn serve as the star complementary characterization of graphs with $\lambda_2 = 1$, as described in Section 4. Finally, two applications, one in the domain of signed graphs and another in the domain of oriented graphs, are given in Section 5.

2 Preliminaries

In the entire paper, we use the standard notation P_n, C_n and $K_{r,n-r}$ to denote the n-vertex path, cycle, and complete bipartite graph, respectively. In particular, $K_{1,n-1}$ is referred to as the *star* graph, and the vertex of maximum degree is called its centre. A *double star* graph, denoted by $D_{r,s}$, is formed by connecting the centres of two stars, $K_{1,r}$ and $K_{1,s}$, with a single edge.

We also write O and J to denote the zero matrix and the all-ones matrix, respectively, with the order indicated by a subscript when needed.

Given a graph G, a partition $V(G) = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ is called an *equitable partition* of the vertex set if every vertex in V_i has the same number of neighbours in V_j, for all i, j. The matrix $Q = (q_{ij})$, where q_{ij} denotes the number of neighbours a vertex in V_i has in V_j, is called the *quotient matrix* (or divisor matrix) of G. The characteristic polynomial of Q divides the characteristic polynomial of the adjacency matrix A_G [1, Theorem 3.9.5].

In what follows, we provide the definition of star complements and list the corresponding results. Our exposition is framed in the context of signed graphs, as this perspective will be essential in the final section. All results remain valid in the unsigned setting, provided that an unsigned graph is interpreted as a special case of a signed graph.

A *star complement* is defined as an induced subgraph $\dot{H}$ of order k in an n-vertex signed graph $\dot{G}$, such that μ is an eigenvalue of $\dot{G}$ with multiplicity $n-k$, while μ is not an eigenvalue of $\dot{H}$. Here, the multiplicity is understood to be zero when μ is not an eigenvalue of $\dot{G}$. We note in passing that, by the eigenvalue interlacing argument, a star complement exists for every signed graph and any its eigenvalue. In this context, $\dot{G}$ is referred to as an *extension* (precisely, a *μ-extension*) of $\dot{H}$. The case $n = k$ is allowed, and we say that $\dot{H}$ is a *proper* star complement for a prescribed eigenvalue μ if there exists at least one one-vertex μ-extension of $\dot{H}$. A μ-extension is *maximal* if it is not a proper induced subgraph of any other μ-extension. We remark that if $\dot{H}$ is a proper star complement for μ in $\dot{G}$, and $\dot{F}$ is an induced subgraph of $\dot{G}$ that contains $\dot{H}$, then $\dot{H}$ is a star complement for μ in $\dot{F}$. In the sequel, we give some basic statements.

Lemma 2.1 ([8]). *Let $\dot{H}$ be a star complement for μ in a signed graph $\dot{G}$, and $S = V(\dot{G}) \setminus V(\dot{H})$.*

(i) If $\mu \neq 0$, then every vertex of S is adjacent to at least one vertex of $V(\dot{H})$.

(ii) If $\mu \notin \{1, 0, -1\}$, then any two vertices of S have distinct neighbourhoods in $V(\dot{H})$.

The next result is known as the Reconstruction Theorem; recognized as crucial in the context of star complements. The proof follows the general line of the original result formulated for unsigned graphs. However, an anonymous referee suggested the following alternative approach, which differs from the original proof and applies in a broader setting, including signed graphs.

Theorem 2.2 ([8], cf. [1, Theorem 5.1.7]). *Given a signed graph $\dot{G}$ with the adjacency matrix*

$$\begin{pmatrix} A_S & B^\intercal \\ B & A_{\dot{H}} \end{pmatrix},$$

where A_S is the adjacency matrix of the subgraph induced by a vertex set S, while $\dot{H}$ is the subgraph induced by $V(\dot{G})\setminus S$. Then $\dot{H}$ is a star complement for μ if and only if μ is not an eigenvalue of $\dot{H}$ and

$$\mu I - A_S = B^\intercal(\mu I - A_{\dot{H}})^{-1}B.$$

Proof. By [4, p. 11], if A_{22} is an invertible $k\times k$ matrix, then an $n\times n$ block-partitioned matrix can be written as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{pmatrix}. \tag{1}$$

In particular,

$$\operatorname{rank}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = k + \operatorname{rank}\big(A_{11} - A_{12}A_{22}^{-1}A_{21}\big). \tag{2}$$

