AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **96(1)** (2026), Pages 1–9

On proper $(1, 3)$-dominating sets in graphs

Paweł Bednarz   Natalia Paja

Rzeszow University of Technology
The Faculty of Mathematics and Applied Physics
Department of Discrete Mathematics
al. Powstańców Warszawy 8, 35-029 Rzeszów
Poland
pbednarz@prz.edu.pl   nbednarz@prz.edu.pl

Abstract

A subset D of vertices is called $(1, k)$-dominating if every vertex v not belonging to the set D has a neighbour in D and also has another vertex in D at a distance of at most k from v. In this paper, we consider the concept of proper $(1, 3)$-dominating sets in graphs, i.e., subsets of vertices that are $(1, 3)$-dominating but not $(1, 2)$-dominating. Proper $(1, 3)$-dominating sets constitute a special case of proper $(1, k)$-dominating sets introduced by Michalski et al. [*Math. Methods Appl. Sci.* 45(11) (2022), 7050–7057]. We provide necessary and sufficient conditions for a graph to have proper $(1, 3)$-dominating sets.

1 Introduction

We follow the standard notation of graph theory as found in [9]. In this paper, we focus on simple, undirected, connected graphs $G = (V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. For vertices $x, y \in V(G)$, we say that x is a *neighbour* of y if there exists an edge $xy \in E(G)$. *The neighbourhood of a vertex* x in G, denoted $N_G(x)$, is a set of all neighbours of x. The set $N_G[x] = N_G(x) \cup \{x\}$ represents the *closed neighbourhood* of x. We write $d_G(x) = |N_G(x)|$ for the *degree of vertex x in G*. *The minimum degree of a graph* G, denoted by $\delta(G)$, is the smallest degree among all vertices in G. A vertex of degree 1 is called a *leaf*. A vertex that is adjacent to a leaf is called a *support vertex*. For any two vertices $x, y \in V(G)$, a *geodesic* from x to y is a path with the minimum possible number of edges among all paths from x to y. The length of a geodesic between x and y is called the *distance* between x and y, and denoted by $d_G(x, y)$. The *diameter* of a connected graph G, $\mathrm{diam}(G)$, is the maximum distance between two vertices of G. The set of vertices that are within distance at most l from a vertex $x \in V(G)$ is denoted by $N^l_G(x)$. For any integer i, we define $U^i_G(x)$ as the set of all vertices whose distance from x is exactly i. Thus, $N^l_G(x)$ is equal to the union of the sets $U^i_G(x)$

ISSN: 2202-3518

for $i = 1, \ldots, l$, that is, $N^l_G(x) = \bigcup_{i=1}^{l} U^i_G(x)$, where the sets $U^i_G(x)$ and $U^j_G(x)$ are pairwise disjoint for all $i, j \in \{1, 2, \ldots, l\}$ and $i \neq j$. The *closed l-neighbourhood* of x is then $N^l_G[x] = N^l_G(x) \cup \{x\}$.

Let n be an integer. By P_n, T_n, $n \geq 1$ and C_n, $n \geq 3$, we denote a *path*, a *tree* and a *cycle* of order n, respectively.

A subset $B \subseteq V(G)$ is called a *dominating set* of G if every vertex outside B has at least one neighbour in B. The *domination number* of G, $\gamma(G)$, is the size of a minimum dominating set of G. The literature presents various types of dominating sets, which are either obtained by adding certain restrictions or by generalizing the classical concept of domination, as discussed in [12]. One of the most extensively studied types is the multiple dominating set, introduced by Fink and Jacobson in [10]. For an integer $p \geq 1$, a subset S is called *p-dominating* if every vertex outside S has at least p neighbours in S. When $p = 1$, this corresponds to the classical dominating set. For $p = 2$, we obtain the concept of 2-dominating sets, which were studied, for example, in [7, 8].

