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Strongly regular graphs from hyperbolic quadrics and their maximal cliques

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Abstract

Let $Q^+(2n+1,q)$ be a hyperbolic quadric of $\mathrm{PG}(2n+1,q)$. Fix a generator Π of the quadric. Define $\mathcal{G}_n$ as the graph with vertex set the points of $Q^+(2n+1,q) \setminus \Pi$ and two vertices adjacent if they either span a secant to $Q^+(2n+1,q)$ or a line contained in $Q^+(2n+1,q)$ meeting Π non-trivially. Then such a construction defines a strongly regular graph, which is the complement of a (non-induced) subgraph of the collinearity graph of $Q^+(2n+1,q)$. In this paper, we directly compute the parameters of $\mathcal{G}_n$, which is cospectral, when $q=2$, to the tangent graph $NO^+(2n+2,2)$, but it is non-isomorphic for $n \geq 3$. We also classify the maximal cliques of $\mathcal{G}_3$ for $q=2$, proving as a by-product the non-isomorphism with the graph $NO^+(8,2)$.

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1 Introduction

A *strongly regular graph* with parameters (v, k, λ, μ) is a graph with v vertices where each vertex has degree k, any two adjacent vertices have λ common neighbours, and any two non-adjacent vertices have μ common neighbours. The *spectrum* of a strongly regular graph is the triple $(k^1, \theta_1^{m_1}, \theta_2^{m_2})$ of its eigenvalues with their respective multiplicities. Therefore, two strongly regular graphs with the same parameters are *cospectral*, that is, they have the same spectrum. Two isomorphic graphs are always cospectral, but the converse is not always true. Even in the family of strongly regular graphs, there are examples with the same parameters but which are not isomorphic. Let $Q^+(2n+1, 2)$ be a non–degenerate hyperbolic quadric of $\mathrm{PG}(2n+1, 2)$. Let $NO^+(2n+2, 2)$ be the graph whose vertices are the points of $\mathrm{PG}(2n+1, 2) \setminus Q^+(2n+1, 2)$ where two vertices P_1 and P_2 are adjacent if the line joining P_1 and P_2 is a line tangent to $Q^+(2n+1, 2)$. The graph $NO^+(2n+2, 2)$ is a strongly regular graph. In this paper, we provide a construction for strongly regular graphs arising from hyperbolic quadrics. Fix a generator Π of $Q^+(2n+1, 2)$, and let $\mathcal{G}_n$ be the graph whose vertices are the points of $Q^+(2n+1, 2) \setminus \Pi$, and two vertices P_1 and P_2 are adjacent if the line joining P_1 and P_2 is either a secant of $Q^+(2n+1, 2)$, or intersects Π. It turns out that $\mathcal{G}_n$ is a strongly regular graph, cospectral to $NO^+(2n+2, 2)$. Note that this construction also works in the case $q > 2$, giving rise to strongly regular graphs for all quadrics $Q^+(2n+1, q)$. For all values of q, the graph $\mathcal{G}_n$ was already known as the complement of the graph defined in [5]. The main goal of this paper is to give a full geometric characterisation of such a graph. The first interesting case is $(n, q) = (3, 2)$, where the graph is cospectral to $NO^+(8, 2)$. The so-called *ovoids* of $Q^+(7, 2)$ are obvious examples of cliques of $\mathcal{G}_3(2)$ (which also turn out to be cliques of maximal size), and this observation motivates us to classify and geometrically characterise all maximal cliques of $\mathcal{G}_n$ for $(n, q) = (3, 2)$. This characterisation includes a full description of all types of maximal cliques in terms of substructures of the underlying quadric. As a by-product, this complete classification gives the non-isomorphism between $\mathcal{G}_n$ and $NO^+(2n+2, 2)$ whenever $n \geq 3$.

The paper is outlined as follows. In Section 2 we give the necessary preliminaries on spectral graph theory of strongly regular graphs and projective geometry of hyperbolic quadrics, while in Section 3 we compute the parameters of $\mathcal{G}_n$. Finally, in Section 4 we compute cliques of $\mathcal{G}_3$, thus proving the non-isomorphism issue.

2 Preliminaries

2.1 Spectra of strongly regular graphs

A graph Γ is an ordered pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a non-empty set of vertices and $E(\Gamma)$ is the set of edges, i.e. unordered pairs of vertices. If $e = uv$ we say that the edge e joins u and v, and u and v are called adjacent vertices or neighbours. A *clique* of the graph Γ is a set of pairwise adjacent vertices. A clique is said to be *maximal* if it is maximal with respect to set theoretical inclusion.

Spectral graph theory is a widely used combinatorial approach to the study of graphs based on the eigenvalues of their adjacency matrix. We refer the reader to [4] for more details on spectral graph theory. Strongly regular graphs were introduced by Bose in [2] in 1963, and ever since then they have been extensively investigated. In particular, the eigenvalues of the adjacency matrix of a strongly regular graph are known; see [6]: a strongly regular graph G with parameters (v, k, λ, μ) has exactly three eigenvalues: k, θ_1 and θ_2 of multiplicity, respectively, 1, m_1 and m_2, where:

$$\theta_1 = \frac{1}{2}\left[(\lambda-\mu) + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right],$$

$$\theta_2 = \frac{1}{2}\left[(\lambda-\mu) - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)}\right],$$

$$m_1 = \frac{1}{2}\left[(v-1) - \frac{2k-(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right],$$

$$m_2 = \frac{1}{2}\left[(v-1) + \frac{2k-(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2 + 4(k-\mu)}}\right].$$

In an analogous way, we can express the parameters as a function of the spectrum. Since the spectrum and the parameters of strongly regular graphs are equivalent, two strongly regular graphs are cospectral if and only if they have the same parameters. Whether the characterisation of strongly regular graphs is determined by their spectrum is still an open problem.

2.2 Projective spaces and hyperbolic quadrics

Let $\mathrm{PG}(r-1, q)$ be the projective space of projective dimension $r-1$ over $\mathrm{GF}(q)$ equipped with homogeneous projective coordinates $X_1, \ldots, X_r$. We will use the term n–space of $\mathrm{PG}(r-1, q)$ to denote an n–dimensional projective subspace of $\mathrm{PG}(r-1, q)$. Let θ_r be the number of points of $\mathrm{PG}(r-1, q)$, then

$$\theta_r = \frac{q^r - 1}{q-1}.$$

We shall find it helpful to represent the projectivities of $\mathrm{PG}(r-1, q)$ by invertible $r \times r$ matrices over $\mathrm{GF}(q)$ and consider the points of $\mathrm{PG}(r-1, q)$ as column vectors, with matrices acting on the left. Let U_i be the points that have 1 in the i–th position and 0 elsewhere.

