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A step toward an S-packing coloring conjecture for subcubic graphs

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Abstract

For a non-decreasing sequence of integers $S = (s_1, s_2, \ldots, s_k)$, an S-packing coloring of G is a partition of $V(G)$ into k subsets $I_1, I_2, \ldots, I_k$ such that the distance between any two distinct vertices $u, v \in I_i$ is greater than s_i, for any i such that $1 \leq i \leq k$. Gastineau and Togni [*Discrete Math.* 339 (2016), 2461–2470] asked whether a subcubic graph is $(1,2,2,2,2,2)$- and $(1,1,2,3)$-packing colorable, except the Petersen graph. Recently, Brešar, Kuenzel, and Rall [*Discrete Math.* 348(8) (2025), 114477] proved that every claw-free subcubic graph is $(1,1,2,2)$-packing colorable. The local girth of a vertex v in a graph is the length of the shortest cycle containing v. Hence, a claw-free subcubic graph is a subcubic graph where every vertex of degree 3 has local girth 3. In this paper, we prove that every subcubic graph such that every vertex of degree 3 has local girth at most 4 is $(1,2,2,2,2,2)$-packing colorable.

1 Introduction

All graphs considered in this paper are finite, simple, and undirected. For a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. The set of neighbors

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of a vertex v is denoted by $N_G(v)$, and the degree of v, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. A vertex v is said to be a $d_G(v)$-vertex. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degrees of G, respectively. A graph G is called a subcubic graph if $\Delta(G) \leq 3$, and it is said to be cubic if $d_G(v) = 3$ for every vertex v in G. A subcubic graph is said to be k-saturated, for $0 \leq k \leq 3$, if every 3-vertex is adjacent to at most k 3-vertices. The local girth of a vertex v in G, denoted by girth(v), is the length of the shortest cycle containing v. Note that every vertex in the Petersen graph has a local girth 5.

The distance between two vertices u and v in G is denoted by $dist_G(u, v)$, or simply $dist(u, v)$. The square of G, denoted by G^2, is the graph obtained by adding an edge between every two vertices u and v such that $dist_G(u, v) = 2$. For $H \subseteq V(G)$, we denote by $G[H]$ the subgraph of G induced by H. The subdivision of G, denoted by $S(G)$, is the graph obtained from G after replacing each edge in G by a path of length two. Given a sequence of positive integers $(s_1, s_2, \dots, s_k)$ where $s_1 \leq s_2 \leq \cdots \leq s_k$, an S-packing coloring of a graph G is a partition of the vertex set $V(G)$ into subsets $V_1,\ V_2, \dots,\ V_k$, such that for any two distinct vertices u and v in the same subset V_i, for $1 \leq i \leq k$, $dist_G(u, v) > s_i$. The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that G is $(1, 2, \dots, k)$-packing colorable. The packing chromatic number was originally introduced by Goddard et al. [9]. Much of the research on packing coloring, mostly focused on subcubic graphs, is detailed in the survey by Brešar et al. [2]. Note that the packing chromatic number in the family of subcubic graphs is unbounded.

Furthermore, Balogh, Kostochka, and Liu [1] proved that for every subcubic graph G, $\chi_\rho(S(G))$ does not exceed 8. Later, Gastineau and Togni [8] asked whether this bound could be reduced to 5, a question that was subsequently presented as a conjecture by Brešar et al. [4]. In the same work, Gastineau and Togni [8] established that it is enough for a subcubic graph G to be $(1, 1, 2, 2)$-packing colorable in order to ensure $\chi_\rho(S(G)) \leq 5$. They also demonstrated that every subcubic graph admits a $(1, 2, 2, 2, 2, 2, 2)$-packing coloring and raised the problem of whether all such graphs, except for the Petersen graph, are $(1, 2, 2, 2, 2, 2)$-packing colorable.