Another widely researched type of dominating set is a generalization in the distance sense, which was introduced by Meir and Moon in [15]. In [11], Hedetniemi et al. extended the concept of multiple domination by combining it with distance domination, defining $(1, k)$-dominating sets, which are also named secondary dominating sets. Let $k \geq 1$ be an integer. A subset S is a $(1, k)$*-dominating set* of G if for every vertex $v \in V(G) \setminus S$, there exist distinct vertices $u, w \in S$ such that $uv \in E(G)$ and $d_G(v, w) \leq k$.

The following result shows a basic relation between (l, k)-dominating sets for different values of k.

Theorem 1.1 ([11]). *Let $k, l \in \mathbb{N}$ and $k > l$. Every $(1, l)$-dominating set is a $(1, k)$-dominating set.*

In [11] it was also proved that studying $(1, k)$-dominating sets makes sense only for $k \in \{1, 2, 3, 4\}$.

Theorem 1.2 ([11]). *Every dominating set of cardinality at least 2 in a connected graph G is a $(1, 4)$-dominating set.*

It should be noted that $(1, 1)$-dominating sets have been studied extensively. Indeed, $(1, 1)$-dominating sets are also known as 2-dominating sets. In particular, many results were obtained for their connections with independent sets; see, for example, [1–5, 18]. Several problems related to $(1, 2)$-dominating sets have been studied in [13, 14, 16, 17].

By Theorem 1.1, any $(1, 1)$-dominating set is also a $(1, k)$-dominating set for $k > 1$. However, not every $(1, 2)$-dominating set is a $(1, 1)$-dominating set. This motivates the introduction of proper $(1, 2)$-dominating sets, studied in [16]. Let $k \geq 2$ be an integer. A subset $D \subset V(G)$ is called a *proper* $(1, k)$*-dominating set* if D is $(1, k)$-dominating but not $(1, k-1)$-dominating. To simplify the notation, we will use the term $(1, \overline{k})$-dset to denote a proper $(1, k)$-dominating set.

The authors in [16] give a necessary and sufficient condition for the existence of $(1,\overline{2})$-dsets in connected graphs.

Theorem 1.3 ([16]). *A connected graph G has a $(1,\overline{2})$-dset if and only if G is not a complete graph.*

This paper is a natural consequence of the previously mentioned articles, where we study $(1,\overline{3})$-dsets and provide a complete characterization of graphs with such sets. Note that D is a $(1,\overline{3})$-dset if D is a $(1,3)$-dominating set and there exists $v \in V(G) \setminus D$ that has exactly one neighbour in D, there are no vertices in D with a distance of 2 from v, and there exists at least one other vertex in D with a distance of 3 from v. Using our previous notation this brings us to the following observation:

Observation 1.4. *If D is a $(1,\overline{3})$-dset of G, then there exists a vertex $v \in V(G) \setminus D$ such that $|U^1_G(v) \cap D| = 1$, $|U^2_G(v) \cap D| = 0$ and $|U^3_G(v) \cap D| \geq 1$, and we say that the vertex v is proper $(1,3)$-dominated or $(1,\overline{3})$-dominated.*

2 Results

In this section, we give a complete characterization of graphs G that have $(1,\overline{3})$-dsets. Let us start with a simple observation: such a graph G must contain a geodesic of length 3, otherwise none of the vertices can be proper $(1,3)$-dominated.

Observation 2.1. *If G has a $(1,\overline{3})$-dset, then* $\operatorname{diam}(G) \geq 3$.

The next theorem gives a necessary condition for the existence of $(1,\overline{3})$-dsets in graphs.