Let $Q^+(2n+1, q)$ be a hyperbolic quadric of $\mathrm{PG}(2n+1, q)$. Up to choice of basis, this quadric is the set of points of the variety given by

$$Q : X_1 X_{2n+2} + X_2 X_{2n+1} + \ldots + X_n X_{n+1} = 0.$$

A projective subspace of maximal dimension contained in $\mathcal{Q}$ is called a *generator* of $\mathcal{Q}$, and the projective dimension of the generators of $\mathcal{Q}$ equals n. The number of points of $Q^+(2n+1, q)$ equals $\frac{(q^n+1)(q^{n+1}-1)}{q-1}$, while the number of generators equals

$\prod_{i=1}^{n+1}(q^{n-i+1}+1) = 2(q+1)(q^2+1)\cdot\cdots\cdot(q^n+1)$. The generators split into two families of size $(q+1)(q^2+1)\cdot\cdots\cdot(q^n+1)$. An important property of the hyperbolic quadric (and all finite classical polar spaces), is the one-or-all property (or axiom), i.e., given a point P and a line l of the polar space with $P \notin l$, then either P is collinear with exactly one point of l or with all points of l. Hence, if P is collinear with two points of l, then the plane $\langle l, P\rangle$ is completely contained in the polar space.

An ovoid of $Q^+(2n+1,q)$ is a set $\mathcal{O}$ of points such that every generator of $Q^+(2n+1,q)$ meets $\mathcal{O}$ in exactly one point. Necessarily, $\mathcal{O}$ contains q^n+1 points. Ovoids in quadrics (and more generally finite classical polar spaces) are rare, but for $n=3$ and $q=2$, there is, up to isomorphism, a unique ovoid of $Q^+(7,2)$, which will play an important role in classifying cliques of $\mathcal{G}_3$ for $q=2$.

For $n=2$, an ovoid of $Q^+(5,q)$ is by Klein correspondence equivalent with a spread of the projective space $\mathrm{PG}(3,q)$. Hence there are many examples of ovoids of $Q^+(5,q)$. An important example for this paper is the elliptic quadric $Q^-(3,q)$, which is the set of points of the variety given by

$$Q : F(X_1, X_2) + X_3X_4 = 0,$$

with $F(X_1,X_2)$ a homogeneous and irreducible quadratic form over the field $\mathrm{GF}(q)$.

The quadratic form $X_1X_{2n+2} + X_2X_{2n+1} + \cdots + X_nX_{n+1}$ defining the quadric has associated polar form f, which is non-degenerate and orthogonal in odd characteristic, symplectic in even characteristic. The form f gives rise to a polarity of the ambient space $\mathrm{PG}(2n+1,q)$, denoted as $\perp$, and with the property that for any d-subspace π contained in $\mathcal{Q}$, the space $\pi^\perp$ is the tangent space to $\mathcal{Q}$ at π. We will rely on the well-known intersection properties of $\pi^\perp$ and the quadric $\mathcal{Q}$ for various subspaces π contained or not contained in $\mathcal{Q}$. For details on hyperbolic quadrics, we refer the reader to [12]. When $n=3$, the quadric $Q^+(7,q)$ is the set of points of the variety given by:

$$Q : X_1X_8 + X_2X_7 + X_3X_6 + X_4X_5 = 0.$$

The quadric contains $(q+1)(q^2+1)(q^3+1)$ points, and the generators of the quadric are the $2(q+1)(q^2+1)(q^3+1)$ solids, which split into two families of size $(q+1)(q^2+1)(q^3+1)$. An ovoid of size q^3+1 is equivalent to a spread of q^3+1 solids, see [7]. If $q=2$, $Q^+(7,2)$ contains 135 points, 270 generator solids, 135 in each family, and the ovoids (and spreads) have size 9.

3 The strongly regular graph $\mathcal{G}_n$

Consider the quadric $Q^+(2n+1,q)$ with $\perp$ its induced polarity of $\mathrm{PG}(2n+1,q)$. Fix a generator Π of $Q^+(2n+1,q)$. The set of vertices V of $\mathcal{G}_n$ consists of the points of $Q^+(2n+1,q)\setminus\Pi$. We define two relations in V. Let $P,Q\in V$, then $P\sim_1 Q$ if and only if the line $\langle P,Q\rangle$ is secant to $Q^+(2n+1,q)$, and $P\sim_2 Q$ if and only if the line $\langle P,Q\rangle$ is contained in $Q^+(2n+1,q)$ and meets the generator Π in a point. The adjacency relation of $\mathcal{G}_n$ is the union of $\sim_1$ and $\sim_2$.

Theorem 3.1. *The graph $\mathcal{G}_n$ is a strongly regular graph with parameters*

$$v = \frac{(q^n+1)(q^{n+1}-1)}{q-1} - \frac{q^{n+1}-1}{q-1} = \frac{q^n(q^{n+1}-1)}{q-1},\ k = q^{2n}-1,\ \lambda = q^{2n-1}(q-1)-2$$

and $\quad \mu = (q^{2n-1}+q^{n-1})(q-1)$.

Proof. Let us start by determining the valency k of $\mathcal{G}_n$.

Let P be a point of $Q^+(2n+1,q) \setminus \Pi$. Then $\Pi \cap P^\perp$ is an $(n-1)$-space, say $\overline{\Pi}$. Let Π' be a $(2n-1)$-space of $P^\perp$ through $\overline{\Pi}$ and not passing through P, then $P^\perp \cap Q^+(2n+1,q)$ is a quadratic cone with as vertex P and as basis a hyperbolic quadric $Q^+(2n-1,q)$ contained in Π' and of which $\overline{\Pi}$ is a totally isotropic space. In order to determine the valency k of $\mathcal{G}_n$, we have:

- to count the number of points $Q^+(2n+1,q)$ not contained in $P^\perp \cap Q^+(2n+1,q)$,
- to delete the number of points of $\Pi \setminus \overline{\Pi}$,
- to add the points of $P^\perp \cap Q^+(2n+1,q)$, different from P, lying on lines through P intersecting the space $\overline{\Pi}$ and not contained in Π'.

Hence, we have

$$\begin{aligned} k &= |Q^+(2n+1,q)| - (q|Q^+(2n-1,q)|+1) - (\theta_n - \theta_{n-1}) + (q-1)\theta_{n-1} \\ &= |Q^+(2n+1,q)| - \theta_n - q|Q^+(2n-1,q)| + q\theta_{n-1} - 1 \\ &= \frac{(q^n+1)(q^{n+1}-1)}{q-1} - \frac{q^{n+1}-1}{q-1} - q\frac{(q^{n-1}+1)(q^n-1)}{q-1} + q\frac{q^n-1}{q-1} - 1 = q^{2n}-1. \end{aligned}$$

Now we want to determine <u>the value of λ</u>.

$\boxed{\text{Case I}}$: Let P_1 and P_2 be two points of $Q^+(2n+1,q)$ not in Π such that the line $l := P_1P_2$ is secant to $Q^+(2n+1,q)$. We want to determine the number of vertices of $\mathcal{G}_n$ adjacent to P_1 and P_2.

Note that $|l \cap \Pi| = 0$, $|l^\perp \cap Q^+(2n+1,q)| = |Q^+(2n-1,q)|$ and $l^\perp \cap \Pi$ is an $(n-2)$-space, say $\overline{\Pi}$, which is totally isotropic of $Q^+(2n-1,q)$.

We first show that there are no point X of $\mathcal{G}_n$ adjacent to P_1 and P_2 such that the lines XP_1 and XP_2 both intersect the space Π. Indeed, in such a case the line P_1P_2 would intersect Π, a contradiction.

