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Volumes of consecutively defined sets

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Abstract

We study a variant of the graph polytopes of a path and of a cycle where we replace the inequality $x_i + x_{i+1} \leq 1$ with the two inequalities $(1-\alpha) \cdot x_i + \alpha \cdot x_{i+1} \leq \alpha$ for $0 \leq x_i \leq \alpha$ and $\alpha \cdot x_i + (1-\alpha) \cdot x_{i+1} \leq \alpha$ for $\alpha \leq x_i \leq 1$. Using a self-adjoint operator and its eigenvalues we obtain convergent series for their volumes. As a corollary we obtain that the volumes of the set associated to a path on n vertices and the set associated to a cycle on $n+1$ vertices are related by a constant factor of α.

1 Introduction

The two polytopes

$$\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_i + x_{i+1} \leq 1 \text{ for } 1 \leq i \leq n-1\}, \tag{1.1}$$

$$\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_i + x_{i+1} \leq 1 \text{ for } 1 \leq i \leq n\}, \tag{1.2}$$

where $x_{n+1} = x_1$, have been well studied in the literature. See the papers [1, 5, 8, 9, 11]. These two polytopes can also be described as the graph polytopes of a path, respectively, a cycle; see the concluding remarks at the end of the paper. They are closely related to alternating permutations and hence the Euler numbers. In this paper we introduce a generalization of these polytopes by replacing the inequality between two adjacent variables with the pair of inequalities

$$\begin{aligned} \frac{1-\alpha}{\alpha} \cdot x_i + x_{i+1} &\leq 1, && \text{for } 0 \leq x_i \leq \alpha, \\ x_i + \frac{1-\alpha}{\alpha} \cdot x_{i+1} &\leq 1, && \text{for } \alpha \leq x_i \leq 1. \end{aligned}$$

See Figure 1 for a visualization of these inequalities. When $\alpha = 1/2$ they yield the inequality $x_i + x_{i+1} \leq 1$ and hence the two polytopes (1.1) and (1.2). Note that when $1/2 < \alpha < 1$ our new sets are also polytopes, whereas for the interval $0 < \alpha < 1/2$ the sets are non-convex.

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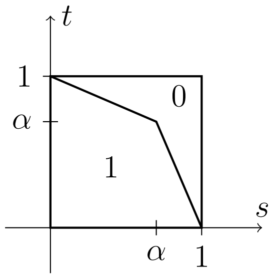

Figure 1: The symmetric function χ.

To be more explicit, let $\ell_1(t)$ and $\ell_2(t)$ be the two affine linear functions

$$\ell_1(t) = 1 - \frac{1-\alpha}{\alpha} \cdot t, \qquad \ell_2(t) = \alpha \cdot \frac{1-t}{1-\alpha}.$$

Observe that they are inverses of each other. Also note that $\ell_1(0) = 1$, $\ell_1(\alpha) = \ell_2(\alpha) = \alpha$ and $\ell_2(1) = 0$. That is, ℓ_1 maps the interval $[0, \alpha]$ to the interval $[\alpha, 1]$. Similary, ℓ_2 maps the interval $[\alpha, 1]$ back to the interval $[0, \alpha]$. Define the self-inverse function $b(t)$ by

$$b(t) = \begin{cases} \ell_1(t) & \text{if } 0 \leq t \leq \alpha, \\ \ell_2(t) & \text{if } \alpha \leq t \leq 1. \end{cases} \tag{1.3}$$

Now define the two n-dimensional sets P_n and Q_n by

$$\begin{aligned} P_n &= \{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_{i+1} \leq b(x_i) \text{ for } 1 \leq i \leq n-1\}, \\ Q_n &= \{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_{i+1} \leq b(x_i) \text{ for } 1 \leq i \leq n\}, \end{aligned}$$

where we use that $x_{n+1} = x_1$ in the definition of the set Q_n. Again note how the set P_n corresponds to a path and Q_n to a cycle.