If μ is not an eigenvalue of $A_{\dot{H}}$, then we can select

$$A_{22} = \mu I - A_{\dot{H}}, \quad A_{11} = \mu I - A_S, \quad A_{12} = -B^\top, \quad A_{21} = -B.$$

With this selection, the matrix in (1) equals $\mu I - A_{\dot{G}}$. Therefore, if m_μ denotes the multiplicity of μ in $A_{\dot{G}}$, then (2) implies that

$$n - m_\mu = \operatorname{rank}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = k + \operatorname{rank}\big(\mu I - A_S - B^\intercal(\mu I - A_{\dot{H}})^{-1}B\big).$$

Hence, $m_\mu = n - k$ if and only if

$$\mu I - A_S = B^\intercal(\mu I - A_{\dot{H}})^{-1}B,$$

as claimed. □

For $k = |V(\dot{H})|$, we introduce the following bilinear form on $\mathbb{R}^k$, for which we keep the previous assumption that μ is not an eigenvalue of $\dot{H}$:

$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^\intercal(\mu I - A_{\dot{H}})^{-1}\mathbf{y}.$$

An efficient method for computing the inverse of $\mu I - A_{\dot{H}}$ is given in [1, Proposition 5.2.1]: If the minimal polynomial of $A_{\dot{H}}$ is $\phi(x) = \sum_{i=0}^{d} c_i x^i + x^{d+1}$ and μ is not an eigenvalue of $\dot{H}$, then

$$\phi(\mu)\big(\mu I - A_{\dot{H}}\big)^{-1} = \sum_{i=0}^{d} a_i A_{\dot{H}}^i, \tag{3}$$

with $a_d = 1$ and $a_{d-j} = \sum_{i=0}^{j-1} c_{d-i}\mu^{j-1-i} + \mu^j$, for $1 \le j \le d$.

We proceed with a direct consequence of the Reconstruction Theorem.

Corollary 2.3 ([1, Corollary 5.1.8]). *If μ is not an eigenvalue of a signed graph $\dot{H}$ with k vertices, then there is a signed graph $\dot{G}$ with n vertices having $\dot{H}$ as a star complement for μ if and only if there exist $(0,1,-1)$-vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n-k} \in \mathbb{R}^k$ such that*

$$\langle \mathbf{b}_i, \mathbf{b}_i \rangle = \mu \quad \text{and} \quad \langle \mathbf{b}_i, \mathbf{b}_j \rangle \in \{1, 0, -1\},$$

for all distinct $i, j \in S = V(\dot{G}) \setminus V(\dot{H})$.

Here, $\mathbf{b}_i$ and $\mathbf{b}_j$ determine the neighbourhoods of i and j in $\dot{H}$, respectively.

Clearly, the vectors $\mathbf{b}_i$ $(1 \leq i \leq n-k)$ form the submatrix B of the Reconstruction Theorem. Also, i and j are not adjacent if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$, joined by a positive edge if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = -1$, and joined by a negative edge if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 1$.

We also mention the following argument.

Theorem 2.4 ([1, Theorem 5.1.6]). *Let μ be an eigenvalue of a connected signed graph $\dot{G}$, and let $\dot{H}$ be a connected induced subgraph of $\dot{G}$ not having μ as an eigenvalue. Then $\dot{G}$ has a connected star complement for μ containing $\dot{H}$.*

Consequently, except when $\dot{G} \cong K_1$ and $\mu = 0$, every connected signed graph contains a connected star complement for any of its eigenvalues.

3 Bipartite graphs with $\lambda_2 < 1$

In this section, we determine all connected bipartite graphs G whose second largest eigenvalue satisfies $\lambda_2 < 1$. Clearly, any disconnected bipartite graph with the same spectral property must be a disjoint union of such a connected graph and a collection of isolated vertices.

We begin by examining trees and unicyclic graphs. The diameter of G does not exceed 3; otherwise, G would contain the path P_5 as an induced subgraph, which is impossible since $\lambda_2(P_5) = 1$. Therefore, if G is a tree, it must be either a star or a double star. For every star $K_{1,s}$, with $s \geq 2$, we have $\lambda_2 = 0 < 1$.

In contrast, the double star $D_{1,s}$ also satisfies the required condition, which can be verified either by applying Schwenk's formula [1, Theorem 2.3.4] or by analyzing the corresponding quotient matrix. Since $\lambda_2(D_{2,2}) = 1$, and by the eigenvalue interlacing theorem, we deduce that this family of double stars exhausts all possible solutions within the class of trees.