The parameter λ is given by $\lambda := x - y + z_1 + z_2$, where

x	is the number of points X of $Q^+(2n+1,q)$ such that the lines P_1X and P_2X are both secant lines to $Q^+(2n+1,q)$
y	is the number of points of $\Pi \setminus (P_1^\perp \cup P_2^\perp)$
z_1	is the number of points of $P_1^\perp \cap Q^+(2n+1,q) \setminus (\{P_1\} \cup \Pi \cup P_2^\perp)$
z_2	is the number of points of $P_2^\perp \cap Q^+(2n+1,q) \setminus (\{P_2\} \cup \Pi \cup P_1^\perp)$.

The integer x is obtained by subtracting from the number of points in $Q^+(2n+1,q) \setminus \{P_1, P_2\}$ the number of points of $l^\perp \cap Q^+(2n+1,q)$ and the size of $(P_i^\perp \cap Q^+(2n+1,q) \setminus \{P_i\}) \setminus P_j^\perp$, where $i,j \in \{1,2\}$ with $i \neq j$. Then

$$\begin{aligned} x &= |Q^+(2n+1,q)| - 2 - |Q^+(2n-1,q)| - 2(q-1)|Q^+(2n-1,q)| \\ &= \frac{(q^n+1)(q^{n+1}-1)}{q-1} - 2 + (1-2q)\frac{(q^{n-1}+1)(q^n-1)}{q-1}. \end{aligned} \tag{3.1}$$

Since $\Pi \cap P_i^\perp$ is an n-space and $P_1^\perp \cap P_2^\perp = l^\perp$ which is an $(n-2)$-space, we get

$$y = \theta_n - q^{n-1} - q^{q^n-1} - \theta_{n-2} = q^{n-1}(q-1). \tag{3.2}$$

The integer z is given by $(P_1^\perp \cap Q^+(2n+1,q)) \setminus (\{P_1\} \cup \Pi \cup l^\perp)$. Since $\Pi \cap P_1^\perp$ is an $(n-1)$-space, $l^\perp \subset P_1^\perp$ and $M := l^\perp \cap \Pi$ is an $(n-2)$-space, any line of $Q^+(2n+1,q)$ through P_1 intersecting M gives a contribution of $q-1$ points and any line of $Q^+(2n+1,q)$ through P_1 intersecting Π also intersects $l^\perp$ and hence gives a contribution of $q-2$ points, we get

$$z_1 = (q-1)\theta_{n-2} + (q-2)q^{n-1}. \tag{3.3}$$

Arguing as above, we get $z_2 = z_1$. Hence from Equations (3.1), (3.2) and (3.3) we have

$$\begin{aligned} \lambda =& \frac{(q^n+1)(q^{n+1-1})}{q-1} - 2 + (1-2q)\frac{(q^{n-1}+1)(q^n-1)}{q-1} - q^{n-1}(q-1) \\ &+ 2(q-1)\frac{q^{n-1}-1}{q-1} + 2(q-2)q^{n-1} = q^{2n-1}(q-1) - 2. \end{aligned}$$

$\boxed{\text{Case II}}$: Let P_1 and P_2 be two points of $Q^+(2n+1,q)$ not in Π such that the line $l := P_1P_2$ intersects Π at a point, say R, and therefore is totally isotropic. We want to determine the number of vertices of $\mathcal{G}_n$ adjacent to P_1 and P_2.

Note that $|l \cap \Pi| = 1$, $l^\perp \cap Q^+(2n+1,q)$ is a quadratic cone with vertex l and basis a hyperbolic quadric $Q^+(2n-3,q)$. Also, $l^\perp \cap \Pi$ is an $(n-1)$-space, say $\overline{\Pi}$.

The parameter λ is given by $\lambda := x - y + z + t$, where

- x is the number of points X of $Q^+(2n+1,q)$ such that the lines P_1X and P_2X are both secant lines to $Q^+(2n+1,q)$
- y is the number of points of $\Pi \setminus (P_1^\perp \cup P_2^\perp)$
- z is the number of points of $l \setminus (\{P_1, P_2, R\}$
- t is the number of points of $\langle l, \overline{\Pi} \rangle \setminus (\overline{\Pi} \cup l)$,

The integer x is obtained by subtracting from the number of points in $Q^+(2n+1,q)$ the number of points of $l^\perp \cap Q^+(2n+1,q)$ and the size of $(P_i^\perp \cap Q^+(2n+1,q) \setminus \{P_i\}) \setminus P_j^\perp$, where $i,j \in \{1,2\}$ with $i \neq j$. Then

$$\begin{aligned} x &= |Q^+(2n+1,q)| - q^2|Q^+(2n-3,q)| - (q+1) - 2(q|Q^+(2n-1,q)| + 1 \\ &\quad - (q^2|Q^+(2n-3,q)| + q + 1)) \\ &= \frac{(q^n+1)(q^{n+1}-1)}{q-1} - (q+1) + q^2\frac{(q^{n-2}+1)(q^{n-1}-1)}{q-1} - 2q\frac{(q^{n-1}+1)(q^n-1)}{q-1} + 2q \\ &= q^{2n-1}(q-1). \end{aligned} \tag{3.4}$$

Since $\Pi \cap P_1^\perp = \Pi \cap P_2^\perp = \Pi \cap l^\perp = \overline{\Pi}$ is an $(n-1)$-space we get

$$y = \theta_n - \theta_{n-1} = q^n. \tag{3.5}$$

It is clear that $z = q - 2$ and since $l \cap \overline{\Pi}$ is the point R we get

$$t = \theta_n - \theta_{n-1} - q = q^n - q.$$

Hence, from the last two equalities and from Equations (3.4) and (3.5) we get

$$\lambda = q^{2n-1}(q-1) - q^n + q - 2 + q^n - q = q^{2n-1}(q-1) - 2.$$

Finally, we want to determine the value of μ.

Let P_1 and P_2 be two points of $\overline{Q^+(2n+1,q)}$ such that the line $l := P_1P_2$ is totally isotropic and it is disjoint from Π. We want to determine the number of vertices of $\mathcal{G}_n$ adjacent to P_1 and P_2.

Note that $l^\perp \cap Q^+(2n+1,q)$ is a quadratic cone with vertex l and basis a hyperbolic quadric $Q^+(2n-3,q)$. Also, $l^\perp \cap \Pi$ is an $(n-2)$-space, say $\overline{\Pi}$.

The parameter μ is given by $\mu := x - y + z_1 - z_2 - z_3 + t_1 - t_2 - t_3$, where

- x is the number of points X of $Q^+(2n+1,q)$ such that the lines P_1X and P_2X are both secant lines to $Q^+(2n+1,q)$
- y is the number of points of $\Pi \setminus (P_1^\perp \cup P_2^\perp)$
- z_1 is the number of points of the quadratic cone with vertex P_1 and basis the $(n-1)$-space $P_1^\perp \cap \Pi$
- z_2 is the number of points of $P_1^\perp \cap \Pi$
- z_3 is the number of points of the quadratic cone with vertex P_1 and basis the $(n-2)$-space $l^\perp \cap \Pi$ not contained in $\overline{\Pi}$
- t_1 is the number of points of the quadratic cone with vertex P_2 and basis the $(n-1)$-space $P_2^\perp \cap \Pi$
- t_2 is the number of points of $P_2^\perp \cap \Pi$
- t_3 is the number of points of the quadratic cone with vertex P_2 and basis the $(n-2)$-space $l^\perp \cap \Pi$ not contained in $\overline{\Pi}$.