In Theorem 3.3 we express the volumes of these sets in terms of convergent series. Our technique is to reformulate the problem into understanding a linear operator T and its spectrum. This linear operator is on the space $L^2([0,1])$ and is the continuous analogue of a transfer matrix. That is, the kernel $\chi(s,t)$ plays the role of the adjacency matrix, T replaces multiplication with this matrix, the inner product $(1, T^{n-1}(1))$ is the continuous analogue of path enumeration and the trace $\mathrm{tr}(T^n)$ is the cycle enumeration. In Section 2 we introduce the operator in a more general case, where we replace the function $b(t)$ with a more general self-inverse function $c(t)$. In Section 3 we obtain the spectrum for our operator in the piecewise linear function case and hence the sought-after volumes of our two sets. A particularly striking consequence is Corollary 3.4 which states that the volumes of P_n and Q_{n+1} differ by a multiplicative constant α. We also use an identity of Lipschitz to give an explicit expression for the volume of the set Q_n. We end the paper with open questions and directions for future research.

2 The general case of a self-inverse function

In this section we take a more general approach in understanding the volumes of the two sets P_n and Q_n. We introduce a linear operator T on the space $L^2([0,1])$, that is, the space of square integrable functions on the interval $[0,1]$. The role of the operator T is analogous to a transfer operator for consecutive constraints. Repeated application of this operator builds admissible sequences of coordinates, so the volume of P_n is obtained from T^{n-1}, while the volume of Q_n is the trace of T^n. Because the kernel is symmetric, T is self-adjoint and can be diagonalized. Furthermore, by the spectral theorem, the operator only has real eigenvalues.

Let $c : [0,1] \longrightarrow [0,1]$ be a decreasing continuous function that is its own inverse. That is, we have $c(0) = 1$, $c(1) = 0$ and $c(c(t)) = t$. Let $\chi(s,t)$ be a function on the unit square $[0,1]^2$ defined by

$$\chi(s,t) = \begin{cases} 1 & \text{if } s \leq c(t), \\ 0 & \text{otherwise.} \end{cases}$$

Apply the decreasing function c to the inequality $s \leq c(t)$ and we obtain $c(s) \geq c(c(t)) = t$. By a second application of the function c we have that the two inequalities are equivalent. Hence the function χ is symmetric, that is, $\chi(s,t) = \chi(t,s)$. Let P_n and Q_n be the two n-dimensional sets defined by

$$\begin{aligned} P_n &= \{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_{i+1} \leq c(x_i) \text{ for } 1 \leq i \leq n-1\}, \\ Q_n &= \{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_{i+1} \leq c(x_i) \text{ for } 1 \leq i \leq n\}. \end{aligned}$$

Note that when the function $c(t)$ is concave down the two sets P_n and Q_n are convex. Observe that the two sets introduced in the introduction is a special case of this setup.

Define the operator T on the space $L^2([0,1])$ by

$$T(f)(t) = \int_0^1 \chi(t,s) \cdot f(s)\, ds = \int_0^{c(t)} f(s)\, ds.$$

The operator T is a Hilbert–Schmidt operator with kernel χ. That should be viewed as the continuous analogue of matrix times vector multiplication where the function $\chi(t,s)$ corresponds to the matrix. Furthermore, since the function χ is symmetric, the operator T is self-adjoint. Again, the discrete analogue is a symmetric matrix. The spectral theorem [3, XIII.4 Theorem 2] implies that all the eigenvalues are real and the eigenfunctions form a complete orthogonal set.

We also observe that 0 is not an eigenvalue of the operator T, since if we have $T(f) = 0$ then differentiating this equation yields $c'(t) \cdot f(c(t)) = 0$ and hence $f(t) = 0$.

The relation between T and the set P_n is demonstrated by the following (small) example, where we let 1 denote the constant function on the interval $[0, 1]$:

$$\begin{aligned}\mathrm{Vol}(P_3) &= \int_0^1 \int_0^1 \int_0^1 \chi(x_1, x_2) \cdot \chi(x_2, x_3)\, dx_3 dx_2 dx_1 \\ &= \int_0^1 \int_0^1 \chi(x_1, x_2) \cdot \left(\int_0^1 \chi(x_2, x_3) \cdot 1\, dx_3 \right) dx_2 dx_1 \\ &= \int_0^1 \int_0^1 \chi(x_1, x_2) \cdot T(1)(x_2)\, dx_2 dx_1 \\ &= \int_0^1 T^2(1)(x_1)\, dx_1 \\ &= (1, T^2(1)),\end{aligned}$$

