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Watchman's walk on complete multipartite digraphs

Sergiy Kozerenko*

Kyiv School of Economics
Mykoly Shpaka str. 3, 03113 Kyiv
Ukraine
kozerenkosergiy@ukr.net

Bohdan Romanchuk

Ukrainian Catholic University
Kozelnytska str. 2A, 79026 Lviv
Ukraine

Abstract

A generalized orientation of a graph is the one where both orientations of edges are allowed at the same time. We prove that complete multipartite graphs are characterized by the following property: any generalized orientation without sources has a watchman's walk. We also characterize graphs satisfying this and a related property in the case of simple orientations. Our main result establishes an upper bound for the watchman's number $w(D)$ for complete multipartite digraphs D without sources: $w(D) \leq \gamma(D) + 2\alpha(D)$, where $\gamma(D)$ is the domination number and $\alpha(D)$ is the independence number of D. This result extends the upper bound previously established for tournaments by Dyer, Howell and Pittman in 2021.

1 Introduction

The watchman's walk problem, introduced in [8], is a variation of the classical domination problem in graphs. While the classical domination problem seeks a minimum dominating set in a graph, the watchman's walk problem involves finding a closed walk for a single guard (watchman) to monitor all rooms in a museum. Formally, the rooms in a museum can be represented as vertices in a graph with the obvious adjacency relation. Hence, the watchman's walk on a graph G is defined as a closed dominating walk (meaning that every vertex in G is either visited by the watchman

* Corresponding author.

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or adjacent to a visited vertex) of minimum possible length. It is evident that any connected graph has such a walk. The goal, of course, is to find its length, known as the watchman's number $w(G)$ of the graph G.

Research on the watchman's walk problem has primarily concentrated on undirected graphs [1, 2, 3, 6], although there have been extensions to directed graphs. Namely, the notions of a watchman's walk and the watchman's number of a digraph were introduced in [5]. One obvious and key difference compared to the undirected case is that not every digraph possesses a watchman's walk, as the vertex set of such a walk must induce a strong subdigraph. Nonetheless, in [5], it was proved that every simple orientation of a complete multipartite graph without sources has a watchman's walk. They also obtained an upper bound on the watchman's number $w(D)$ for tournaments D and investigated tournaments and digraphs with small watchman's numbers. In [4], the authors dealt with de Bruijn graphs, including an explicit calculation of the watchman's number for the de Bruijn graph of a de Bruijn sequence of a given order and alphabet size.

This paper extends the results from [5]. We demonstrate that complete multipartite graphs are precisely those graphs in which every generalized orientation without sources has a watchman's walk (see Theorem 3.1). We also examine the same conditions for simple orientations (see Theorems 3.3 and 3.4). Additionally, we establish an upper bound for the watchman's walk in complete multipartite digraphs without sources (see Theorems 3.7 and 3.8).

2 Definitions and preliminary results

2.1 Undirected graphs

In this paper, *graphs* will refer to finite simple undirected graphs. For convenience, the edge between two vertices u, v will be denoted as uv instead of $\{u, v\}$. Two vertices $u, v \in V(G)$ in a graph G are *adjacent* provided $uv \in E(G)$. Similarly, two distinct edges e_1, e_2 are *adjacent* if they share a common vertex. For a set $S \subseteq V(G)$, by $E(S)$ we denote all the edges in G with endvertices from S. The *neighborhood* of a vertex $u \in V(G)$ is the set $N_G(u) = \{x \in V(G) : ux \in E(G)\}$, and the *closed neighborhood* of u is defined as $N_G[u] = N_G(u) \cup \{u\}$.

A graph is *complete* provided any two of its vertices are adjacent. A graph G is *complete multipartite* if its vertex set can be partitioned into several sets $V(G) = V_1 \sqcup \cdots \sqcup V_m$ such that $uv \in E(G)$ if and only if u and v belong to different sets V_i, V_j.