The integer x is obtained by subtracting from the number of points in $Q^+(2n+1,q)$ the number of points of $l^\perp \cap Q^+(2n+1,q)$ and the size of

$(P_i^\perp \cap Q^+(2n+1,q) \setminus \{P_i\}) \setminus P_j^\perp$, where $i,j \in \{1,2\}$ with $i \neq j$. Then, as in the previous case, we get

$$\begin{aligned} x =& \|Q^+(2n+1,q)| - q^2|Q^+(2n-3,q)| - (q+1) - 2(q|Q^+(2n-1,q)| + 1 \\ &- (q^2|Q^+(2n-3,q)| + q + 1)) \\ =& \frac{(q^n+1)(q^{n+1}-1)}{q-1} - (q+1) + q^2\frac{(q^{n-2}+1)(q^{n-1}-1)}{q-1} - 2q\frac{(q^{n-1}+1)(q^n-1)}{q-1} + 2q \\ =& q^{2n-1}(q-1). \end{aligned} \tag{3.6}$$

Since $\Pi \cap P_1^\perp$ and $\Pi \cap P_2^\perp$ are two $(n-1)$-spaces and $\overline{\Pi} = \Pi \cap l^\perp$ is an $(n-2)$-space we get

$$y = \theta_n - 2\theta_{n-1} + \theta_{n-2} = q^{n-1}(q-1). \tag{3.7}$$

It can be easily seen that

$$\begin{aligned} z_1 &= t_1 = q\theta_{n-1} + 1, \\ z_2 &= t_2 = \theta_{n-1}, \\ z_3 &= t_3 = (q-1)\theta_{n-2} + 1, \end{aligned}$$

and hence $z_1 - z_2 - z_3 = t_1 - t_2 - t_3 = q^{n-1}(q-1)$. From these equalities and from Equations (3.6) and (3.7), we get

$$\mu = q^{2n-1}(q-1) - q^{n-1}(q-1) + 2q^{n-1}(q-1) = (q^{2n-1} - q^{n-1})(q-1).$$

□

Remark 3.2. Let $\overline{\mathcal{G}}_n$ be the complement of $\mathcal{G}_n$. Then the vertices of $\overline{\mathcal{G}}_n$ are the points of $Q^+(2n+1,q) \setminus \Pi$ and two vertices are adjacent in $\overline{\mathcal{G}}_n$ whenever the line joining them is contained in $Q^+(2n+1,q)$ and skips Π. Therefore $\overline{\mathcal{G}}_n$ is a (not-induced) subgraph of the collinearity graph of $Q^+(2n+1,q)$. Hence, the strong regularity was already stated in [5]. The argument above reported involves the complementary graph and it provides a full proof of [5, Theorem 1] by the following identity: the complement of a $srg(v,k,\lambda,\mu)$ is strongly regular with parameters $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$.

4 The graphs $NO^+(2n+2,2)$ and $\mathcal{G}_n$ for $q=2$

Consider again the quadric $Q^+(2n+1,2)$ embedded in $\mathrm{PG}(2n+1,2)$. The graph $NO^+(2n+2,2)$ has vertex set the set of points of $\mathrm{PG}(2n+1,2) \setminus Q^+(2n+1,2)$, and two vertices P,Q are adjacent if and only if $P \in Q^\perp$, equivalently, if and only if the line $\langle P,Q\rangle$ is a tangent line to $Q^+(2n+1,2)$. The family of graphs $NO^+(2n+2,2)$ (for $n \geq 1$) is described in [6], they are strongly regular with parameters $v = 2^{2n+1} - 2^n$, $k = 2^{2n} - 1$, $\lambda = 2^{2n-1} - 2$, $\mu = 2^{2n-1} + 2^{n-1}$, the same parameters as the graphs $\mathcal{G}_n$ for $q=2$. One can ask whether the graphs are isomorphic or not.

Proposition 4.1. *The graph $\mathcal{G}_n$ is isomorphic to $NO^+(2n+2,2)$ if $n = 0,1,2$.*

# of cliques	Size	Adjacencies	Geometric description
10752	5	$\sim_1$	$Q^-(3,2) \subseteq Q^+(7,2)$ not meeting Π
960	8	$\sim_1$	Cameron-Praeger ovoid [7]
15	8	$\sim_2$	Generators of $Q^+(7,2)$ meeting Π on a plane
840	8	mixed	Cones $PQ^-(3,2)$, $P \in \Pi$, meeting Π in a line
210	8	mixed	Cones $lQ^+(1,2)$, $l \in \Pi$, meeting Π in l

Table 1: List of maximal cliques of $\mathcal{G}_3$. In each case we consider all points in the geometric description *out* of Π.

Proof. For $n = 0$ both graphs consist of one vertex, for $n = 1$ both graphs are the complete bipartite graph $K_{3,3}$. The first non-trivial case is $n = 2$, for which it is shown in [14] that both graphs are isomorphic. □

Now consider the case $n = 3$. We will show that both graphs have maximal cliques of different sizes, which shows that they are non-isomorphic. Recall that a *clique* is a set of vertices in which every pair of vertices is adjacent. A clique is *maximal* if it cannot be extended to a clique of larger size. Recall the following well known theorem on the largest possible size of cliques.

Result 4.2 (**Delsarte clique bound**, [9, Section 3.3.2])**.** *Let Γ be a strongly regular graph with regularity k and smallest eigenvalue λ. Then the size of a clique in Γ is at most $1 - \frac{k}{\lambda}$.*

Corollary 4.3. *The size of a clique in $NO^+(8,2)$ and $\mathcal{G}_3$ is at most 8.*

Proof. Use Result 4.2 with $k = 63$ and $\lambda = -9$. □

Note that a maximal clique is not necessarily a clique of largest possible size. In [6] the maximal cliques of $NO^+(2n+2,2)$ are characterised and appear as follows. Let π be an n-space of $\mathrm{PG}(2n+2,q)$ meeting $Q^+(2n+1,q)$ in an $(n-1)$-space. The 2^n points of $\pi \setminus Q^+(2n+1,q)$ are a clique of size 2^n, necessarily maximal, and no other maximal cliques exist in $NO^+(2n+2,2)$.

4.1 Classification of maximal cliques of $\mathcal{G}_3$

It is a relatively easy exercise to set up the graph $\mathcal{G}_3$ for $q = 2$ using GAP and FinInG ([10, 1]) and to obtain a complete classification of all maximal cliques of $\mathcal{G}_3$. Furthermore, the results obtained by computer allow us to better understand the geometrical structure of the cliques and inspire us to give a computer-free proof of the classification, including a geometrical characterisation in terms of substructures of the quadric. We have created a github archive containing all necessary code and results; see [8]. The computational approach gives rise directly to the classification results in Table 1, which lists the five orbits that the automorphism group of the graph has on the 12777 maximal cliques.