Let G be a bipartite unicyclic graph, that is, a connected bipartite graph with cyclomatic number equal to 1. Since $\lambda_2(C_n) \geq 1$ for all $n \geq 6$, it follows that the unique induced cycle in G must be C_4.

Moreover, adding an edge between a vertex of C_4 and an end of the path P_2 results in $\lambda_2 > 1$. Hence, only pendant vertices are allowed to be attached to C_4. A simple analysis shows that at most two such pendant vertices can be attached, either both at the same vertex or at two adjacent vertices of C_4, as P_5 is forbidden.

From this point onward, we assume that G is at least bicyclic, and denote its colour classes by V_1 and V_2.

Suppose that G contains at least two pendant vertices both belonging to V_1. They must share the same neighbour, as otherwise P_5 would appear as an induced subgraph. Moreover, this neighbour is adjacent to all vertices in V_1, since otherwise $2K_2$ (with $\lambda_2 = 1$) would be an induced subgraph. Because G is at least bicyclic, at least one colour class contains three non-pendant vertices. Since $D_{2,2}$ and $2K_2$ are forbidden, this cannot be V_2. Thus, V_2 has exactly two such vertices. By the previous discussion on unicyclic graphs, neither V_1 nor V_2 contains an additional pendant vertex, leading to the unique remaining structure: G is obtained by attaching two pendant vertices to a single vertex of degree s in $K_{2,s}$. The quotient matrix is

$$\begin{pmatrix} 0 & 0 & 2 & s \\ 0 & 0 & 0 & s \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix},$$

giving the minimal polynomial $x(x^4 - 2(s+1)x^2 + 2s)$ being negative in $x = 1$, which means that $\lambda_2 < 1$ holds for every s. Here and henceforth, we use the fact that the remaining eigenvalues of G are all zero, which follows from inspecting vertices that share the same neighbourhoods.

Suppose that G contains at least two pendant vertices belonging to distinct colour classes. We have already established that each of their neighbours must be adjacent to all vertices in the opposite class. If, apart from these fixed pendant vertices, G contains two additional non-adjacent vertices belonging to distinct classes, then G would contain $D_{2,2}$ as an induced subgraph. Therefore, G is obtained from $K_{r,s}$ by attaching a pendant vertex to a vertex in one colour class and another pendant vertex to a vertex in the other colour class. In addition, the second largest eigenvalue of this graph is less than 1, as follows by considering the corresponding quotient matrix.

In what follows, we consider the remaining scenarios but provide less detail, since the arguments closely mirror the previous ones and repeating them would render the exposition unnecessarily verbose. In particular, we assume throughout that the diameter of G is at most 3, and that neither $D_{2,2}$ nor $2K_2$ appears as an induced subgraph.

Suppose now that G contains exactly one pendant vertex. Following the same general line of reasoning as in the previous case (though not identically, as slight adjustments are required) one can deduce that G is obtained either by attaching a pendant vertex to $K_{r,s}$, or by removing an edge that is not adjacent to the newly added pendant edge in the resulting graph. For clarity, removing any other edge would produce a forbidden unicyclic subgraph.

Suppose that G has no pendant vertices. If each colour class contains at least two vertices that are not adjacent to every vertex of the opposite class, then G contains an induced subgraph obtained by removing two parallel edges from $K_{4,4}$. This is impossible, since the second largest eigenvalue of this particular subgraph equals 1, as confirmed by direct computation.

The remaining scenario assumes that at most one vertex of V_1 is non-adjacent to all vertices of V_2. If no such vertex exists, then G is the complete bipartite graph $K_{r,s}$. Otherwise, denote this vertex by u. We distinguish the following cases.