Theorem 2.2. *Let G be a connected graph such that* $\operatorname{diam}(G) \geq 3$. *If G has a $(1,\overline{3})$-dset then there exist two vertices $x, y \in V(G)$ such that $N_G[x] \subset N_G[y]$ and x is $(1,\overline{3})$-dominated.*

Proof. Let G be a graph with a $(1,\overline{3})$-dset D. Then, from Observation 1.4 we know that there exist a vertex $v \in V(G) \setminus D$ and vertices $t, w \in D$ such that $N_G(v) \cap D = \{t\}$, $d_G(v,w) = 3$, and there is no vertex $w^* \in D$ such that $d_G(v,w^*) = 2$. Assume by contradiction, that every two distinct vertices $x, y \in V(G)$ satisfy the condition $N_G[x] \not\subset N_G[y]$. The vertex v is not a leaf because then $N_G[v] \subset N_G[t]$. Thus $d_G(v) = k$, where $k \geq 2$. Let $t_1, \ldots, t_{k-1}$ be vertices adjacent to v and not belonging to D. Since $N_G[v] \not\subset N_G[t]$ there exists a vertex t_i, $i \in \{1, \ldots, k-1\}$ such that $tt_i \notin E(G)$. Since D is dominating, there is a vertex $z \in D$, such that $z \neq t$ and z is adjacent to t_i. Then $d_G(v,z) = 2$, a contradiction. □

The converse implication of Theorem 2.2 is not true. That is, the existence of two vertices x, y such that $N_G[x] \subset N_G[y]$ does not necessarily imply that G has a $(1,\overline{3})$-dset. Let us consider the graph G_1 from Figure 1.

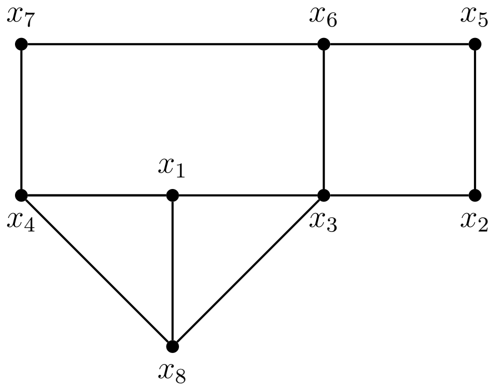

Figure 1: Graph G_1.

Then $\mathrm{diam}(G_1) = 3$ and $N_{G_1}[x_8] = N_{G_1}[x_1]$. It is easy to check that in every construction of a dominating set in this graph, we have to pick at least two vertices from the set $N^2_{G_1}(x_i)$, for every vertex x_i. This means that each x_i is $(1, 2)$-dominated for any dominating set of G_1.

Let $u \in V(G)$. If u is proper $(1, 3)$-dominated by the set $D \subset V(G)$ then there exists a vertex $t \in D$ such that the set $N^2_G(u) \cap D = \{t\}$ and $ut \in E(G)$. This leads us to the following sufficient condition for the existence of $(1, \overline{3})$-dsets in graphs.

Theorem 2.3. *Let G be a connected graph and* $\mathrm{diam}(G) \geq 3$. *Let $t, u \in V(G)$ be vertices of the graph G such that t is adjacent to every vertex from the set $N^2_G[u] \setminus \{t\}$. Then G has a $(1, \overline{3})$-dset and, in particular, u is $(1, \overline{3})$-dominated.*

Proof. Let $u \in V(G)$, $U = N^2_G[u]$, and $t \in V(G)$ be a vertex adjacent to every vertex from the set $U \setminus \{t\}$. First, we will show that $V(G) \setminus U \neq \emptyset$. Assume by contradiction that $V(G) \setminus U = \emptyset$. Then for every pair of vertices $u', u'' \in V(G)$, $d(u', u'') \leq 2$. This means that $\mathrm{diam}(G) \leq 2$, a contradiction. Let us consider the set $D = (V(G) \setminus U) \cup \{t\}$. Since $V(G) \setminus U \neq \emptyset$, then there exists a vertex $v \in V(G) \setminus U$, such that $d(u, v) = 3$. Hence because $ut \in E(G)$ and $|N^2_G(u) \cap D| = 1$, the vertex u is proper $(1, 3)$-dominated by the set D. Moreover, for every vertex $t' \neq t$ from the set U, t' is adjacent to t and $d(t', v) \leq 3$, so t' is $(1, 3)$-dominated by t and v. □

Considering the above theorem, we can see that if G has diameter at least 3 and contains a support vertex t, then for a leaf $x \in N_G(t)$ the condition from Theorem 2.3 is met and we can create a $(1, \overline{3})$-dset. This leads us to the following corollaries.