For a graph G and a subset $S \subseteq V(G)$, by $G[S]$ we will denote the corresponding *induced subgraph*. Also, we put $G - S = G[V(G) \setminus S]$ for the subgraph of G which is obtained after the deletion of the vertices from S. Similarly, $G - E'$ denotes the subgraph of G obtained by deleting the edges in $E' \subseteq E(G)$.

A *walk* in a graph is a finite sequence of vertices $u_1, \ldots, u_m$ such that $u_i u_{i+1} \in E(G)$ for $1 \leq i \leq m-1$. A graph is *connected* if every pair of its vertices is joined by a walk. A set of vertices $S \subseteq V(G)$ in a graph G is called *connected* provided $G[S]$ is connected. The vertex set $V(G)$ of any connected graph is endowed with the natural metric d_G, where $d_G(u,v)$ equals the length of a shortest walk between $u, v \in V(G)$. For a subset $S \subseteq V(G)$ and a vertex $u \in V(G)$, the distance from u to S is defined as $d_G(u,S) = \min\{d_G(u,x) : x \in S\}$.

Let $k \in \mathbb{N}$. A connected graph G is called *k-edge-connected* if G remains connected after the deletion of any set of fewer than k edges. In other words, for any $E' \subseteq E(G)$ with $|E'| < k$, the subgraph $G - E'$ is connected. Similarly, a set $S \subseteq V(G)$ is *k-edge-connected* if the induced subgraph $G[S]$ is k-edge-connected.

A set of vertices $A \subseteq V(G)$ is *dominating* if every $x \in V(G) \setminus A$ is adjacent to some vertex $a \in A$. The following folklore result is useful when dealing with complete multipartite graphs.

Lemma 2.1. *A graph is complete multipartite if and only if every edge is dominating.*

A *watchman's walk* on a graph G is a minimum closed walk W such that its vertex set $V(W)$ is dominating in G (see [8]). It is clear that any connected graph has a watchman's walk: indeed, taking a spanning tree of G and performing a depth-first traversal yields a closed walk whose vertex set coincides with $V(G)$, and hence is dominating.

The *watchman's number* $w(G)$ of a graph G is the length (i.e., the number of edges) of a watchman's walk in G. The number $w(G)$ has been studied for classes of graphs such as trees [1], block intersection graphs of Steiner triple systems [2], and several graph products [3]. Also, a time-constrained variant of the watchman's walk problem, where the length of the walk is bounded by a given time budget, was studied in [6].

2.2 Directed graphs

All *digraphs* in this paper are simple, finite, and without loops (but may include 2-cycles). The existence of an arc $(u,v) \in A(D)$ in a digraph D will also be denoted as $u \to v$.

For a digraph D, by $[D]$ we denote the corresponding undirected graph with $V([D]) = V(D)$ and $E([D]) = \{uv : u \to v \text{ or } v \to u \text{ in } D\}$. Clearly, $[D]$ is obtained from D by ignoring all the orientations of arcs and 2-cycles.

A digraph D is called a *generalized orientation* of a graph G provided $[D] = G$. A digraph D is called a *simple orientation* (or just an *orientation*) of G if $[D] = G$ and D has no 2-cycles. In a simple orientation, the edges of G can be oriented in only one direction, whereas in a generalized orientation, edges can be oriented in both directions, forming 2-cycles.

For a subset of vertices $\Gamma \subseteq V(D)$ in a digraph D, by $D[\Gamma]$ we denote the subdigraph in D induced by Γ.

For a vertex $u \in V(D)$ in a digraph D, by $N_D^+(u) = \{v \in V(D) : u \to v \text{ in } D\}$ we denote *out-neighborhood* of u. We also define $N_D^+[u] = N_D^+(u) \cup \{u\}$ and $N_D^+[\Gamma] = \cup_{u\in\Gamma} N_D^+[u]$ for a subset $\Gamma \subseteq V(D)$.