Yang and Wu [20] proved that every 3-irregular subcubic graph admits a $(1, 1, 3)$-packing coloring, with a simplified proof given later in [17]. Brešar et al. [4] further showed that a generalized prism of a cycle is $(1, 1, 2, 2)$-packing colorable if and only if it is distinct from the Petersen graph. In another direction, Tarhini and Togni [19] proved that every cubic Halin graph admits a $(1, 1, 2, 3)$-packing coloring. Moreover, Liu et al. [11] established that subcubic graphs with maximum average degree less than $\frac{30}{11}$ are $(1, 1, 2, 2)$-packing colorable, while Brešar, Kuenzel, and Rall [5] confirmed the same property for all claw-free cubic graphs; a shorter proof was later provided in [13]. More recently, Liu, Zhang, and Zhang [12] proved that every subcubic graph is $(1, 1, 2, 2, 3)$-packing colorable, which implies that the packing chromatic number of the subdivision of any subcubic graph is at most 6. Finally, the authors [6] proved that every non-regular subcubic graph is $(1, 1, 2, 2)$-packing colorable, implying that every subcubic graph is $(1, 1, 2, 2, k)$-packing colorable, for arbitrary integer k.

The S-packing coloring of k-saturated subcubic graphs has also been studied in [14, 15, 16, 18], where a summary of results, sharpness, and open questions is provided in [18]. In addition, outerplanar subcubic graphs are known to be $(1, 1, 2)$-packing colorable whenever (i) the graph is triangle-free [3], or (ii) it is 2-connected [10]. On the other hand, [7] presents results on the $(1, 2, 2, 2, 2)$- and $(1, 2, 2, 2, 2, 2)$-packing colorability of certain subcubic graphs.

In this paper, we prove that every subcubic graph such that every 3-vertex has local girth at most 4 is $(1, 2, 2, 2, 2, 2)$-packing colorable.

2 Main Result

Theorem 1 *Every subcubic graph such that every* 3*-vertex has local girth at most* 4 *is* $(1, 2, 2, 2, 2, 2)$*-packing colorable.*

Proof. Let G be a subcubic graph such that every 3-vertex has local girth at most 4. Without loss of generality, suppose that G is connected. Let I be a maximum independent set in G and $J = V(G) \setminus I$. Clearly, every vertex in J has a neighbor in I. Then, $\Delta(G[J]) \leq 2$. Our plan is to show that $\Delta(G^2[J]) \leq 5$ and that $G^2[J]$ contains no K_6 unless $G = G_{12}$, where G_{12} is represented in Figure 4. For the case where $G \neq G_{12}$, that is, $G^2[J]$ does not contain a copy of K_6, we will argue as follows. By Brooks' theorem, one can color the vertices of J using five colors such that each two vertices of the same color are not adjacent in G^2, that is, are at a distance of at least 3 in G. Therefore, one can color the vertices of J using five 2-colors (in G) and the vertices of I using the color 1 to obtain a $(1, 2, 2, 2, 2, 2)$-packing coloring of G. Finally, we will show a $(1, 2, 2, 2, 2, 2)$-packing coloring of G_{12}. This completes the proof.

Let x be a vertex in J. For abbreviation, we denote by $J(x)$ the set of vertices in J that are at a distance 1 or 2 from x. Clearly, if x is a 1-vertex (respectively 2-vertex), then $d_{G^2[J]}(x) \leq 3$ (respectively 4). Now, we study the 3-vertices.

Claim 1.1 *Let* $x \in J$ *be a* 3*-vertex such that girth*$(x) = 3$*. Then,* $d_{G^2[J]}(x) \leq 5$*, and the equality occurs if and only if* x *is of type i, for some* $i \in \{1, 2\}$ *(see Figure 1).*