where $(\cdot, \cdot)$ denotes the inner product on the space $L^2([0, 1])$. Recall that the trace of a matrix is the sum of the entries on the diagonal. Similarly for a Hilbert–Schmidt operator U with kernel κ on the space $L^2([0, 1])$, that is, $U(f)(t) = \int_0^1 \kappa(t, s) \cdot f(s)\, ds$, the trace is defined by

$$\mathrm{tr}(U) = \int_0^1 \kappa(t, t)\, dt.$$

We now express the volume of the set Q_3 in terms of the trace of the operator T^3. First note that T^3 is given by

$$T^3(f)(x_1) = \int_0^1 \int_0^1 \int_0^1 \chi(x_1, x_2) \cdot \chi(x_2, x_3) \cdot \chi(x_3, x_4) \cdot f(x_4)\, dx_4 dx_3 dx_2$$

and hence we have

$$\mathrm{Vol}(Q_3) = \int_0^1 \int_0^1 \int_0^1 \chi(x_1, x_2) \cdot \chi(x_2, x_3) \cdot \chi(x_3, x_1)\, dx_3 dx_2 dx_1 = \mathrm{tr}(T^3).$$

Recall that the trace of a matrix is also given by the sum of the eigenvalues. An operator is trace class if the sum of its eigenvalues converges absolutely. In this case the trace is also given by the sum of the eigenvalues.

Theorem 2.1. *Let n be greater than or equal to 2. Let K be an index set for the eigenfunctions φ_k of the operator T with eigenvalues λ_k. Then the volume of the set P_n is given by*

$$\mathrm{Vol}(P_n) = \sum_{k \in K} \frac{(1, \varphi_k)^2}{(\varphi_k, \varphi_k)} \cdot \lambda_k^{n-1} \tag{2.1}$$

Under the assumption that the series $\sum_{k \in K} \lambda_k^n$ converges absolutely the volume of the set Q_n is given by

$$\mathrm{Vol}(Q_n) = \sum_{k \in K} \lambda_k^n. \tag{2.2}$$

Proof. Begin by observing that

$$T^m(f)(t) = \int_{[0,1]^m} \chi(t, s_m) \cdot \chi(s_m, s_{m-1}) \cdots \chi(s_2, s_1) \cdot f(s_1)\, ds_1 \cdots ds_m.$$

Expand the constant function 1 in the complete orthogonal basis $\{\varphi_k\}_{k \in K}$:

$$1 = \sum_{k \in K} \frac{(1, \varphi_k)}{(\varphi_k, \varphi_k)} \cdot \varphi_k.$$

Now the volume of the set P_n is given by

$$\begin{aligned}
\mathrm{Vol}(P_n) &= \int_0^1 \int_{[0,1]^{n-1}} \chi(t, s_{n-1}) \cdot \chi(s_{n-1}, s_{n-2}) \cdots \chi(s_2, s_1)\, ds_1 \cdots ds_{n-1}\, dt \\
&= (1, T^{n-1}(1)) \\
&= \left(1, \sum_{k \in K} \frac{(1, \varphi_k)}{(\varphi_k, \varphi_k)} \cdot \lambda_k^{n-1} \cdot \varphi_k\right) \\
&= \sum_{k \in K} \frac{(1, \varphi_k)^2}{(\varphi_k, \varphi_k)} \cdot \lambda_k^{n-1}.
\end{aligned}$$

Recall that T^n is a Hilbert–Schmidt operator. Since the sum $\sum_k \lambda_k^n$ converges absolutely by the assumption of the theorem, the operator T^n is a trace class operator; see Section XI.8, Exercise 49 in [3]. The trace of the operator is given by the convergent sum

$$\mathrm{tr}(T^n) = \sum_{k \in K} \lambda_k^n.$$

By part (c) of the above mentioned exercise in [3], the trace of T^n is also given by the following integral:

$$\mathrm{tr}(T^n) = \int_{[0,1]^n} \chi(s_1, s_2) \cdots \chi(s_{n-1}, s_n) \cdot \chi(s_n, s_1)\, ds_1 \cdots ds_n = \mathrm{Vol}(Q_n). \qquad \square$$

Note that equation (2.1) is reminiscent of Theorem 1.1 in [7]. However, since the operator here is self-adjoint we obtain an exact result. Equation (2.2) is similar to Theorem 3.2 in [4].