Corollary 2.4. *Let G be a connected graph and* $\mathrm{diam}(G) \geq 3$. *If $\delta(G) = 1$, then G has a $(1, \overline{3})$-dset.*

Corollary 2.5. *Let $n \geq 4$ be an integer. Let T_n be a tree such that* $\mathrm{diam}(T_n) \geq 3$. *Then T_n has a $(1, \overline{3})$-dset.*

In particular, we can see that paths of order at least 4 are graphs that have a $(1, \overline{3})$-dset.

Corollary 2.6. *Let $n \geq 4$. Then P_n has a $(1, \overline{3})$-dset.*

From Theorem 2.2, we find that cycles are a class of graphs that do not have $(1, \overline{3})$-dsets. For $n \geq 4$, we have $N_{C_n}[x] \not\subset N_{C_n}[y]$ for any two vertices $x, y \in V(C_n)$. Furthermore, since $\mathrm{diam}(C_3) = 1$, this leads to the following corollary.

Corollary 2.7. *Let $n \geq 3$ be an integer. Then C_n does not have a $(1, \overline{3})$-dset.*

By Theorem 2.3, we can see that a vertex u of a graph of diameter at least 3 is proper $(1, 3)$-dominated if it has a neighbour t adjacent to all vertices which are at distance at most 2 from u. This condition can be weakened if a vertex at distance 2 from u has a neighbour in the set of vertices at distance 3 from u. Taking this into account, we obtain the next result.

Theorem 2.8. *Let G be a connected graph with $\mathrm{diam}(G) \geq 3$. The graph G has a $(1, \overline{3})$-dset if and only if there exist vertices t and u such that the set $D' = \{t\} \cup U^3_G(u)$ is a dominating set of the subgraph induced by $N^3_G[u]$.*

Proof. Let G be a connected graph and $\mathrm{diam}(G) \geq 3$. We begin by proving the necessity of the condition. Let us assume that the graph G has a $(1, \overline{3})$-dset and let us denote this set by D. Therefore, there is a vertex $u \in V(G)$, such that $u \notin D$ and u is $(1, \overline{3})$-dominated by D. Hence $|D \cap N_G[u]| = 1$. Let $t \in D$ and $t \in N_G(u)$. Then, $tu \in E(G)$ and by Theorem 2.2 we know that $N_G[u] \subset N_G[t]$. So each vertex $z^1 \in N_G(u)$ is dominated by t. Now let z^2 be an arbitrary vertex of the set $U^2_G(u)$. Then since the vertex u is proper $(1, 3)$-dominated, $z^2 \notin D$. Then at least one of the following possibilities holds:

- **(a)** $z^2 t \in E(G)$,
- **(b)** there is a vertex $v \in D$, $v \neq t$, such that $z^2 v \in E(G)$.

For each vertex v in case **(b)**, $d(u, v) = 3$ holds. Therefore, the set $D' = \{t\} \cup U^3_G(u)$ is a dominating set of the subgraph induced by $N^3_G[u]$.

We now prove the sufficiency of the condition. Suppose that there exist vertices $t, u \in V(G)$ such that $tu \in E(G)$ and the set $\{t\} \cup U^3_G(u)$ is a dominating set of the subgraph of G induced by $N^3_G[u]$. For simplicity let us denote $D' = \{t\} \cup U^3_G(u)$ and let G' be the subgraph of G induced by $N^3_G[u]$. Firstly we will show that $U^3_G(u) \neq \emptyset$. Suppose by contrary that $U^3_G(u) = \emptyset$. Then since G is connected, $G = G'$ and $N_G[t] = V(G)$, and so $\mathrm{diam}(G) \leq 2$, a contradiction with the assumption that $\mathrm{diam}(G) \geq 3$. Thus $U^3_G(u) \neq \emptyset$ and $|D'| \geq 2$. Moreover, since D' dominates G', $N_G[u] \subseteq N_G[t]$.