Proof. Let x_1, x_2, x_3 be the neighbors of x such that $x_1x_2 \in E(G)$ and let y_1 (respectively y_2) be the third neighbor of x_1 (respectively x_2). Let y_3 and y_4 be the neighbors of x_3 other than x. Note that some of the defined vertices may not exist, and two or more vertices may be the same vertex. In such cases, the study will be easier. First, suppose that $x_1, x_2 \in J$. Then, $y_1, y_2 \in I$. Therefore, $J(x) \subseteq \{x_1, x_2, y_3, y_4\}$. Thus, $d_{G^2[J]}(x) \leq 4$. Note that x_1 and x_2 cannot be both in I. Hence, we will now assume that $x_2 \in I$. If $x_3 \in J$, then either y_3 or y_4 is in I, say y_3. Thus, $J(x) \subseteq \{x_1, x_3, y_1, y_2, y_4\}$ and the equality occurs if and only if x is of type 1. Finally, suppose that $x_3 \in I$. Here, $J(x) \subseteq \{x_1, y_1, y_2, y_3, y_4\}$ and the equality occurs if and only if x is of type 2. ■

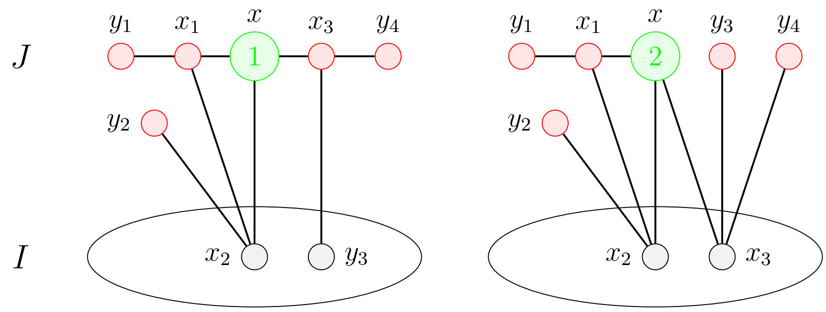

Figure 1: Vertices of type 1 and 2.

Claim 1.2 *Let $x \in J$ be a 3-vertex such that girth$(x) = 3$ and Q be a clique of order 6 in $G^2[J]$. Then, $x \notin V(Q)$.*

Proof. Suppose to the contrary that $x \in V(Q)$. Here, $d_{G^2[J]}(x) = 5$. Then, by Claim 1.1, x is of type 1 or of type 2. Let $x_1, x_2, x_3, y_1, y_2, y_3, y_4$ be the vertices defined as in Claim 1.1. First, suppose that x is of type 2. Let x_1 be the unique neighbor of x in J. Note that girth$(x_1) = 3$ and x_1 has two neighbors in J, namely x and y_1. But $x_1 \in V(Q)$, then $|J(x_1)| = 5$. Thus, x_1 is of type 1. Therefore, x has a neighbor in J other than x_1, a contradiction. So, we may assume that x is of type 1. Clearly, x_1 is of type 1. As $y_4 \in J(x_1)$, the only path of length 2 from x_1 to y_4 should pass through y_4. Hence, $y_1 y_4 \in E(G)$. Note that y_1 should be adjacent to a vertex $z_1 \in I$. Now, since $y_2 \in J(y_1)$, it follows that y_1 and y_2 have a common neighbor. Moreover, the third neighbor of y_4 belongs to I, implying that y_2 and y_4 are not adjacent. Thus, the common neighbor of y_1 and y_2 is z_1. Recall that x_1 and y_4 have no common neighbor other than y_1. In addition, z_1 has no common neighbor with x_1 (respectively y_4) other than y_1. Hence, girth$(y_1) \neq 4$. Thus, girth$(y_1) = 3$. That is, $y_4 z_1 \in E(G)$. Therefore, girth$(x_3) \geq 5$, a contradiction. ■

In light of Claim 1.2, 3-vertices with local girth 3 cannot belong to a clique of order 6 in $G^2[J]$. Consequently, any such clique must consist only of vertices with local girth 4. In what follows, we focus on these vertices.