A *walk* in a digraph D is a finite sequence of vertices $u_1, \ldots, u_m \in V(D)$ with $(u_i, u_{i+1}) \in A(D)$ for all $1 \leq i \leq m-1$. A *path* in a digraph is a walk where all vertices are distinct. A *Hamiltonian path* in a digraph D is a path that includes every vertex of D.

A *tournament* is a simple orientation of a complete graph, while a *semi-complete digraph* is a generalized orientation of a complete graph. The following basic result regarding the Hamiltonicity of tournaments is known as Redei's theorem.

Theorem 2.2. [7, p. 206, Theorem 16.10] *Every tournament has a Hamiltonian path.*

Note that this theorem also implies that every semi-complete digraph has a Hamiltonian path.

A digraph D is *weak* if $[D]$ is connected. A *weak component* in a digraph is its maximal weak subdigraph. A digraph D is *strong* if every pair of its vertices can be joined by a walk. A *strong component* in a digraph is its maximal strong subdigraph. The following lemma provides a standard characterization of strong digraphs.

Lemma 2.3. [7, p. 99, Theorem 16.1] *A digraph D is strong if and only if D has a closed walk that contains all of its vertices.*

It is clear that a graph G has a strong generalized orientation if and only if G is connected. The case of simple orientations is addressed by the well-known Robbins' theorem.

Theorem 2.4. [7, p. 210, Exercise 16.21] *A graph G has a strong simple orientation if and only if G is 2-edge-connected.*

The *condensation* of a digraph D is a digraph D^* whose vertices are the vertex sets of strong components in D, with the existence of an arc $X \to Y$ in case $x \to y$ in D for some $x \in X$ and $y \in Y$. It is easy to see that the condensation is always an acyclic digraph.

Since we will be dealing with dominating walks in digraphs, we introduce the following refinement of the condensation construction. The *surjective condensation* of a digraph D is a digraph D^*_{sur} with the same vertex set as D^*, but with arcs of the form $X \to Y$ only for pairs X, Y in which X *dominates* Y (meaning that for every $y \in Y$, there is $x \in X$ with $x \to y$ in D). It is clear that D^*_{sur} is a subdigraph of D^*.

A set of vertices $\Gamma \subseteq V(D)$ in a digraph D is *dominating* provided for every $v \in V(D) \setminus \Gamma$, there is $u \in \Gamma$ with $u \to v$ in D. A vertex $u \in V(D)$ is *universal* if the singleton $\{u\}$ is dominating. The *domination number* $\gamma(D)$ of a digraph D is the cardinality of a smallest dominating set in D. A set of vertices $\Gamma \subseteq V(D)$ is called *minimum dominating* if $|\Gamma| = \gamma(D)$.

A set $I \subseteq V(D)$ is *independent* if there are no arcs between the vertices of I. The *independence number* $\alpha(D)$ of a digraph D is the cardinality of the largest independent set in D.

In [5], the notion of a watchman's walk was extended to digraphs. Specifically, a *watchman's walk* in a digraph D is a minimum closed dominating walk. It can be easily seen that not every digraph has a watchman's walk.

The following basic result is new and justifies the definition of the surjective condensation.

Lemma 2.5. *A digraph D has a watchman's walk if and only if its surjective condensation D^*_{sur} has a universal vertex.*

Proof. If D has a watchman's walk W, then, being a closed walk, W lies within some strong component C of D (see Lemma 2.3). It is clear that $V(C)$ is a universal vertex in D^*_{sur}. Conversely, if there is a strong component C of D which corresponds to a universal vertex in D^*_{sur}, then by Lemma 2.3, there is a closed walk W in C which visits every vertex of C. Clearly, W is a dominating walk in D. □

If D has a watchman's walk, its length is denoted by $w(D)$, and is called the *watchman's number* of D. The watchman's problem and the number $w(D)$ were initially studied in [5] for general digraphs, tournaments, and orientations of complete multipartite graphs. In [4] watchman's walks were also considered for the class of de Bruijn graphs.