By Theorem 1.2, each vertex $v \in N^2_G[u] \setminus \{t\}$ is $(1, 4)$-dominated by the set D'. Therefore for each vertex $v \in N^2_G[u] \setminus \{t\}$ there are two vertices from D' such that one of them is adjacent to v and the other one is within the distance at most 4 from v. We consider whether D' $(1, 3)$-dominates every vertex in $N^2_G[u] \setminus \{t\}$ or not.

Case 1. Let us assume that each vertex in $N_G^2[u] \setminus \{t\}$ is $(1,3)$-dominated by the set D'. Let $D = (V(G) \setminus N_G^2[u]) \cup \{t\}$. Note that $U_G^3(u) \subset D$. The vertex u is adjacent to t, and there exists a vertex $w \in U_G^3(u)$ such that $d(u,w) = 3$, thus u is $(1,\overline{3})$-dominated by the set D. Then every vertex from the set $V(G) \setminus D$ is $(1,3)$-dominated by D and u is $(1,\overline{3})$-dominated by D. Hence D is $(1,\overline{3})$-dset of G.

Case 2. Let us assume that there exists a vertex $v \in N_G^2[u] \setminus \{t\}$ which is $(1,4)$-dominated but not $(1,3)$-dominated by D'. Let U^* be the set of such vertices, and note that $|U_G^1(v) \cap D'| = 1$, $|U_G^2(v) \cap D'| = 0$, $|U_G^3(v) \cap D'| = 0$ and $|U_G^4(v) \cap D'| \geq 1$. As the vertex u is $(1,\overline{3})$-dominated by D', it follows that $u \notin U^*$. Each vertex $v \in U^*$ is adjacent to t. Otherwise, since D' is dominating, v would be adjacent to some $w \in U_G^3(u)$, and as $d(v,u) \leq 2$ and $ut \in E(G)$, it would follow that $d(v,t) \leq 3$, which would mean that v is $(1,3)$-dominated by D', a contradiction.

Let $w \in U_G^3(u)$ and $P = (p_0, p_1, p_2, p_3, p_4)$ be a v–w geodesic, with $v = p_0$ and $w = p_4$. Note $V(P) \subset N_G^3[u]$. We define $a_i = d(u, p_i)$ for $i = 0, \ldots, 4$, with consecutive terms differing by at most 1. Since $v \in N_G^2[u] \setminus \{t, u\}$, we have $a_0 \in \{1,2\}$, and $a_4 = 3$ because $p_4 = w \in U_G^3(u)$. Because consecutive values of a_i differ by at most 1, and $p_3 \in N_G^3[u]$, $a_3 \in \{2,3\}$. Suppose $a_3 = 3$. Thus $p_3 \in U_G^3(u) \subset D'$ with $d(v,p_3) = 3$, a contradiction to $|U_G^3(v) \cap D'| = 0$. Therefore, $a_3 = 2$ and $p_3 \in U_G^2(u)$. Let $W^* = \{x \in U_G^2(u) : d(v,x) = 3, v \in U^*\}$. Note that $p_3 \in W^*$ so $W^* \neq \emptyset$. We define $D^* = D' \cup W^*$. We will show that D^* is a $(1,\overline{3})$-dset of G. Let $z \in V(G) \setminus (D^* \cup U^*)$. Since z is $(1,3)$-dominated by D', then it is also $(1,3)$-dominated by D^*, since $D' \subset D^*$. Let $v \in U^*$, then by definitions of D^* and W^*, we have that $|U_G^1(v) \cap D^*| = |\{t\}| = 1$, $|U_G^2(v) \cap D^*| = 0$ and $|U_G^3(v) \cap D^*| = |W^*| \geq 1$. Hence, v is $(1,\overline{3})$-dominated by D^*, and finally, D^* is a $(1,\overline{3})$-dset of G. □

Now it is easy to see that the graph from Figure 1 does not have a $(1,\overline{3})$-dset since there do not exist two adjacent vertices satisfying the condition from Theorem 2.8.