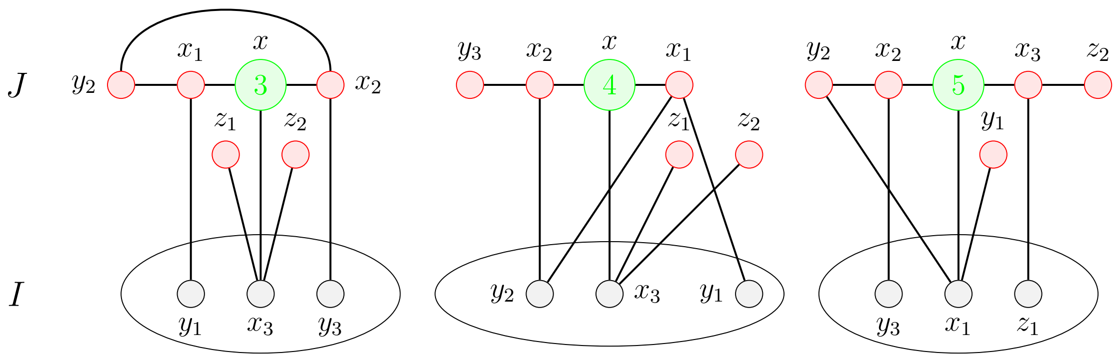

Figure 2: Vertices of type 3, 4, and 5.

Claim 1.3 *Let $x \in J$ be a 3-vertex such that girth$(x) = 4$. Then, $d_{G^2[J]}(x) \leq 5$, and the equality occurs if and only if x is of type i, for some $i \in \{3, \ldots, 8\}$ (see Figures 2 and 3).*

Proof. Let x_1, x_2, x_3 be the neighbors of x such that x_1 and x_2 have a common neighbor. Let y_1 and y_2 be the neighbors of x_1 other than x such that y_2 is also a neighbor of x_2, and let y_3 be the third neighbor of x_2. Let z_1 and z_2 be the neighbors of x_3 other than x. Note that some of the defined vertices may not exist, and two or more vertices may be the same vertex. In such cases, the study will be easier.

Case 1: $x_1, x_2 \in J$.
If $y_2 \in J$, then $y_1, y_3 \in I$. Hence, $J(x) \subseteq \{x_1, x_2, y_2, z_1, z_2\}$ with equality if and only if x is of type 3 (see Figure 2). Otherwise, $y_2 \in I$. Here, either y_1 or y_3 must be in I, since, otherwise, $(I \setminus \{y_2\}) \cup \{x_1, x_2\}$ is an independent set of cardinality greater than that of I, a contradiction. Without loss of generality, suppose that $y_1 \in I$. Thus, $J(x) \subseteq \{x_1, x_2, y_3, z_1, z_2\}$ with equality if and only if x is of type 4 (see Figure 2).

Case 2: $|\{x_1, x_2\} \cap J| = 1$.
Without loss of generality, suppose that $x_1 \in I$ and $x_2 \in J$. Clearly, $y_1, y_2 \in J$. Then, $y_3 \in I$. If $x_3 \in J$, then, since I is a maximum independent set, either z_1 or z_2 is in I. Without loss of generality, suppose that $z_1 \in I$. Thus, $J(x) \subseteq \{x_2, y_1, y_2, x_3, z_2\}$ with equality if and only if x is of type 5 (see Figure 2). Otherwise, $x_3 \in I$. Here, $J(x) \subseteq \{x_2, y_1, y_2, z_1, z_2\}$ with equality if and only if x is of type 6 (see Figure 3).