Recall that the adjacency of the graph $\mathcal{G}_n$ is the union of the two relations $\sim_1$ (vertices defining a secant line) and $\sim_2$ (vertices defining a line contained in the quadric and meeting the fixed generator Π in a point). Vertices in a clique of $\mathcal{G}_3$ can hence be in one of the relations $\sim_1$ and $\sim_2$. If all pairs of a clique $\mathcal{C}$ are in $\sim_1$, $\sim_2$ respectively, then we say that $\mathcal{C}$ is of type 1, respectively type 2, otherwise we call $\mathcal{C}$ of *mixed* type.

Lemma 4.4. *All the numbers in Table 1 are correct.*

Proof. In $Q^+(7,2)$ we have 24192 elliptic sections $Q^-(3,2)$, 896 through each point. As $|\Pi| = 15$, we obtain the 10752 $Q^-(3,2)$ not intersecting Π as $24192 - (15 \cdot 896)$. The number of ovoids is proven in [7], while the 15 listed configurations of type 2 are in one-to-one correspondence to the 15 planes in Π. We now want to count mixed type configurations. Fix a line $\bar{l} \subseteq \Pi$. Then $P^\perp$ is a 6-space that meets the quadric $Q^+(7,2)$ in a cone $PQ^+(5,2)$, which intersects Π in $\bar{l}$. The subquadrics $Q^-(3,2) \subseteq Q^+(5,2)$ correspond to the 56 lines in $\mathrm{PG}(5,2)$ skew to $Q^+(5,2)$. The 56 quadrics $Q^-(3,2)$ are exactly 8 on each point, and we choose the 8 quadrics through the point $\bar{l} \cap \mathrm{PG}(5,2)$. Then, since Π contains 35 lines and each line contains 3 points, we get the $35 \cdot 3 \cdot 8 = 840$ such configurations $PQ^-(3,2)$. Now fix a line l in Π, then $l^\perp$ is a 5-space that meets the quadric $Q^+(7,2)$ in a cone $lQ^+(3,2)$. The basis $Q^+(3,2)$ meets Π in a line r. The bases $Q^+(1,2)$ needed correspond to the 6 secant lines in $Q^+(3,2)$ that do not intersect the fixed r. Hence, we have $210 = 35 \cdot 6$ configurations $lQ^+(1,2)$, as Π contains 35 lines. □

In the remainder of this section we will prove in a geometric way that Table 1 provides the whole classification of the cliques of $\mathcal{G}_3$.

Lemma 4.5. *An ovoid of $Q^+(7,2)$ gives rise to a maximal clique of size 8 of $\mathcal{G}_3$.*

Proof. Let $\mathcal{O}$ be an ovoid of $Q^+(7,2)$. Then every pair of points $P, Q \in \mathcal{O}$ determines necessarily a secant to $Q^+(7,2)$. The ovoid $\mathcal{O}$ meets the generator S in a unique point, hence $\mathcal{O}$ contains exactly 8 vertices that are mutually in relation $\sim_1$. This is a clique of size 8, the largest possible size by Corollary 4.3. □

A *partial ovoid* of $Q^+(2n+1,q)$ is a set $\mathcal{B}$ of points such that no two points of $\mathcal{B}$ are collinear on the quadric. A partial ovoid is *maximal* if it cannot be extended to a larger partial ovoid. Obviously, an ovoid is a maximal partial ovoid, any subset of an ovoid is a partial ovoid which is not maximal.

Lemma 4.6. *A partial ovoid of $Q^+(7,2)$ gives rise to a clique in the relation $\sim_1$.*

Proof. Similar to the proof of Lemma 4.5. □

It is not a priori clear whether cliques from maximal partial ovoids are maximal or can be extended by adding vertices that are in relation $\sim_2$ with the vertices of the initial partial ovoid. However, we can easily construct cliques of size 5 in $\sim_1$.

Lemma 4.7. *An elliptic quadric $Q^-(3,q)$ contained in $Q^+(7,q)$ is a maximal partial ovoid of $Q^+(7,q)$.*

Proof. Assume that some solid π_3 of PG$(7,q)$ meets $Q^+(7,q)$ in an elliptic quadric $Q^-(3,q)$. The points of $Q^-(3,q)$ are a partial ovoid of $Q^+(7,q)$. Now, let $P \in Q^+(7,q) \setminus \pi_3$. Then $P^\perp$ meets π_3 in a plane, which meets $Q^-(3,q)$ either in a point or in a conic. Hence, at least one point of $Q^-(3,q)$ is collinear in $Q^+(7,q)$ with P, so no point of $Q^+(7,q) \setminus Q^-(3,q)$ can be added to extend $Q^-(3,q)$ as a partial ovoid. □

Lemma 4.8. *Let $q = 2$ and let $Q^-(3,q)$ be an elliptic quadric contained in $Q^+(7,q)$, not meeting* Π*. Then the points of $Q^-(3,q)$ are a maximal clique of size* 5*.*

Proof. By Lemma 4.7 the points of $Q^-(3,q)$ are a maximal partial ovoid of size 5 contained in the vertex set of $\mathcal{G}_3$, and hence by Lemma 4.6 this gives a clique of size 5 in relation $\sim_1$. Assume that a vertex T extends $Q^-(3,q)$ to a larger clique. It is not possible that T extends this clique to a larger clique in $\sim_1$, since this would extend $Q^-(3,q)$ to a larger partial ovoid, a contradiction by the maximality of $Q^-(3,q)$ as partial ovoid. Hence $T \sim_2 P$ for some point $P \in Q^-(3,q)$. But then the line $\langle P, T\rangle$ meets Π in a point S, hence the space $\langle T, Q^-(3,q)\rangle$ meets $Q^+(7,q)$ in a cone $SQ^-(3,q)$. Now the base of this cone is an ovoid of the base of the cone $S^\perp \cap Q^+(7,q) = SQ^+(5,q)$, so Π contains exactly one line of the cone $SQ^-(3,q)$, and hence $Q^-(3,q)$ meets Π, a contradiction. So, the points of $Q^-(3,q)$ are a maximal clique of $\mathcal{G}_3$ □

Before dealing with cliques of mixed type, we give a final lemma proving that small partial ovoids determine either an elliptic quadric (hence a maximal partial ovoid of size 5) or a complete ovoid in a unique way.

Lemma 4.9. *Let $q = 2$. A partial ovoid $\mathcal{P}$ of $Q^+(7,q)$ of size* 4 *can be extended in a unique way to an ovoid.*

Proof. Let $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$. We show that these four points determine uniquely 5 points extending the partial ovoid to an ovoid. Define the following sets for $j \in \{2,3,4\}$ and $1 \le i \le j-1$

$$A_i^j := P_j^\perp \cap P_i^\perp \cap Q^+(7,q) = \langle P_i, P_j\rangle^\perp \cap Q^+(7,q)\,.$$

Hence, the set A_i^j is the set of points collinear with both P_i and P_j.