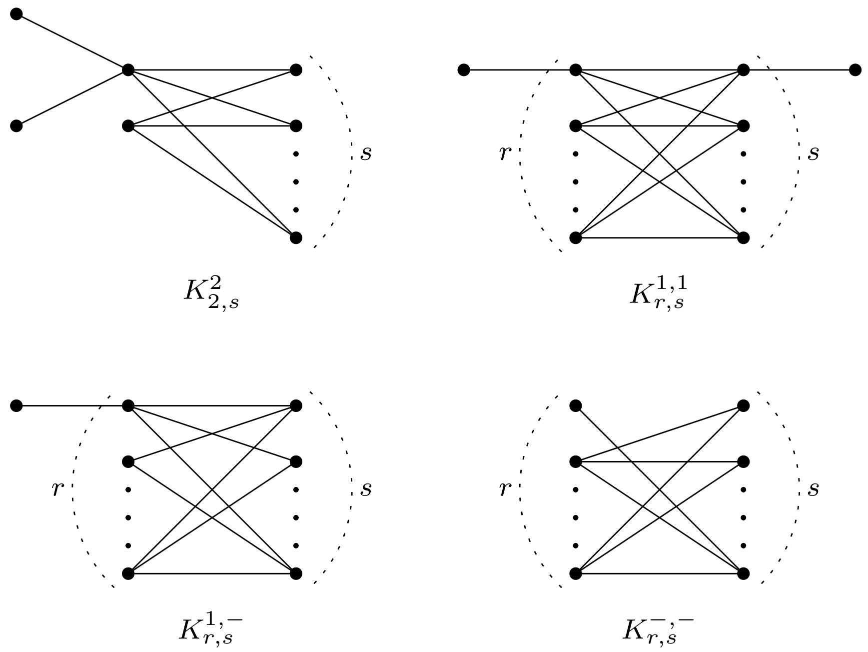

Figure 2: Graphs for Theorem 3.1.

If u is adjacent to exactly one vertex of V_2, then G is obtained from $K_{r,s}$ by attaching an additional pendant vertex.

If u is adjacent to all but one vertex of V_2, then $\lambda_2 < 1$, as follows from considering the quotient matrix.

The quotient matrix argument is also used in the remaining cases. If u is adjacent to all but two vertices of V_2, then:

- for $|V_2| \leq 4$, the set V_1 may be arbitrary;
- for $|V_2| \geq 5$, we have $|V_1| \leq 2$, and in this case V_2 may be arbitrary.

Finally, the scenario in which u is adjacent to all but three or more vertices of V_2 is impossible under the condition that G is at least bicyclic, as it would then contain a C_4 with three vertices attached at a single vertex–an already excluded configuration.

Gathering the conclusions from the previous discussion, we arrive at the following result.

Theorem 3.1. *The second largest eigenvalue of a connected bipartite graph G is less than* 1 *if and only if G is an induced subgraph of at least one of the following graphs:*

(i) $K^2_{2,s}$, *obtained by attaching two pendant vertices at a vertex of degree s of $K_{2,s}$;*

(ii) $K^{1,1}_{r,s}$, *obtained by attaching a pendant vertex at a vertex of degree r and a pendant vertex at a vertex of degree s of $K_{r,s}$;*

(iii) $K^{1,-}_{r,s}$, *obtained by attaching a pendant vertex at a vertex of degree s of $K_{r,s}$ and removing an edge which is non-adjacent to the previously added edge;*

(iv) $K^{-,-}_{r,s}$, *obtained by removing two edges incident with a vertex of degree s in $K_{r,s}$, where either $r \leq 2$ or $s \leq 4$.*

Proof. The preceding discussion is exhaustive: it first considers trees and unicyclic graphs, and then divides all remaining graphs according to the number and distribution of their pendant vertices. Each case leads to an induced subgraph of one of the graphs in (i)–(iv), including the complete bipartite graphs and double stars. Conversely, the computations above show that all four families have the required spectral property, which is inherited by their induced subgraphs. □

The graphs of the previous theorem are visualized in Figure 2.

4 Star complementary characterization of bipartite graphs with $\lambda_2 \leq 1$

Theorem 4.1. *Let G be a connected bipartite graph. Then $\lambda_2(G) = 1$ holds if and only if G contains, as a proper induced subgraph, a graph of Theorem 3.1 as a star complement for* 1.

Proof. Suppose that $\lambda_2(G) = 1$. Then G contains a star complement for this particular eigenvalue. Accordingly, the second largest eigenvalue of a star complement is less than 1. By Theorem 2.4, there is a connected star complement. The claim follows since Theorem 3.1 establishes all connected graphs with the desired spectral property.

The converse follows directly from the definition of a star complement. □

We note that the graphs described in Theorem 3.1 belong to only a few structural types, each obtained from a complete bipartite graph by the addition of at most two vertices and/or the deletion of at most two edges. This observation reinforces the robustness of the preceding characterization.