Proposition 2.2. *Assume that $c(t)$ is a piecewise twice differentiable function. Let $\varphi(t)$ be an eigenfunction of the operator T with eigenvalue λ. Then the eigenfunction $\varphi(t)$ satisfies the second order linear differential equation*

$$\varphi''(t) - \frac{c''(t)}{c'(t)} \cdot \varphi'(t) - \frac{c'(t)}{\lambda^2} \cdot \varphi(t) = 0,$$

and satisfies the boundary conditions $\varphi(1) = 0$ and $\varphi'(0) = 0$.

Proof. The eigenfunction equation states that

$$\lambda \cdot \varphi(t) = \int_0^{c(t)} \varphi(s)\, ds. \tag{2.3}$$

Differentiate this equation twice

$$\lambda \cdot \varphi'(t) = c'(t) \cdot \varphi(c(t)), \tag{2.4}$$
$$\lambda \cdot \varphi''(t) = c''(t) \cdot \varphi(c(t)) + c'(t)^2 \cdot \varphi'(c(t)). \tag{2.5}$$

Substitute $t \longmapsto c(t)$ in equation (2.4) and we obtain $\lambda \cdot \varphi'(c(t)) = c'(c(t)) \cdot \varphi(c(c(t)))$ $= c'(c(t)) \cdot \varphi(t)$. Applying this identity and (2.4) in equation (2.5) yields

$$\lambda \cdot \varphi''(t) = c''(t) \cdot \frac{\lambda}{c'(t)} \cdot \varphi'(t) + c'(t)^2 \cdot \frac{c'(c(t))}{\lambda} \cdot \varphi(t).$$

The derivative of $c(c(t)) = t$ is $c'(t) \cdot c'(c(t)) = 1$. Applying this last identity yields the second order differential equation. Finally, setting t to be 1 in (2.3) and t to be 0 in (2.4) yields the two boundary conditions. $\square$

3 The operator

We now return to our piecewise linear function $b(t)$ from equation (1.3) in the introduction. We determine the eigenvalues explicitly for its associated operator T. The main tool is the second order differential equation in Proposition 2.2. In our case it breaks into two second order differential equations with constant coefficients.

Proposition 3.1. *The eigenvalues of the operator T are given by*

$$\lambda_k = \frac{\sqrt{(1-\alpha)\cdot\alpha}}{\arctan\left(\sqrt{\frac{1-\alpha}{\alpha}}\right) + \pi \cdot k}, \tag{3.1}$$

where k is an integer. The eigenfunction associated with the eigenvalue λ_k is

$$\varphi_k(t) = \begin{cases} \frac{1}{\sqrt{\alpha}} \cdot \cos\left(\sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\lambda_k} \cdot t\right) & \text{if } 0 \leq t \leq \alpha, \\ \frac{1}{\sqrt{1-\alpha}} \cdot \sin\left(\sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda_k} \cdot (1-t)\right) & \text{if } \alpha \leq t \leq 1. \end{cases}$$

Furthermore, the algebraic multiplicity of each of the non-zero eigenvalues is 1.

Proof. Let φ be an eigenfunction, that is, $\lambda \cdot \varphi = T(\varphi)$. Assume that φ is of the form

$$\varphi(t) = \begin{cases} p(t) & \text{if } 0 \leq t \leq \alpha, \\ q(t) & \text{if } \alpha \leq t \leq 1. \end{cases}$$

The derivative of $b(t)$ is given by

$$b'(t) = \begin{cases} -(1-\alpha)/\alpha & \text{for } 0 \leq t \leq \alpha, \\ -\alpha/(1-\alpha) & \text{for } \alpha \leq t \leq 1. \end{cases}$$