3 Main results

3.1 The existence of a watchman's walk for several classes of (generalized) orientations

It was proven in [5] that every simple orientation of a complete multipartite graph without sources has a watchman's walk. We show that the same result holds for generalized orientations, providing a nice characterization of complete multipartite graphs.

Theorem 3.1. *Let G be a connected graph. The following conditions are equivalent:*

1. *G is a complete multipartite graph;*
2. *every generalized orientation of G without sources has a watchman's walk;*
3. *every generalized orientation of G without singleton strong components has a watchman's walk.*

Proof. 1 ⇒ 2: Let G be a complete multipartite graph and D be a generalized orientation of G without sources. Consider a condensation D^* and let $S \subseteq V(D)$ denote its source. If $|S| = 1$, then D contains a source, which contradicts our assumption. Hence, $|S| \geq 2$. Since S induces a strong component in D, the set S is connected in G. In particular, there exists an edge $uv \in E(S)$. Since G is complete multipartite, the set $\{u, v\}$ is dominating by Lemma 2.1. Thus, for every $x \in V(G) \setminus S$ we have $x \in N_G(u) \cup N_G(v)$. Let us fix such a vertex $x \in V(G) \setminus S$. Without loss of generality, assume $xu \in E(G)$. In D, this edge can only be oriented as $u \to x$ (since u is in the source S in D^*). Therefore, S is also a universal vertex in D^*_{sur}. By Lemma 2.5, D has a watchman's walk.

2 ⇒ 3: This implication is straightforward because every source in a digraph induces a singleton strong component.

3 ⇒ 1: To the contrary, assume G is not a complete multipartite graph. Then by Lemma 2.1, there exists a non-dominating edge $uv \in E(G)$. This means there is a vertex $x \in V(G)$ with $x \notin N_G[u] \cup N_G[v]$.

We now construct a generalized orientation of G without singleton strong components that does not have a watchman's walk. To do this, we partition the edge set $E(G)$ into three classes:

$$E_0 = \{uv\} \cup \{e \in E(G) : e \text{ is not adjacent to } uv\},$$
$$E_1 = \{ux \in E(G) : N_G(x) \subseteq \{u, v\}\} \cup \{vy \in E(G) : N_G(y) \subseteq \{u, v\}\},$$
$$E_2 = E(G) \setminus (E_0 \cup E_1).$$

We orient all edges from $E_0 \cup E_1$ in both directions, and the edges $ux, vy \in E_2$ will be oriented as $u \to x$, $v \to y$.

The obtained generalized orientation D of G does not have singleton strong components. This is because each vertex in G is incident to at least one edge oriented in both ways. Moreover, D has no watchman's walk due to the following facts:

- the source in D^*_{sur} is precisely the set $S = \{u, v\} \cup \{z \in N_G(u) \cup N_G(v) : N_G(z) \subseteq \{u, v\}\}$;
- the set $\{u, v\}$ is not dominating in G.

This makes the set S non-dominating in G. Thus, by Lemma 2.5, D does not have a watchman's walk. The obtained contradiction proves the theorem. □

Example 3.2. We illustrate the construction of the generalized orientation in the proof of the implication 3 ⇒ 1 from Theorem 3.1. Consider a graph G with $V(G) = \{1, \ldots, 7, u, v\}$ and $E(G) = \{1v, 2u, 3u, 3v, 4u, 47, 5v, 56, 67, uv\}$. Then the edge $uv \in E(G)$ is non-dominating in G. The corresponding edge set $E_2 = \{4u, 5v\}$. Thus, all the other edges are oriented in both directions, and the edges $4u$, $5v$ are oriented as $u \to 4$ and $v \to 5$, respectively (see Figure 1). It is clear that this generalized orientation D has no watchman's walk.