Recently, Bednarz and Pirga in [6] introduced and studied the existence of perfect $(1,2)$-dominating sets. A subset $D \subset V(G)$ is a *perfect* $(1,2)$-*dominating set* of G if every vertex x outside D has exactly one neighbour in the set D and there is a vertex $u \in D$ such that $d_G(x,u) = 2$. The following theorem shows that an analogous definition for the $(1,3)$-dominating set makes no sense.

Theorem 2.9. *If D is a $(1,\overline{3})$-dset of a graph G, then there exists a vertex of G outside D which is not $(1,\overline{3})$-dominated by D.*

Proof. Let G be a graph, and D be a $(1,\overline{3})$-dset of G. Then by Observation 2.1, $\operatorname{diam}(G) \geq 3$. By the definition of a $(1,\overline{3})$-dset, there exists a vertex $v \in V(G) \setminus D$ which is $(1,\overline{3})$-dominated by D, and so there are vertices $d_1, d_2 \in D$ such that $d_1 v \in E(G)$ and $d(v, d_2) = 3$. Consider a $v - d_2$ geodesic: v, v_1, v_2, d_2. Note $v_1 \in V(G) \setminus D$, otherwise v is not $(1,\overline{3})$-dominated. By the definition of a geodesic $d(v_1, d_2) = 2$ and so v_1 is not $(1,\overline{3})$-dominated. □

2.1 Application of $(1,\overline{3})$-dsets

Imagine a scenario in which we want to place two types of heat generating machines in a hall. Each machine needs two sensors: one very close (up to distance 1) and one farther away (up to distance 3) as a backup. Type one machines work at low temperatures and use standard sensors. Type two machines are much hotter and require a specialised sensor; standard ones cannot be placed near them.

Spatial rules:

- Distance ≤ 1: at least one sensor for every machine.
- Distance 2: no standard sensors near type two machines.
- Distance ≤ 3: at least one backup sensor.

Economically, type two machines are far more profitable, so the company wants to place as many of them as possible while respecting the constraints. If this is feasible, the set of sensors corresponds to $(1,\overline{3})$-dset; otherwise to $(1,3)$-dsets, where:

- every machine has a neighbour in the dominating set,
- no vertex from the dominating set is at distance 2 from a type two machine,
- at least one vertex from the dominating set is at distance 3.

Assume the analysed layout is shown in Figure 2. The company aims to maximize

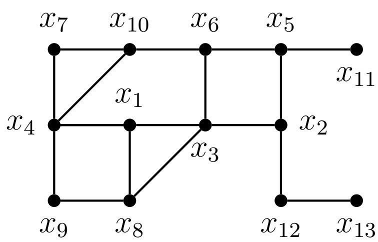

Figure 2: Graph model representing potential machine placement locations.

the number of type two machines. A possible arrangement is shown in Figure 3, where type two machines are marked with red squares at x_7, x_{11}, x_{13} and type one machines are marked with blue ones at the remaining vertices. Figure 4 shows the corresponding sensor placement: blue circles represent standard sensors, and red circles represent heat resistant sensors. In this layout, high temperature resistant sensors must be installed at positions x_4, x_5, and x_{12}, while a standard sensor can be placed at x_3. Each machine is monitored by a nearby sensor (distance 1 or the same position) and by a second sensor placed at distance at most 3. In particular, every type two machine has one sensor in its neighbourhood and another exactly at distance 3, satisfying the required monitoring constraints.

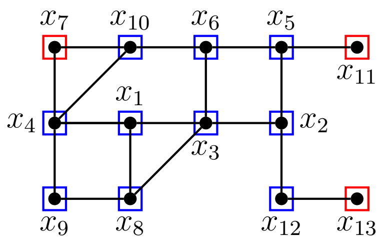

Figure 3: Graph model representing example of machine placement locations.