If we make a choice for a point P_j, the next point P_{j+1} cannot be collinear with P_j. Hence, from the list of possible points to extend the partial ovoid to an ovoid, we always have to remove the points collinear with P_j. Clearly, we will have already removed points collinear with P_i, $1 \le i \le j-1$.

Now consider P_1. The points collinear with P_1 (including P_1) are the points of $P_1^\perp \cap Q^+(7,q) = P_1Q^+(5,q)$. Hence, there are $135 - 71 = 64$ possible choices for P_2. Now consider P_2. We have to remove the points of $P_2Q^+(5,q)$. The line $l = \langle P_1, P_2\rangle$ is a projective line of hyperbolic type, so $l^\perp \cap Q^+(7,q) = Q^+(5,q)$, hence $|A_1^2| = 35$, and these points have been removed already after choosing P_1. So in total, we have to remove $71 - 35 = 36$ points. This leaves $64 - 36 = 28$ choices for P_3. Now consider P_3. The plane $\langle P_1, P_2, P_3\rangle$ is of parabolic type, since none of the three points P_1, P_2, P_3 are collinear. Therefore, $\pi = \langle P_1, P_2, P_3\rangle^\perp$ is of parabolic type, i.e. $\pi \cap Q^+(7,q) = Q(4,q) = A_1^3 \cap A_2^3$. The number of points already removed after choosing P_1 and P_2 equals $|A_1^3 \cup A_2^3| = |A_1^3| + |A_2^3| - |A_1^3 \cap A_2^3| = 35 + 35 - 15 = 55$. So in total, we have to remove $71 - 55 = 16$ points. This leaves $28 - 16 = 12$ choices for P_4. The solid $\pi_3 := \langle P_1, P_2, P_3, P_4 \rangle$ is of elliptic type, so $\pi_3 \cap Q^+(7,q) = Q^-(3,q) = A_1^4 \cap A_2^4 \cap A_3^4$. The number of points already removed after choosing P_1, P_2, and P_3 equals $|A_1^4 \cup A_2^4 \cup A_3^4| = |A_1^4| + |A_2^4| + |A_3^4| - |A_1^4 \cap A_2^4| - |A_1^4 \cap A_3^4| - |A_2^4 \cap A_3^4| + |A_1^4 \cap A_2^4 \cap A_3^4| = 35 + 35 + 35 - 15 - 15 - 15 + 5 = 65$. So in total, we have to remove $71 - 65 = 6$ points. This leaves a set $\mathcal{R}$ of $12 - 6 = 6$ remaining points. The solid π_3 intersects $Q^+(7,q)$ in an elliptic quadric $Q^-(3,q)$. Its unique point different from $P_1, \ldots, P_4$ belongs to $\mathcal{R}$. If we add this unique point to the partial ovoid, then it becomes a maximal partial ovoid by Lemma 4.7. Hence, if we want to extend $\mathcal{S}$ to an ovoid, we can remove this unique point from $\mathcal{R}$. Then $\mathcal{R}$ is a set of 5 points.

Now assume that two points $S_1, S_2 \in \mathcal{R}$ are collinear. The line $m = \langle S_1, S_2 \rangle$ is contained in $Q^+(7,q)$. But this line cannot be contained in $P_i^\perp$ for $i \in \{1,2,3,4\}$. So, necessarily, $P_i^\perp$ meets m in a point not in $\mathcal{R}$, $m \setminus \mathcal{R} = \{R\}$. This means that $P_i \in r^\perp$, $i \in \{1,2,3,4\}$. Now $r^\perp \cap Q^+(7,q) = RQ^+(5,q)$. The four points P_i, $i \in \{1,2,3,4\}$ are necessarily projected from R onto a partial ovoid $\mathcal{R}'$ of $Q^+(5,q)$. Hence, they span a solid π_3 meeting this $Q^+(5,q)$ in an elliptic quadric $Q^-(3,q)$. The line m meets $Q^+(5,q)$ in a point, without loss of generality we may assume this is S_1. Then $S_1^\perp \cap Q^-(3,q)$ is necessarily a conic. Since $Q^-(3,q)$ contains 5 points in total, of which only one is not in $\mathcal{R}'$, $S_1^\perp$ must contain at least one point of $\mathcal{R}'$. Hence, there exists at least one plane of $Q^+(7,q)$ through the line m containing one of the points P_i, a contradiction since none of these points is collinear with S_1 and S_2. We conclude that no two points of $\mathcal{R}$ are collinear. So, the set $\mathcal{R} \cup \{P_1, P_2, P_3, P_4\}$ is an ovoid of $Q^+(7,q)$. □

The strategy to classify maximal cliques in $\mathcal{G}_3$ for $q = 2$ will be to characterise very small cliques of mixed type in a geometric way. This will restrict the possibilities to extend small cliques to larger cliques.

Lemma 4.10. *If $\mathcal{C} = \{P, Q, R\}$ is a clique of mixed type in $\mathcal{G}_3$, then there are exactly two $\sim_1$-adjacencies and one $\sim_2$-adjacency, and $\mathcal{C}$ spans a plane meeting Π in a point S.*

Proof. Let $P, Q \in \mathcal{C}$ with $P \sim_1 Q$, then the line $\langle P, Q \rangle$ is a secant to $Q^+(7,q)$. We may assume w.l.o.g that $P \sim_2 R$. If $Q \sim_2 R$ as well, then the lines $\langle P, R \rangle$, respectively $\langle Q, R \rangle$ are contained in $Q^+(7,q)$ and meet Π in the respective points S and T. So, the line $\langle S, T \rangle$ is contained in Π but also in the plane $\langle P, Q, R \rangle$, hence $\langle S, T \rangle$ also meets the line $\langle P, Q \rangle$ in a point of $Q^+(7,q)$, different from P and Q, a contradiction. Hence $Q \sim_1 R$. But now it is not possible that the plane $\langle P, Q, R \rangle$ meets Π in a line m, since then the line $\langle Q, R \rangle$ would also meet m in a point not in $Q^+(7,q)$, a contradiction. □

Now we will restrict the argument to the case $q = 2$.

Lemma 4.11. *Assume that $\mathcal{C}$ is a $\sim_2$-clique in $\mathcal{G}_3$. Let $\alpha := \langle \mathcal{C} \rangle$. Then α is contained in $Q^+(7,q)$, and hence α is a plane or a solid, meeting S in a line or in a plane respectively. No $\sim_2$-clique can span a space of dimension larger than 3.*

Proof. Let $P, Q, R \in \mathcal{C}$. Since $q = 2$, $\langle P, Q, R\rangle$ is a plane. Each of the lines $\langle P, Q\rangle$, $\langle P, R\rangle$, and $\langle Q, R\rangle$ meets Π in a different point, hence $\langle P, Q, R\rangle$ meets Π in a line m. If there exists another point $S \in \mathcal{C}\setminus\langle P, Q, R\rangle$, then each of the lines connecting S with one of the points P, Q, or R meets Π in a point not on m. Hence $\langle P, Q, R, S\rangle$ is then a solid, contained in $Q^+(7, q)$, meeting Π in a plane. A similar argument for a point $T \in \mathcal{C} \setminus \langle P, Q, R, S\rangle$ gives rise to a 4-space contained in $Q^+(7, q)$, a contradiction. □