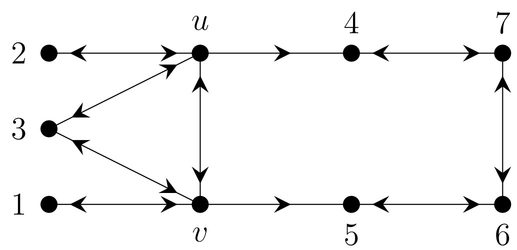

Figure 1: The non-dominating edge uv induces a generalized orientation without watchman's walk.

We now turn our attention back to the simple orientations and examine the relationships between the conditions in Theorem 3.1 for them. In this context, we observe an almost trivial chain of implications: 1 ⇒ 2 ⇒ 3. However, these two implications are strict. For example, the path P_4 does not allow orientations without sources; hence, the second condition from Theorem 3.1 reformulated for simple orientations is vacuously true for P_4. But clearly, P_4 is not a complete multipartite graph.

Similarly, consider the graph G depicted in the first part of Figure 2. This graph does not allow orientations without singleton strong components, as the vertex x always ends up in such a component. Thus, the third condition from Theorem 3.1 for simple orientations is vacuously true for G. However, G can be oriented in such a way that it has no sources but still lacks a watchman's walk (see the second part of Figure 2). The following result highlights the difference between the second condition as formulated for simple orientations and its counterpart for generalized orientations.

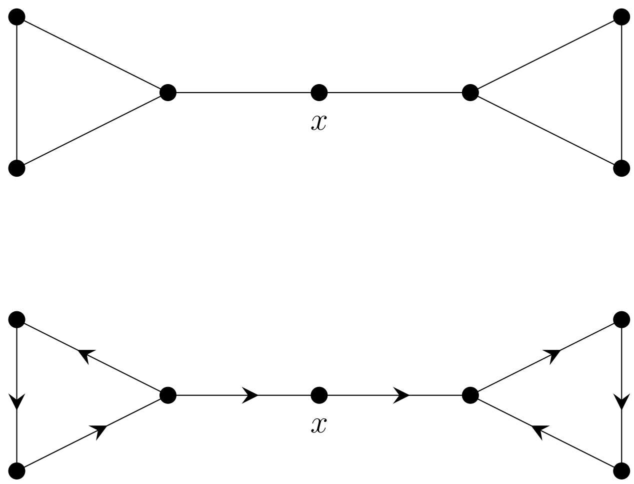

Figure 2: A graph and its simple orientation without sources that does not have a watchman's walk.

Theorem 3.3. *Let G be a connected graph. The following conditions are equivalent:*

1. *the vertex set of any cycle in G is dominating;*
2. *every orientation of G without sources has a watchman's walk;*
3. *every 2-edge-connected non-singleton set of vertices in G is dominating.*

Proof. $1 \Rightarrow 2$: We draw inspiration from the proof of this implication in Theorem 3.1. Namely, let D be an orientation of G without sources. Fix a source $S \subseteq V(G)$ in the condensation D^*. Then $|S| \geq 3$; otherwise, D would clearly contain a source. Further, since $|S| \geq 3$ and S induces a strong subdigraph in D, this subdigraph must contain a cycle C. Moreover, as D is a simple orientation, C induces a cycle in G as well. By the first condition, $V(C)$ is dominating in G. Hence, S is a dominating set as well. Finally, as S is a source in D^*, it is a universal vertex in D^*_{sur}. By Lemma 2.5, D must have a watchman's walk.

$2 \Rightarrow 3$: To the contrary, assume there is a 2-edge-connected non-singleton set $S \subseteq V(G)$ that is not dominating in G. By Theorem 2.4, the subgraph $G[S]$ admits a strong orientation D'. We extend D' to an orientation D of the whole graph G as follows: any edge $uv \in E(G)$ with $d_G(u, S) \neq d_G(v, S)$ obtains an orientation $u \to v$ provided $d_G(u, S) < d_G(v, S)$ (in this case, of course, $d_G(u, S) = d_G(v, S) - 1$).