Note also that $b''(t) = 0$. Hence Proposition 2.2 implies that

$$p''(t) + \frac{1}{\lambda^2} \cdot \frac{1-\alpha}{\alpha} \cdot p(t) = 0, \qquad 0 \leq t \leq \alpha,$$
$$q''(t) + \frac{1}{\lambda^2} \cdot \frac{\alpha}{1-\alpha} \cdot q(t) = 0, \qquad \alpha \leq t \leq 1.$$

These two second order linear differential equations have the general solutions

$$p(t) = A \cdot \cos\left(\sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\lambda} \cdot t\right) + B \cdot \sin\left(\sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\lambda} \cdot t\right), \tag{3.2}$$

$$q(t) = C \cdot \cos\left(\sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda} \cdot t\right) + D \cdot \sin\left(\sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda} \cdot t\right). \tag{3.3}$$

It remains to find the four constants A, B, C and D and an equation for the eigenvalue λ.

Note that $q(1) = \varphi(1) = 0$ and $p'(0) = \varphi'(0) = 0$ by Proposition 2.2. This last observation implies $B = 0$. Now see that $A = 0$ would imply that $\varphi(t) = 0$. Hence we obtain that the constant A is non-zero. Furthermore, setting $t = 1$ in equation (3.3) yields

$$\tan\left(\sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda}\right) = -\frac{C}{D}.$$

Thus we may select

$$C = \frac{1}{\sqrt{1-\alpha}} \cdot \sin\left(\sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda}\right) \quad \text{and} \quad D = -\frac{1}{\sqrt{1-\alpha}} \cdot \cos\left(\sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda}\right).$$

We are selecting the coefficients such that the eigenfunction will have unit length; see Lemma 3.2. Now $q(t)$ has the expression

$$q(t) = \frac{1}{\sqrt{1-\alpha}} \cdot \sin\left(\sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda} \cdot (1-t)\right). \tag{3.4}$$

Since $\varphi(t)$ is continuous we have the condition $p(\alpha) = q(\alpha)$. From equations (3.2) and (3.4) we have

$$A \cdot \cos\left(\sqrt{(1-\alpha) \cdot \alpha} \cdot \frac{1}{\lambda}\right) = \frac{1}{\sqrt{1-\alpha}} \cdot \sin\left(\sqrt{(1-\alpha) \cdot \alpha} \cdot \frac{1}{\lambda}\right).$$

Hence the constant A is given by the tangent function

$$A = \frac{1}{\sqrt{1-\alpha}} \cdot \tan\left(\sqrt{(1-\alpha) \cdot \alpha} \cdot \frac{1}{\lambda}\right).$$

Next we compute the integral

$$\begin{aligned} \int_0^\alpha p(t)\, dt &= A \cdot \int_0^\alpha \cos\left(\sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\lambda} \cdot t\right) dt \\ &= A \cdot \sqrt{\frac{\alpha}{1-\alpha}} \cdot \lambda \cdot \sin\left(\sqrt{(1-\alpha) \cdot \alpha} \cdot \frac{1}{\lambda}\right). \end{aligned}$$

Since $\lambda \cdot p(\alpha) = \lambda \cdot \varphi(\alpha) = \int_0^\alpha \varphi(t)\, dt = \int_0^\alpha p(t)\, dt$ we have

$$\lambda \cdot A \cdot \cos\left(\sqrt{(1-\alpha)\cdot\alpha} \cdot \frac{1}{\lambda}\right) = A \cdot \sqrt{\frac{\alpha}{1-\alpha}} \cdot \lambda \cdot \sin\left(\sqrt{(1-\alpha)\cdot\alpha} \cdot \frac{1}{\lambda}\right).$$

Canceling the factor $A \cdot \lambda$ we obtain

$$\sqrt{\frac{1-\alpha}{\alpha}} = \tan\left(\sqrt{(1-\alpha)\cdot\alpha} \cdot \frac{1}{\lambda}\right).$$

This yields a more explicit expression $1/\sqrt{\alpha}$ for the constant A. Furthermore we can now solve for the eigenvalue λ

$$\arctan\left(\sqrt{\frac{1-\alpha}{\alpha}}\right) + \pi \cdot k = \sqrt{(1-\alpha)\cdot\alpha} \cdot \frac{1}{\lambda}.$$