Lemma 4.12. *Let $\mathcal{C} = \{P, Q, R\}$ be a clique of mixed type in $\mathcal{G}_3$. Assume that $T \notin \langle P, Q, R\rangle$ extends $\mathcal{C}$ to a larger clique such that T is $\sim_2$-adjacent with at least one of the vertices of $\mathcal{C}$. Then $\gamma = \langle P, Q, R, T\rangle$ is a 3-dimensional space and $\gamma \cap Q^+(7, q) = lQ^+(1, q)$, with the line l contained in Π, and there is only one way to extend $\{P, Q, R, T\}$ to a maximal clique of size 8, which is then the set of vertices of $\mathcal{G}_3$ in $lQ^+(1, q)$*

Proof. By Lemma 4.10 we may assume that $P \sim_2 Q$ and $P \sim_1 R \sim_1 Q$, and the plane $\alpha := \langle P, Q, R\rangle$ meets Π in a point S. Hence $\alpha \cap Q^+(7, q) = SQ^+(1, q)$. The line $\langle R, S\rangle$ contains one more point R' not in Π, for which clearly $R \sim_2 R'$ and $P \sim_1 R' \sim_1 Q$.

We have assumed that T extends $\mathcal{C}$ to a larger clique. Assume first that $T \sim_2 R$. Then the line $\langle T, R\rangle$ meets Π in a point $S' \neq S$. By Lemma 4.10, it is not possible that $T \sim_1 R'$. Hence the line $\langle T, R'\rangle$ is contained in $Q^+(7, q)$ and in the plane $\langle T, R, R'\rangle$, which meets Π in the line $\langle S, S'\rangle$. Hence the line $\langle T, R'\rangle$ also meets Π in a point of the line $\langle S, S'\rangle$, so we conclude that $T \sim_2 R'$.

It is not possible that $P \sim_2 T \sim_2 R$, since $P \sim_1 R$ and using Lemma 4.10. Hence, the assumption that T is $\sim_2$-adjacent with at least one of the points of $\mathcal{C}$ implies that $T^\perp$ contains exactly one of the lines of $\alpha \cap Q^+(7, q) = SQ^+(1, q)$. W.l.o.g we may assume now that $T \sim_2 P$, $T \sim_2 Q$ and $T \sim_1 R$. Now define $\beta := \langle T, P, Q\rangle$. The clique $\{P, Q, T\}$ is a $\sim_2$-clique and by Lemma 4.11, the plane β is contained in $Q^+(7, q)$ and meets Π in a line l, containing S. But then $\gamma := \langle \alpha, \beta\rangle$ is a solid containing a plane β of $Q^+(7, q)$ and a point $R \notin \beta$. Hence $\gamma \cap Q^+(7, q) = lQ^+(1, q)$,

Consider now a vertex $U \notin \gamma$ and assume that U extends $\{P, Q, R, T\}$ to a larger clique. If $U^\perp$ meets l in one point, then U is collinear in $Q^+(7, q)$ with at least one of the points P, Q or T. However, the connecting line cannot meet Π, so there can be no adjacency $\sim_2$ or $\sim_1$ with at least one of the points P, Q, or T. Hence, if U extends $\{P, Q, R, T\}$ to a larger clique, then $l \subseteq U^\perp$. Hence, the 4-space $\delta := \langle\{P, Q, R, T, U\}\rangle$ contains the planes β, β' and $\beta'' := \langle U, l\rangle$, so $\delta \cap Q^+(7, q) = lQ$, a cone with vertex l and base Q, which is either a conic or a degenerate quadric in a plane. Now $U \sim R$ and $P \sim U \sim Q$. But $l^\perp \cap Q^+(7, q) = lQ^+(3, q)$, and since $q = 2$, the points of either β or β' are collinear with all the points of β'', including U. But the line connecting such a point not on l with U will not meet Π. Hence, there can be no adjacency between U and either the vertices in β or in β'. Hence, the only possibility to extend further the clique $\{P, Q, R, T\}$ is by adding vertices in the planes β and β', and it is easy to see that the vertices in both planes indeed make a clique of size 8, the largest possible size. Finally, note that there would be no difference if the point R' was used first to extend the initial clique $\{P, Q, R\}$. □

Lemma 4.13. *Let $\mathcal{C} = \{P, Q, R\}$ be a clique of mixed type in $\mathcal{G}_3$. Assume that $T \notin \langle P, Q, R\rangle$ extends $\mathcal{C}$ to a larger clique such that T is $\sim_1$-adjacent to all the vertices of $\mathcal{C}$. Then $\gamma = \langle P, Q, R, T\rangle$ is a 3-dimensional space and $\gamma \cap Q^+(7, q) = SQ(2, q)$, with $S \in \Pi$, $Q(2, q)$ a conic not meeting Π and there is only one way to extend $\{P, Q, R, T\}$ to a maximal clique of size 8, which is then the set of vertices in a cone $SQ^-(3, q)$ that meets Π in a line.*

Proof. By Lemma 4.10 we may assume that $P \sim_2 Q$ and $P \sim_1 R \sim_1 Q$, and the plane $\alpha := \langle P, Q, R\rangle$ meets Π in a point S. Hence $\alpha \cap Q^+(7, q) = SQ^+(1, q)$. Since $P \sim_1 T \sim_1 Q$, $T^\perp$ meets the line $\langle P, Q\rangle$ in the point S, and since $T \sim_1 R$, it is now clear that $\gamma = \langle P, Q, R, T\rangle$ meets $Q^+(7, q)$ in a cone $SQ(2, q)$. W.l.o.g we may even assume that the base of this cone is the conic consisting of the points $\{P, R, T\}$.

Now assume that $U \notin \gamma$ extends $\{P, Q, R, T\}$ to a larger clique. If U is $\sim_2$-adjacent with one of the vertices of $\{P, Q, R, T\}$ then we can apply Lemma 4.12. Assume e.g. that $U \sim_2 R$, then apply Lemma 4.12 on the mixed clique $\{P, Q, R\}$ and the vertex U to conclude that $\gamma := \langle U, P, Q, R\rangle$ meets $Q^+(7, q)$ in a cone $lQ^+(1, q)$. However, this cone cannot contain the vertex T, while T extends the clique $\{U, P, Q, R\}$ to a larger clique, a contradiction by Lemma 4.12. This argument can be repeated for any $\sim_2$-adjacency between U and a vertex of $\{P, Q, R, T\}$. Hence, we must conclude that U is $\sim_1$-adjacent with all vertices of $\{P, Q, R, T\}$. But then the points $\{P, R, T, U\}$ span a 3-dimensional space meeting $Q^+(7, q)$ in a set of mutually non-collinear points, hence $\langle P, R, T, U\rangle \cap Q^+(7, q) = Q^-(3, q)$. As we concluded that U must be $\sim_1$ adjacent to P and Q, necessarily $U^\perp$ meets the line $\langle P, Q\rangle$ in the point S. Hence $\langle P, Q, R, T, U\rangle \cap Q^+(7, q) = SQ^-(3, q)$. Now the base of this cone is $Q^-(3, q)$, which is an ovoid of the base of the cone $S^\perp \cap Q^+(7, q) = SQ^+(5, q)$, so Π contains exactly one line of the cone $SQ^-(3, q)$. Now we assume that V extends the clique $\{P, Q, R, T, U\}$ further. We can apply Lemma 4.12 again to conclude that a $\sim_2$-adjacency between V and any vertex of $\{P, Q, R, T, U\}$ is not possible. But now the vertices $\{P, R, T, U, V\}$ are a $\sim_1$-clique of size 5. Either this is maximal, or can be extended only with other $\sim_1$-adjacencies to a larger clique by Lemma 4.8, a contradiction now with the fact that the clique $\{P, Q, R, T, U, V\}$ contains the adjacency $P \sim_2 Q$. Hence, the only way to extend the clique $\{P, Q, R, T, U\}$ is to add the remaining vertices on the lines of the cone $SQ^-(3, q)$, which extends the clique to a clique of maximal size. □