Other edges $uv \in E(G)$ (with $u, v \notin S$) can be oriented arbitrarily. The resulting orientation D of G has no sources by construction. Moreover, S is clearly the source in D^*. However, S is not a universal vertex in D^*_{sur} because it is a non-dominating set in G. This contradiction proves that our assumption is false, thus establishing the implication.

3 ⇒ 1: This implication is trivial as the vertex set $V(C)$ of any cycle C in G is 2-edge-connected with $|V(C)| \geq 3$. Therefore, $V(C)$ must be dominating in G. □

The next result provides a criterion for graphs that satisfy the third condition from Theorem 3.1 for simple orientations. We call a vertex u in a graph G *acyclic* if u does not lie on any cycle in G.

Theorem 3.4. *Every orientation of a connected graph G without singleton strong components has a watchman's walk if and only if for each 2-edge-connected non-singleton set $S \subseteq V(G)$, either S is dominating or the subgraph $G - S$ has an acyclic vertex.*

Proof. **Necessity.** Suppose, to the contrary, that there exists a 2-edge-connected non-singleton set $S \subseteq V(G)$ that is not dominating in G, and $G - S$ has no acyclic vertices. We consider the subgraph $G - S$ and partition its vertex set $V(G) \setminus S$ into maximal 2-edge-connected subsets. By assumption, each $x \in V(G) \setminus S$ lies in such a subset of cardinality at least 3. By Robbins' theorem, there are strong orientations of S and all of the maximal 2-edge-connected subsets in $G - S$. We orient the edges within these sets according to these strong orientations. Further, edges of the form $ab \in E(G)$, where $a \in S$ and $b \notin S$, will obtain orientations $a \to b$. The other edges (between different maximal 2-edge-connected subsets in $G - S$) can be oriented arbitrarily. The resulting simple orientation D of G has no singleton strong components, and a source S in D^*_{sur} is non-universal. This means that D has no watchman's walk, which is a contradiction.

Sufficiency. Let D be an orientation of G without singleton strong components, and let $S \subseteq V(G)$ be a source in D^*. Because D is a simple orientation, S is a 2-edge-connected set in G. Also, $|S| \geq 3$ because D is a simple orientation of G and does not contain singleton strong components. We aim to show that S is dominating. Thus, let us prove that the subgraph $G - S$ contains no acyclic vertices.

Assume, for the sake of contradiction, that there exists an acyclic vertex $x \in V(G) \setminus S$ in $G - S$. If x is acyclic in the whole graph G, then x would form a singleton strong component in D, contradicting our assumption. Therefore, x must lie on a cycle in G. Moreover, any such cycle must intersect S.

Further, since $\{x\}$ is not a strong component in D, x must lie on a (directed) cycle C in D. Given that D is a simple orientation, C corresponds to an (undirected) cycle C' in G. Since C' intersects S, so does C, implying that x lies in the same strong component as the vertices from S.

This contradiction confirms that S is a dominating set in G. Consequently, S is a universal vertex in D^*_{sur}, implying that D has a watchman's walk. □

Note that the assumption in Theorem 3.4 cannot be weakened by replacing "2-edge-connected non-singleton set" with "cycle". As an example, consider the graph depicted in Figure 3.

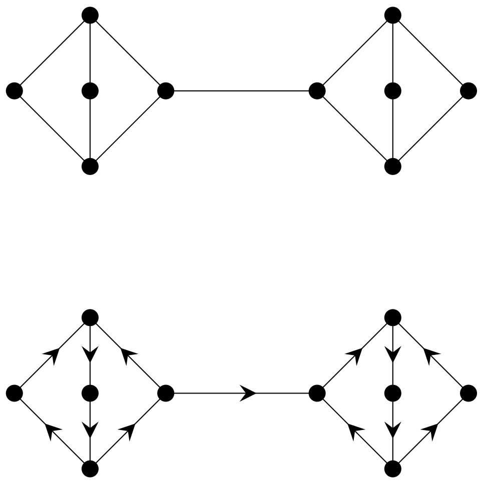

Figure 3: A graph G in which every cycle C has the property that $G - C$ contains an acyclic vertex, and its simple orientation that lacks both singleton strong components and a watchman's walk.