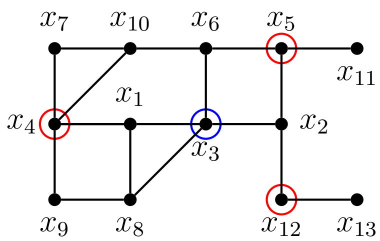

Figure 4: Graph model representing sensor placement locations.

Acknowledgements

The authors wish to thank the referees for a through review and very useful suggestions which improved the rewriting of this paper.

References

[1] P. Bednarz, On $(2-d)$-kernels in the tensor product of graphs, *Symmetry* 13(2) (2021), 230.

[2] P. Bednarz, C. Hernández-Cruz and I. Włoch, On the existence and the number of (2-d)-kernels in graphs, *Ars Combin.* 121 (2015), 341–351.

[3] P. Bednarz and N. Paja, On $(2-d)$-kernels in two generalizations of the Petersen graph, *Symmetry* 13(10) (2021), 1948.

[4] P. Bednarz and I. Włoch, An algorithm determining (2-d)-kernels in trees, *Util. Math.* 102 (2017), 215–222.

[5] P. Bednarz and I. Włoch, On $(2-d)$-kernels in the cartesian product of graphs, *Ann. Univ. Mariae-Curie-Skł-odowska Sect. A–Math.* 70(2) (2016), 1–8.

[6] U. Bednarz and M. Pirga, $(1,2)$-PDS in graphs with the small number of vertices of large degrees, *Opuscula Math.* 45(1) (2025), 53–62.

[7] M. Blidia, M. Chellali and O. Favaron, Independence and 2-domination in trees, *Australas. J. Combin.* 33 (2005), 317–327.

[8] M. Chellali, Bounds on the 2-domination number in cactus graphs, *Opuscula Math.* 26(1) (2006), 5–12.

[9] R. Diestel, Graph Theory, 3rd ed. *Graduate Texts in Mathematics*, 173, 2005.

[10] J. F. Fink and M. S. Jacobson, n-domination in graphs, in: *Graph Theory with Applications to Algorithms and Computer Science*, John Wiley & Sons, Inc., New York, USA, 1985, pp. 283–300.

[11] S. M. Hedetniemi, S. T. Hedetniemi, J. Knisely and D. F. Rall, Secondary domination in graphs, *AKCE Int. J. Graphs Combin.* 5 (2008), 103–115.

[12] T. W. Haynes, S. Hedetniemi and P. Slater, *Fundamentals of domination in graphs*, Monographs and Textbooks in Pure and Applied Mathematics, 208, Dekker, New York, 1998.

[13] K. Kayathri and S. Vallirani, $(1, 2)$-domination in graphs, in: *Theoretical Computer Science and Discrete Mathematics*, ICTCSDM 2016, (Eds.: S. Arumugam, J. Bagga, L. Beineke and B. Panda), Springer, Cham, 2017.

[14] A. Kosiorowska, A. Michalski and I. Włoch, On minimum intersections of certain secondary dominating sets in graphs, *Opuscula Math.* 43(6) (2023), 813–827.

[15] A. Meir and J. W. Moon, Relations between packing and covering numbers of a tree, *Pacific J. Math.* 61(1) (1975), 225–233.

[16] A. Michalski, I. Włoch, M. Dettlaff and M. Lemańska, On proper $(1, 2)$-dominating sets, *Math. Methods Appl. Sci.* 45(11) (2022), 7050–7057.

[17] J. Raczek, Polynomial Algorithm for Minimal $(1, 2)$-Dominating Set in Networks, *Electronics* 11(3) (2022), 300.

[18] A. Włoch, On 2-dominating kernels in graphs, *Australas. J. Combin.* 53 (2012), 273–284.

(Received 7 Mar 2025; revised 12 Sep 2025, 1 May 2026)