Theorem 4.14. *Table 1 lists all maximal cliques of the graph $\mathcal{G}_3$.*

Proof. First, consider a set $\mathcal{C}$ of 4 points in general position and assume that $\mathcal{C}$ is a $\sim_1$ clique. Then $\langle \mathcal{C}\rangle \cap Q^+(7, 2) = Q^-(3, q)$. If $\langle \mathcal{C}\rangle$ meets Π in a point, then this point is unique, and by Lemma 4.9 we can extend $\mathcal{C}$ uniquely to an ovoid of $Q^+(7, 2)$, which determines a maximal clique of size 8. If $\langle \mathcal{C}\rangle$ does not meet Π, then either adding the unique point of $Q^-(3, q) \setminus \mathcal{C}$ gives rise to a maximal clique of size 5, by Lemma 4.8, or, by Lemma 4.9, there is a unique extension of $\mathcal{C}$ to an ovoid, which again determines a clique of size 8.

By Lemma 4.11, any $\sim_2$-clique $\mathcal{C}$ must be contained in a subspace of the quadric. Hence, any $\sim_2$-clique can be extended to a $\sim_2$-clique of size 8 consisting of all the vertices contained in $\langle \mathcal{C}\rangle$.

Finally, assume that $\mathcal{C}$ is a clique of mixed type. Then it is sufficient to consider a subset $\{P, Q, R\} \subset \mathcal{C}$ which is a clique of mixed type and to use Lemmas 4.12 and 4.13 to conclude that a maximal clique of mixed type must be one of the two examples listed in Table 1. □

4.2 The isomorphism issue

We are now ready to state the non-isomorphism between $NO^+(2n+2,2)$ and $\mathcal{G}_n$, when $n \geq 3$. The next theorem collects all results about maximal cliques of $\mathcal{G}_3$, in order to distinguish $NO^+(8,2)$ and $\mathcal{G}_3$.

Theorem 4.15. *The graphs $\mathcal{G}_n$ and $NO^+(2n+2,2)$ are not isomorphic when $n \geq 3$.*

Proof. Firstly, while $n = 3$, $NO^+(8,2)$ contains only maximal 8-cliques, while $\mathcal{G}_3$ contains a family of maximal 5-cliques, and they are not isomorphic. Now, let $n > 3$ and consider a 7-space Σ of $\mathrm{PG}(2n+1,2)$ such that $\Sigma \cap Q^+(2n+1,2) = Q^+(7,2)$, which meets the fixed generator Π in a solid. Then the induced subgraphs by $NO^+(2n+2,2)$ and $\mathcal{G}_n$ on Σ are $NO^+(8,2)$ and $\mathcal{G}_3$, in which we have maximal cliques of different sizes, by Theorem 4.14. □

Remark 4.16. Note that for asymptotically large values of n the graph $NO^+(2n+2,2)$ contains a subgraph $\mathcal{G}_3$ and the graph $\mathcal{G}_n$ contains a subgraph $NO^+(8,2)$. More generally, let H be a graph with s vertices and r edges. The number of induced subgraphs H in a random graph with v vertices and with edge-probability (i.e. ratio between the number of edges and the number of unordered pair of vertices) equal to p is

$$(1+o(1))p^r(1-p)^{\binom{s}{2}-r}\frac{v^s}{|\mathrm{Aut}(H)|}; \tag{4.1}$$

see [13, Section 4.4]. Now we compute $|\mathrm{Aut}(NO^+(8,2)| = |PGO^+(8,2)| = 348364800$, $|\mathrm{Aut}(\mathcal{G}_3)| = 1290240$. Both $NO^+(8,2)$ and $\mathcal{G}_3$ have $s = 120$ vertices and $r = 3360$ edges, while both $NO^+(2n+2,2)$ and $\mathcal{G}_n$ have $s = 2^{2n+1} - 2^n$ vertices and edge-probability $p = \frac{2^{2n}-1}{2^{2n+1}-2^n-1}$. For small n, Equation (4.1) gives asymptotically 0 subgraphs. On the other hand, each hyperbolic section $PG(8,2)$ of $PG(2n+2,2)$ gives rise to a $NO^+(8,2)$ in $NO^+(2n+2,2)$, while there is an induced $\mathcal{G}_3$ in $\mathcal{G}_n$ for any hyperbolic section cutting the fixed generator in a 3-space. Hence Theorem 4.15 holds. In fact, repeated inductive arguments show that $NO^+(2n+2,2)$ does not contain an induced $\mathcal{G}_{n-1}$, and $\mathcal{G}_n$ does not contain an induced $NO^+(2n,2)$, where the induction basis is still Theorem 4.14.

Remark 4.17. According to the statement of [5, Theorem 1], the graph $\mathcal{G}_n$ for any q has automorphism group $q^{\frac{n(n+1)}{2}}\Gamma L(n+1,q)$, extension of $\Gamma L(n+1,q)$ by a normal p-subgroup of order $q^{\frac{n(n+1)}{2}}$. Thus, the non-isomorphism would belong to the group size. The proof provided in Theorem 4.15 provides pure combinatorial and self-contained arguments that show the novelty of the results presented.

Remark 4.18. An alternative combinatorial proof for the non-isomorphism would take into account the induced subgraphs $\Gamma_2(v)$ on vertices at distance 2 from a fixed vertex $v \in \Gamma$. Then $NO^+(8,2)_2(v)$ is known to be distance-regular of diameter 3 while $\mathcal{G}_{3,2}(v)$ has diameter 2 from the construction in Theorem 3.1. We presented the geometric proof in order to give a finite geometric interpretation of both strongly regular graphs.

5 Conclusion

In this paper, we provided the proof of [5, Theorem 1], giving a construction of strongly regular graphs in hyperbolic quadrics $Q^+(2n+1,q)$. We have seen that, while $q=2$, such a family gives examples of graphs that are cospectral to $NO^+(2n+2,2)$, but non-isomorphic whenever $n \geq 3$. In [3], there are some examples of graphs cospectral to $NO^+(2n+2,2)$. One can ask what are the isomorphism class and the switching class of $\mathcal{G}_n$ beneath its cospectral graphs, and a further research line would be to study the isomorphisms and the geometric characterisation.

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