3.2 An upper bound for the watchman's number for complete multipartite digraphs

In this section, we present an upper bound for the watchman's number of complete multipartite digraphs. The corresponding upper bound for tournaments was obtained in [5].

Theorem 3.5. [5] *Let D be a tournament with $n \geq 3$ vertices. Then $w(D) \leq \gamma(D) + 1$.*

As a direct corollary of Theorem 3.5, the inequality $w(D) \leq \gamma(D) + 1$ holds for any semi-complete digraph D.

The following definition naturally arises when dealing with minimum dominating sets in digraphs. Let D be a digraph and $\Gamma \subseteq V(D)$ be a subset of its vertices. The set of *uniquely dominated vertices* by a vertex $v \in \Gamma$ is defined as:

$$U_\Gamma(v) := N_D^+[v] \setminus N_D^+[\Gamma \setminus \{v\}].$$

Lemma 3.6. *Every vertex v in a minimum dominating set Γ has a non-empty set of uniquely dominated vertices $U_\Gamma(v)$.*

Proof. To the contrary, assume that there exists a vertex $v \in \Gamma$ such that $U_\Gamma(v) = \emptyset$. Then, the set $\Gamma \setminus \{v\}$ would be a smaller dominating set. This contradiction establishes the lemma. □

Before proving the main result, we first outline the key ideas in a restricted case similar to tournaments.

Theorem 3.7. *Let D be a complete multipartite digraph without sources. If there exists a minimum dominating set $\Gamma \subseteq V(D)$ that induces a semi-complete subdigraph, then*

$$w(D) \leq \gamma(D) + 2.$$

Proof. Let $\Gamma = \{v_1, v_2, \ldots, v_t\}$, where $t = \gamma(D)$. Then Γ induces a semi-complete subdigraph, implying that there exists a Hamiltonian path in $D[\Gamma]$. We can assume that this Hamiltonian path is $v_1 \to v_2 \to \cdots \to v_t$. Note, that if additionally, $(v_t, v_1) \in A(D)$, then clearly $w(D) = \gamma(D)$.

In the general case, according to Lemma 3.6, the set of uniquely dominated vertices $U_\Gamma(v_t)$ is non-empty. If there is a vertex $y \in U_\Gamma(v_t)$ that lies in a different part than v_1, then $(y, v_1) \in A(D)$. In this case, $w(D) \leq \gamma(D) + 1$ (see Figure 4).

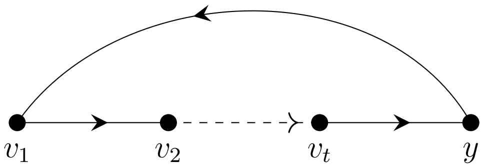

Figure 4: A dominating walk of length $\gamma(T) + 1$.

Let us consider the case when all $y \in U_\Gamma(v_t)$ and v_1 belong to the same part. Since D has no sources, there exists a vertex $x \in V(D)$ with $(x, v_1) \in A(D)$. Clearly, x is adjacent to every vertex $y \in U_\Gamma(v_t)$. If $(v_t, x) \in A(D)$, then we obtain a path of length 2 from v_t to v_1. Similarly, if $(y, x) \in A(D)$ for some $y \in U_\Gamma(v_t)$, then there is a path of length 3 from v_t to v_1. Thus, in these cases, $w(D) \leq \gamma(D) + 2$. Otherwise, assume that $(v_t, x) \notin A(D)$ and $(x, y) \in A(D)$ for all $y \in U_\Gamma(v_t)$. Then $U_\Gamma(v_t) \subseteq N_D^+(x)$, implying that $x \notin \Gamma$. In this case, we can modify the minimum dominating set as follows:

$$\Gamma' := (\Gamma \setminus \{v_t\}) \cup \{x\}.$$

For the new minimum dominating set Γ', the set $U_{\Gamma'}(v_{t-1})$ is also non-empty (see Figure 5).