Solving for λ yields (3.1). Finally, since we obtain a unique eigenfunction, the geometric multiplicity is 1 of the non-zero eigenvalues. Since there are no generalized eigenfunctions, the algebraic multiplicity is the same as the geometric multiplicity. □

Lemma 3.2. *The following two identities hold:*

$$(1, \varphi_k) = \frac{\lambda_k}{\sqrt{\alpha}}, \qquad\qquad (\varphi_k, \varphi_k) = 1.$$

Proof. Setting $t = 0$ in (2.3) yields the expression for the inner product $(1, \varphi_k)$. Next we have

$$\begin{aligned}
(\varphi_k, \varphi_k) &= \int_0^1 \varphi_k(t)^2\, dt = \int_0^\alpha p(t)^2\, dt + \int_\alpha^1 q(t)^2\, dt \\
&= \frac{1}{\alpha} \cdot \left[\frac{t}{2} + \frac{\sin\left(2 \cdot \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\lambda_k} \cdot t\right)}{4 \cdot \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\lambda_k}}\right]_0^\alpha + \frac{1}{1-\alpha} \cdot \left[\frac{t}{2} + \frac{\sin\left(2 \cdot \sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda_k} \cdot (1-t)\right)}{4 \cdot \sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda_k}}\right]_\alpha^1 \\
&= 1 + \frac{1}{\alpha} \cdot \left[\frac{\sin\left(2 \cdot \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\lambda_k} \cdot t\right)}{4 \cdot \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\lambda_k}}\right]_0^\alpha + \frac{1}{1-\alpha} \cdot \left[\frac{\sin\left(2 \cdot \sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda_k} \cdot (1-t)\right)}{4 \cdot \sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda_k}}\right]_\alpha^1 \\
&= 1 + \frac{1}{\alpha} \cdot \frac{\sin\left(2 \cdot \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\lambda_k} \cdot \alpha\right)}{4 \cdot \sqrt{\frac{1-\alpha}{\alpha}} \cdot \frac{1}{\lambda_k}} - \frac{1}{1-\alpha} \cdot \frac{\sin\left(2 \cdot \sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda_k} \cdot (1-\alpha)\right)}{4 \cdot \sqrt{\frac{\alpha}{1-\alpha}} \cdot \frac{1}{\lambda_k}} \\
&= 1 + \frac{\sin\left(2 \cdot \sqrt{\alpha \cdot (1-\alpha)} \cdot \frac{1}{\lambda_k}\right)}{4 \cdot \sqrt{\alpha \cdot (1-\alpha)} \cdot \frac{1}{\lambda_k}} - \frac{\sin\left(2 \cdot \sqrt{\alpha \cdot (1-\alpha)} \cdot \frac{1}{\lambda_k}\right)}{4 \cdot \sqrt{\alpha \cdot (1-\alpha)} \cdot \frac{1}{\lambda_k}} = 1,
\end{aligned}$$

where in the fourth step we use that $1/\alpha \cdot [t/2]_0^\alpha + 1/(1-\alpha) \cdot [t/2]_\alpha^1 = 1$ and in the fifth step setting $t = 0$ and $t = 1$ in the respective antiderivatives yields 0. □

Theorem 3.3. *The volume of the two n-dimensional sets P_n and Q_n, where $n \geq 2$, is given by the convergent series*

$$\mathrm{Vol}(P_n) = \frac{1}{\alpha} \cdot \sum_{k=-\infty}^{\infty} \left(\frac{\sqrt{(1-\alpha)\cdot\alpha}}{\arctan\left(\sqrt{\frac{1-\alpha}{\alpha}}\right) + \pi \cdot k} \right)^{n+1},$$

$$\mathrm{Vol}(Q_n) = \sum_{k=-\infty}^{\infty} \left(\frac{\sqrt{(1-\alpha)\cdot\alpha}}{\arctan\left(\sqrt{\frac{1-\alpha}{\alpha}}\right) + \pi \cdot k} \right)^{n}.$$

Proof. Both results follow from Theorem 2.1. For the second result observe that the series converges absolutely. □

Corollary 3.4. *The volumes of the n-dimensional set P_n and the $(n+1)$-dimensional set Q_{n+1} satisfy the identity $\alpha \cdot \mathrm{Vol}(P_n) = \mathrm{Vol}(Q_{n+1})$ for $n \geq 2$.*

The Lipschitz identity [10] is the following identity.