Case 3: $x_1, x_2 \in I$.
If $x_3 \in J$, then, without loss of generality, we may suppose that $z_1 \in I$. Thus, $J(x) \subseteq \{x_3, y_1, y_2, y_3, z_2\}$ with equality if and only if x is of type 7 (see Figure 3). Otherwise, $x_3 \in I$. Here, $J(x) \subseteq \{y_1, y_2, y_3, z_1, z_2\}$ with equality if and only if x is of type 8 (see Figure 3). ∎

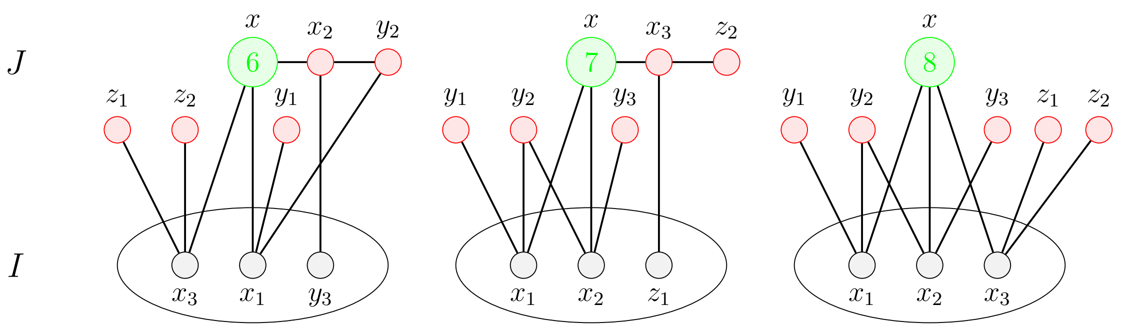

Figure 3: Vertices of type 6, 7, and 8.

Claim 1.4 *If $G^2[J]$ contains a clique of order 6, then each vertex of this clique is of type 8.*

Proof. Suppose $G^2[J]$ contains a clique of order 6, say Q, and let q be an arbitrary vertex from $V(Q)$. Obviously, q is a 3-vertex. By Claim 1.2, girth$(q) = 4$. Now, by

Claim 1.3, q is of type k, for some $k \in \{3, \dots, 8\}$. Moreover, $V(Q) = J(q) \cup \{q\}$. Consider a vertex $x \in V(Q)$ such that x is of type i and i is minimum.

Case 1: $i = 3$.
Here, $J(x) = \{x_1, x_2, y_2, z_1, z_2\}$. Thus, x_1 (respectively x_2, y_2) has a common neighbor, in I, with z_1 and z_2. Then, y_1 (respectively y_3) is adjacent to z_1 and z_2. In addition, y_1 and y_3 are distinct since otherwise $(I \setminus \{y_1\}) \cup \{x_1, x_2\}$ is an independent set of cardinality greater than that of I, a contradiction. Now, since each of y_1, y_3, and x_3 has three neighbors, y_2 should be adjacent to a vertex $u \in I \setminus \{y_1, y_3, x_3\}$ which is adjacent to z_1 and z_2. Thus, $d_G(z_1) \geq |\{u, y_1, y_3, x_3\}| = 4$, a contradiction.

Case 2: $i = 4$.
In this case $J(x) = \{x_1, x_2, y_3, z_1, z_2\}$. Since $\{z_1, z_2\} \subseteq J(x_2)$, then $\{z_1, z_2\} \subseteq N(y_2) \cup N(y_3)$. Moreover, y_2 is adjacent to at most one vertex among z_1 and z_2. In addition, as y_3 has a neighbor in I, it is adjacent to at most one vertex among z_1 and z_2. Then, without loss of generality, we may suppose that $y_2z_2, y_3z_1 \in E(G)$. Moreover, $y_1y_3, y_1z_1 \in E(G)$ since $\{y_3, z_1\} \subseteq J(x_1)$. Thus, girth$(y_3) = 3$, which contradicts Claim 1.2.