Recalling from Section 2 that a graph H is a star complement for a prescribed eigenvalue μ even if there is no any one-vertex μ-extension of H, we arrive at the following corollary, in fact a characterization of bipartite graphs whose second largest eigenvalue does not exceed 1.

Corollary 4.2. *Let G be a connected bipartite graph. Then $\lambda_2(G) \leq 1$ holds if and only if G contains a graph of Theorem 3.1 as a star complement for* 1.

In what follows, we consider the two particular cases, that is, bipartite graphs that admit either the complete bipartite graph $K_{r,s}$ or its subgraph $K_{r,s}^-$, obtained by removing a single edge, as a star complement for the eigenvalue 1.

Theorem 4.3. *For $rs > 1$, the graph $K_{r,s}$ is a proper star complement for* 1 *in a bipartite graph if and only if $(r, s) = (5, 1)$ or $(r, s) = (5, 2)$. In addition, the maximal bipartite extensions having $K_{5,1}$ (resp. $K_{5,2}$) as a star complement for* 1 *are the graph G_2 with* 9 *vertices and G_6 (G_5 and G_7) of Figure 1.*

Proof. The minimal polynomial of $K_{r,s}$ is $\phi(x) = x(x^2 - rs)$. By employing (3), we compute

$$(1 - rs)(I - A_{K_{r,s}})^{-1} = \begin{pmatrix} sJ_{r,r} & J_{r,s} \\ J_{s,r} & rJ_{s,s} \end{pmatrix} + (1 - rs)I.$$

Since we are interesting in bipartite extensions, without loss of generality we set

$$\mathbf{b} = (\underbrace{1, 1, \dots, 1}_{p}, \underbrace{0, 0, \dots, 0}_{r+s-p})^{\intercal},$$

for $p \leq r$. From $\mathbf{b}^{\intercal}(I - A_{K_{r,s}})^{-1}\mathbf{b} = 1$, we get

$$\mathbf{b}^{\intercal}\left(\begin{pmatrix} sJ_{r,r} & J_{r,s} \\ J_{s,r} & rJ_{s,s} \end{pmatrix} + (1 - rs)I\right)\mathbf{b} = 1 - rs.$$

This leads to $(p-1)(1-rs)+p^2s = 0$. The case $p = 1$ opens the impossible scenario $s = 0$. Otherwise, by solving in r, we obtain $r = 1 + p + \frac{1}{p-1} + \frac{1}{s}$. Since r is an integer, we have either $(p, r, s) = (2, 5, 1)$ or $(p, r, s) = (3, 5, 2)$. In conclusion, $K_{r,s}$ has at least one one-vertex extension for $\mu = 1$ if and only if $(r, s) \in \{(5, 1), (5, 2)\}$ (up to symmetry between r and s).

Henceforth, let $\mathbf{b}_i, \mathbf{b}_j$ be the columns of the matrix B that appears in the Reconstruction Theorem.

For $K_{5,1}$, we compute that $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$ holds if and only if the corresponding vectors agree in exactly one position; that is, i and j share exactly one common neighbour in the larger colour class. The case $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = -1$ is not of interest, as it necessarily leads to a non-bipartite graph. There exist exactly two maximal collections of 2-vertex subsets of a 5-vertex set such that every pair of subsets intersects in exactly one vertex. The first collection contains three such subsets and leads to the graph G_2 with nine vertices (the ellipse in the figure contains a single vertex). The second collection consists of four subsets, all sharing a fixed vertex, which corresponds to the graph G_6.

For $K_{5,2}$, $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$ holds if and only if i and j share exactly two neighbours. Using the similar argument, we arrive at two maximal collections of 3-vertex subsets each sharing two vertices. They give rise to G_5 and G_7. □

The assumption $rs > 1$ in the previous theorem eliminates the case $K_{1,1}$ which has 1 in the spectrum. We have a similar situation in the next result eliminating a disconnected star complement.