Theorem 3.5 (Lipschitz)**.** *For a real variable $0 < x < 1$ and n a positive integer greater than 1 the following identity holds:*

$$\sum_{k=-\infty}^{\infty} \frac{1}{(x+k)^n} = (-1)^{n-1} \cdot \frac{\pi}{(n-1)!} \cdot \frac{d^{n-1}}{dx^{n-1}} \cot(\pi \cdot x). \tag{3.5}$$

Corollary 3.6. *The volume of the n-dimensional set Q_n is given by*

$$\mathrm{Vol}(Q_n) = \frac{(-1)^{n-1}}{(n-1)!} \cdot ((1-\alpha)\cdot\alpha)^{n/2} \cdot \cot^{(n-1)}\left(\arctan\left(\sqrt{\frac{1-\alpha}{\alpha}}\right)\right)$$

for $n \geq 2$.

Proof. By substituting $y = \pi \cdot x$ in the Lipschitz identity, equation (3.5), becomes

$$\sum_{k=-\infty}^{\infty} \frac{1}{(y+\pi\cdot k)^n} = \frac{(-1)^{n-1}}{(n-1)!} \cdot \cot^{(n-1)}(y),$$

where $\cot^{(n-1)}(y)$ denotes the $(n-1)$st derivative of the cotangent function. The result follows now by setting $y = \arctan(\sqrt{(1-\alpha)/\alpha})$. □

4 Concluding remarks

Is there a direct explanation of Corollary 3.4 without first determining the volumes of the two sets? Most elegant would be a dissection proof between the set Q_{n+1} and the Cartesian product $[0, \alpha] \times P_n$.

When $1/2 < \alpha < 1$, can the f-vectors of the polytopes P_n and Q_n be determined? The original case $\alpha = 1/2$ was studied for P_n in the two papers [2, 6], whereas it was examined for Q_n in the paper [5]. When the parameter α is rational, can the Ehrhart quasi-polynomials of P_n and Q_n be described?

The graph polytope of a graph G on the vertex set $\{1, 2, \ldots, n\}$ is the polytope $\{\vec{x} \in \mathbb{R}^n_{\geq 0} : \forall ij \in E(G) \;\; x_i + x_j \leq 1\}$. Hence for a self-inverse function $c(t)$ on the unit interval, what can be said about the set

$$\{\vec{x} \in \mathbb{R}^n_{\geq 0} : \forall ij \in E(G) \;\; x_i \leq c(x_j)\}?$$

There are other natural self-inverse functions on the interval $[0, 1]$ to consider. Here we present three examples.

(i) Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be m positive real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_m = 1$. Let β_k be the partial sum $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ such that the interval $[0, 1]$ is the union of the intervals $[\beta_0, \beta_1] \cup [\beta_1, \beta_2] \cup \cdots \cup [\beta_{m-1}, \beta_m]$. Furthermore let $\rho_k = \alpha_{m+1-k}/\alpha_k$. Note that $\rho_{m+1-k} = \rho_k^{-1}$. Define the linear function $\ell_k(t)$ such that $\ell_k(t)$ maps the interval $[\beta_{k-1}, \beta_k]$ to the interval $[\beta_{m-k}, \beta_{m+1-k}]$ by

$$\ell_k(t) = \rho_k \cdot (\beta_{k-1} - t) + \beta_{m+1-k}.$$

Note that $\ell_k(t)$ is a decreasing function. Define the piecewise linear function $c(t)$ by $c(t) = \ell_k(t)$ on the interval $[\beta_{k-1}, \beta_k]$.

(ii) The Möbius tranformation $c(t) = \frac{1-t}{1+a \cdot t}$ for $-1 < a$.

(iii) Finally, the function $c(t) = \sqrt[q]{1 - t^q}$.

All three of these examples are self-inverse functions. Can the volumes of the associated sets P_n and Q_n be computed in these cases?

Finally, are there other functions $c(t)$ such that the volumes of the two sets P_n and Q_{n+1} are related as in Corollary 3.4?

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