Case 3: $i \in \{5, 6, 7\}$.
If $i = 5$, then $V(Q) = \{x_2, x_3, y_1, y_2, z_2\}$. Note that $x_2 \in V(Q)$ and x_2 is of type 4, contradicting the minimality of i. If $i = 6$, then $V(Q) = \{x_2, y_1, y_2, z_1, z_2\}$. Note that $x_2 \in V(Q)$ and x_2 is of type 4, contradicting the minimality of i. If $i = 7$, then $V(Q) = \{x_3, y_1, y_2, y_3, z_2\}$. Note that x_3 is of type j, for some $j \in \{3, 4, 5\}$, contradicting the minimality of i. ■

Claim 1.4 implies that every vertex of a clique of order 6 in $G^2[J]$ should be of type 8. Now, if $G^2[J]$ does not contain a clique of order 6, then we are done. Otherwise, let Q be a clique of order 6 in $G^2[J]$. By Claim 1.4, every vertex in Q is of type 8. Then, $G = G_{12}$ (see Figure 4). Indeed, let $x \in V(G)$ and let $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2$ be the vertices defined as represented in Figure 3. As y_2 is of type 8, the third neighbor u_1 of y_2 belongs to I and is adjacent to z_1 and z_2. Again, as z_1 is of type 8, the third neighbor u_2 of z_1 belongs in I and is adjacent to y_1 and y_3. Similarly, the third neighbor u_3 of z_2 belongs to I and is adjacent to y_1 and y_3. Hence, $G = G_{12}$ and the result follows. ■

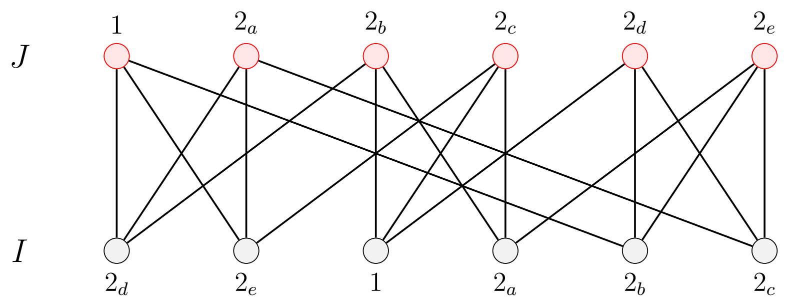

Figure 4: Graph G_{12} and its $(1, 2, 2, 2, 2, 2)$-packing coloring.

3 Conclusion and Open Problems

It is obvious that every 3-vertex in the Petersen graph has local girth 5. Moreover, the Petersen graph is not $(1, 2, 2, 2, 2, 2)$-packing colorable. This shows that the condition on the local girth cannot be relaxed in general.

A $(1, 2, 2, 2, 2, 2)$-packing coloring of the graph G_{12} cannot be obtained by assigning the color 1 to the vertices of a maximum independent set. Indeed, I and J represent the only two maximum independent sets in G_{12}. In fact, any pair of vertices in I (respectively J) do not have three common neighbors in J (respectively I); it follows that any independent set containing vertices from both I and J cannot be of cardinality greater than 4. Moreover, any pair of vertices in I (respectively J) are at a distance 2 from each other; this implies that neither of I and J can be partitioned into five 2-packings.

On the other hand, one can obtain a $(1, 2, 2, 2, 2)$-packing coloring of G_{12} by replacing the color 2_a by 1 in Figure 4. In this case, also, every maximum independent set contains a vertex colored by a 2-color. It seems that the studied class of subcubic graphs is $(1, 2, 2, 2, 2)$-packing colorable but the proof cannot be obtained by coloring the vertices of a maximum independent set by the color 1 only. This limitation is closely related to the method used in the proof of Theorem 1. Indeed, our approach relies on assigning color 1 to a maximum independent set and then coloring the remaining vertices via a coloring of the square of the induced subgraph. While this yields a $(1, 2, 2, 2, 2, 2)$-packing coloring, it does not seem sufficiently flexible to reduce the number of 2-colors to four. The graph G_{12} illustrates this obstruction, as it admits a $(1, 2, 2, 2, 2)$-packing coloring, but not one compatible with this strategy. This suggests that a different approach may be required to settle the problem.