Theorem 4.4. *For $r, s \geq 2$ and $r + s \neq 7$, the graph $K^-_{r,s}$ is a proper star complement for* 1*. Its maximal extension is obtained by removing a perfect matching from $K_{r+s-1,\, r+s-1}$.*

Proof. The case $(r, s) = (2, 2)$ is verified directly. Otherwise, enumerate vertices of the first colour class by $1, 2, \dots r$, the vertices of the second class by $r+1, r+2, \dots, r+s$ and suppose that 1 and $r + 1$ are non-adjacent. The corresponding quotient matrix is

$$\begin{pmatrix} 0 & 0 & 0 & s-1 \\ 0 & 0 & 1 & s-1 \\ 0 & r-1 & 0 & 0 \\ 1 & r-1 & 0 & 0 \end{pmatrix},$$

meaning that the minimal polynomial is $\phi(x) = x(x^4 + (1 - rs)x^2 + rs - r - s + 1)$. Accordingly, the coefficients of (3) are $a_4 = a_3 = 1$, $a_2 = a_1 = 2-rs$ and $a_0 = 3-r-s$.

It remains to compute the powers (up to the fourth) of the adjacency matrix $A = A_{K^-_{r,s}}$. However, since we are interested in bipartite extensions to obtain admissible vectors $\mathbf{b}$, it suffices to compute only the top-left block of each matrix power. This is because $\mathbf{b}$ is defined by the connections from new vertices to the original ones, and the relevant structure lies in the interactions among the vertices in one colour class. Clearly, the top-left block is the zero matrix in both A and A^3, due to the bipartite nature of the graph. For A^2, each entry in the first row and the first column is $s-1$, while the remaining entries of the block are equal to s. Finally, for A^4, the structure of the top-left block is slightly more involved, and is computed as follows:

$$\begin{pmatrix} (s-1)^2r & (s-1)(sr-1) & (s-1)(sr-1) & \cdots & (s-1)(sr-1) \\ (s-1)(sr-1) & s^2r-2s+1 & s^2r-2s+1 & \cdots & s^2r-2s+1 \\ (s-1)(sr-1) & s^2r-2s+1 & s^2r-2s+1 & \cdots & s^2r-2s+1 \\ \vdots & & & \vdots & \vdots \\ (s-1)(sr-1) & s^2r-2s+1 & s^2r-2s+1 & \cdots & s^2r-2s+1 \end{pmatrix}.$$

The same block in $A^4 + A^3 + (2 - rs)(A^2 + A) + (3 - r - s)I$ is

$$\begin{pmatrix} 1+s-rs & (s-1) & (s-1) & \cdots & (s-1) \\ (s-1) & 4-r-s & 1 & \cdots & 1 \\ (s-1) & 1 & 4-r-s & \cdots & 1 \\ \vdots & & & \vdots & \vdots \\ (s-1) & 1 & 1 & \cdots & 4-r-s \end{pmatrix}.$$

From $\phi(1) = 3 - r - s$, we deduce that $\phi(1)\langle \mathbf{b}, \mathbf{b}\rangle = 3 - r - s$ must hold.

Supposing that $\mathbf{b} = (0, \underbrace{1, 1, \dots, 1}_{p}, 0, 0, \dots, 0)^\intercal$, we obtain $p^2-(p-1)(r+s-3) = 0$ without solutions in $\mathbb{Z}^+$ for p unless $r + s = 7$, but this possibility has been excluded in the statement formulation.

For $\mathbf{b} = (\underbrace{1, 1, \dots, 1}_{p+1}, 0, 0, \dots, 0)^\intercal$, we arrive at $(p + 2 - r)s = 0$, giving $p = r - 2$. We proceed to determine the compatibility between the corresponding vertices.

If i and j belong to the same colour class, then $\langle \mathbf{b}_i, \mathbf{b}_j\rangle = 0$ holds whenever they have distinct neighbourhoods (i.e., the corresponding vectors agree at p positions), as can be verified directly. On the other hand, if i and j belong to different colour classes, then $\langle \mathbf{b}_i, \mathbf{b}_j\rangle \neq 0$, whereas $\langle \mathbf{b}_i, \mathbf{b}_j\rangle = -1$ holds unconditionally. The latter can be confirmed either by directly computing the bilinear form (which requires the full inverse matrix $(I - A)^{-1}$ and not just its top-left block) or by computing the quotient matrix after extending the given star complement by adding vertices i and j (which involves less computational effort).

Finally, to obtain the maximal extension, we add $r-1$ vertices, each non-adjacent to exactly one vertex from the set $\{2, 3, \dots, r\}$, and analogously, a set of $s-1$ vertices corresponding to the other colour class. The subgraph induced by these additional vertices forms a complete bipartite graph, which completes the construction of the desired maximal extension. □

The case $r+s=7$ gives rise to several maximal extensions, depicted in Figure 1. We omit a detailed analysis, as it is comparatively less interesting and involves numerous technical cases, and merely observe that, for example, G_4 arises as one such extension for $(r,s)=(3,4)$.