Problem 1 *Is every subcubic graph such that every* 3*-vertex has local girth at most* 4 $(1, 2, 2, 2, 2)$*-packing colorable?* $(1, 1, 2, 3)$*-packing colorable?*

The result in this paper, together with some earlier results, suggests that the question posed by Gastineau and Togni [8] about the $(1, 2, 2, 2, 2, 2)$-packing colorability of every subcubic graph, with the exception of the Petersen one, has an affirmative answer.

Conjecture 1 *Every subcubic graph, except the Petersen one, is* $(1, 2, 2, 2, 2, 2)$*-packing colorable.*

Acknowledgments

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References

[1] J. Balogh, A. Kostochka and X. Liu, Packing Chromatic Number of Subdivisions of Cubic Graphs, *Graphs Combin.* 35(2) (2019), 513–537.

[2] B. Brešar, J. Ferme, S. Klavž and D. F. Rall, A survey on packing colorings, *Discuss. Math. Graph Theory* 40(4) (2020), 923–970.

[3] B. Brešar, N. Gastineau and O. Togni, Packing colorings of subcubic outerplanar graphs, *Aequationes Math.* 94(5) (2020), 945–967.

[4] B. Brešar, S. Klavžar, D. F. Rall and K.Wash, Packing chromatic number, $(1, 1, 2, 2)$-colorings, and characterizing the Petersen graph, *Aequationes Math.* 91 (2017), 169–184.

[5] B. Brešar, K. Kuenze, and D. 'F. Rall, i Claw-free cubic graphs are $(1, 1, 2, 2)$-colorable, *Discrete Math.* 348(8) (2025), 114477.

[6] A. El Zein and M. Mortada, Advances on the packing coloring conjectures of subcubic graphs, `https://arxiv.org/abs/2503.20239`.

[7] A. El Zein and M. Mortada, On the $(1, 2^k)$-Packing Coloring of Subcubic Graphs, (submitted).

[8] N. Gastineau and O. Togni, S-packing colorings of cubic graphs, *Discrete Math.* 339 (2016), 2461–2470.

[9] W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, J. M. Harris and D. F. Rall, Broadcast chromatic numbers of graphs, *Ars Combin.* 86 (2008), 33–49.

[10] A. Kostochka and X. Liu, Packing $(1, 1, 2, 4)$-coloring of subcubic outerplanar graphs, *Discrete Appl. Math.* 302 (2021), 8–15.

[11] R. Liu, X. Liu, M. Rolek and G. Yu, Packing $(1, 1, 2, 2)$-coloring of some subcubic graphs, *Discrete Appl. Math.* 283 (2020), 626–630.

[12] X. Liu, X. Zhang, and Y. Zhang, Every subcubic graph is packing $(1, 1, 2, 2, 3)$-colorable, *Discrete Math.* 348 (11) (2025), 114610.

[13] M. Mortada and A. El Zein, A short proof that every claw-free cubic graph is $(1, 1, 2, 2)$-packing colorable, *Discrete Appl. Math.* 391 (2026), 24–27.

[14] M. Mortada and A. El Zein, On the $(1, 1, 2, 3)$-packing coloring of some subcubic graphs, *Discrete Appl. Math.* 388 (2026), 96–101.

[15] M. Mortada and O. Togni, About S-Packing Coloring of Subcubic Graphs, *Discrete Math.* 347(5) (2024), 113917.

[16] M. Mortada and O. Togni, On S-packing coloring of 2-saturated subcubic graphs, *Austral. J. Combin.* 90(2) (2024), 155–167.

[17] M. Mortada, About S-Packing Coloring of 3-Irregular Subcubic Graphs, *Discrete Appl. Math.* 359 (2024), 16–18.

[18] M. Mortada and O. Togni, Further results and questions on S-packing coloring of subcubic graphs, *Discrete Math.* 348(4) (2025), 114376.

[19] B. Tarhini and O. Togni, S-Packing Coloring of Cubic Halin Graphs, *Discrete Appl. Math.* 349(31) (2024), 53–58.

[20] W. Yang and B. Wu, On packing S-colorings of subcubic graphs, *Discrete Appl. Math.* 334(2023), 1–14.

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