It often occurs that a graph of Figure 1 contains more than one star complements for 1. Say, in the case of G_4, they are $K_{2,4}$ with a pendant vertex attached to vertex of degree 4, $K_{3,4}$ with an edge removed, $K_{3,4}^{--}$ and $K_{2,3}^{1,1}$ (with the notation of Theorem 3.1).

5 Signed and oriented case

Since the second largest eigenvalue of an unbalanced cycle $\dot{C}_{2n}$ is $2\cos\frac{\pi}{2n}$ (see [6]), by employing the eigenvalue interlacing we deduce that every bipartite signed graph with $\lambda_2 \leq 1$ is balanced. Therefore, the statements of the previous section are transferred directly. In particular, we have the following characterization.

Theorem 5.1. *Let $\dot{G}$ be a connected bipartite signed graph. Then $\lambda_2(\dot{G}) \leq 1$ holds if and only if $\dot{G}$ contains a switching of a graph of Theorem 3.1 as a star complement for* 1.

We turn to the oriented graphs. The following results are taken from [9]. For a signed graph $\dot{G}=(G,\dot{\sigma})$ and an oriented graph $G'=(G,\sigma')$, the signature $\dot{\sigma}$ is *associated* with the orientation σ' (and also $\dot{G}$ is *associated* with G') if

$$\dot{\sigma}(ik)\dot{\sigma}(jk)=s_{ik}s_{jk} \text{ holds for every pair of edges } ik \text{ and } jk, \tag{4}$$

where s_{ik} is the (i,k)-entry of the skew adjacency matrix of G' defined in the introductory section. Being associated is a symmetric relation, and we write $\sigma \sim \sigma'$ to designate that a signature and a orientation defined on a fixed graph are mutually associated. For a graph G and an orientation σ', there exists a signature $\dot{\sigma}$ associated with σ' if and only if G is bipartite. Moreover, for a fixed signature $\dot{\sigma}$, the orientation σ' is obtained as follows: Start by assigning an arbitrary orientation to one edge. Then, the orientations of all remaining edges are uniquely determined by the rule (4). In practice, adjacent edges of G' are oriented either both toward or both away from their common vertex if and only if the corresponding edges in $\dot{G}$ have the same sign.

The following result establishes the fundamental spectral relationship between the associated graphs.

Theorem 5.2. *For a bipartite graph G, let $G'=(G,\sigma')$ and $\dot{G}=(G,\dot{\sigma})$. If $\dot{\sigma} \sim \sigma'$, then $\pm\mathrm{i}\lambda_1, \pm\mathrm{i}\lambda_2, \dots, \pm\mathrm{i}\lambda_k, 0^{n-2k}$ are the skew eigenvalues of G' if and only if $\pm\lambda_1, \pm\lambda_2, \dots, \pm\lambda_k, 0^{n-2k}$ are the eigenvalues of $\dot{G}=(G,\dot{\sigma})$.*

A bipartite oriented graph is *oriented uniformly* if every edge is oriented from a fixed colour class to the other colour class.

Theorem 5.3. *Let G' be a connected bipartite oriented graph with at least 3 vertices. All but two skew eigenvalues of G' lie in the interval $[-\mathrm{i}, \mathrm{i}]$ if and only if the underlying graph of G' is one of the graphs described in Corollary 4.2, and G' is switching equivalent to a uniformly oriented graph.*

Proof. If exactly two skew eigenvalues lie outside the given interval of purely imaginary numbers, then exactly two eigenvalues of an associated signed graph lie outside the interval $[-1, 1]$. By Theorem 5.1, such a signed graph switches to a graph of Corollary 4.2. The pathological cases, consisting of signed graphs with at most two vertices, are excluded since in these cases all eigenvalues lie within the given interval. The desired result follows since the signed graph with the all-positive signature is associated with a uniformly oriented graph. This completes the argument in one direction.

The converse is proved by reversing the previous reasoning. Specifically, if a uniformly oriented graph corresponds to a signed graph that switches to one of the graphs in Corollary 4.2, with exactly two eigenvalues outside $[-1, 1]$, then the original oriented graph has exactly two skew eigenvalues outside $[-\mathrm{i}, \mathrm{i}]$. □

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