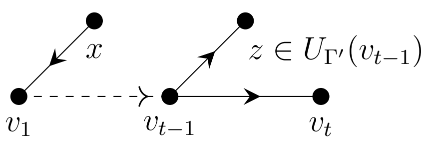

Figure 5: The start of an "oblique" dominating walk.

Since $x \notin \Gamma$ and $(v_t, x) \notin A(D)$, there must be a vertex $w \in \Gamma \cap \Gamma'$ such that $(w, x) \in A(D)$. Therefore, there is a vertex $z \in U_{\Gamma'}(v_{t-1})$ with either $(z, x) \in A(D)$ or $(z, w) \in A(D)$. Thus, we have constructed an "oblique" dominating walk that

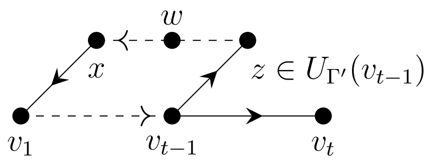

Figure 6: An "oblique" dominating walk of length $\leq \gamma(D)+2$.

preserves all uniquely dominated vertices of v_t. The length of this walk is either $\gamma(D)+1$ or $\gamma(D)+2$, respectively (see Figure 6).

□

We are now ready to state and prove the main result of the paper.

Theorem 3.8. *Let D be a complete multipartite digraph without sources. Then*

$$w(D) \leq \gamma(D) + 2\alpha(D).$$

Proof. Let $\Gamma \subseteq V(D)$ be a minimum dominating set in D. It is clear that Γ induces a complete multipartite digraph in D.

Let m denote the number of vertices in the largest part of $D[\Gamma]$. We have $m = \alpha(D[\Gamma]) \leq \alpha(D)$. Thus, Γ can be decomposed into m disjoint sets, each of which induces a semi-complete subdigraph in $D[\Gamma]$. We will denote these subdigraphs by $T_1, T_2, \ldots, T_m$ and refer to them as *dominating subdigraphs*.

We also assume that the order of the first dominating subdigraph T_1 is equal to the number of parts in $D[\Gamma]$. The order of T_2 is equal to the number of parts in $D\left[\Gamma \setminus V(T_1)\right]$. Similarly, the order of T_k is equal to the number of parts in $D\left[\Gamma \setminus \left(\bigcup_{i=1}^{k-1} V(T_i)\right)\right]$, and so on.

To prove the statement of the theorem, we will construct transitions between the dominating subdigraphs T_i and T_{i+1} for $1 \leq i \leq m-1$, and from T_m to T_1. Here, by a *transition* from T_i to T_{i+1}, we mean a walk W that starts at a vertex of T_i and ends at a vertex of T_{i+1}, traversing almost all vertices of these two dominating subdigraphs such that W either traverses or dominates all vertices in $N_D^+[V(T_i) \cup V(T_{i+1})]$.

The possible transitions are classified as follows:

1. transition from a dominating subdigraph with at least 2 vertices to any other dominating subdigraph;

2. transition between two singleton dominating subdigraphs;

3. transition from a singleton dominating subdigraph to any other dominating subdigraph.

Let us first demonstrate the existence of a transition from a dominating subdigraph T_i with at least 2 vertices to the next dominating subdigraph T_{i+1}. Suppose $|V(T_i)| = t \geq 2$ and $|V(T_{i+1})| = s \geq 1$. Let:

$$V(T_i) = \{u_1, u_2, \ldots, u_t\},$$
$$V(T_{i+1}) = \{v_1, \ldots, v_s\}.$$

Recall that every dominating subdigraph is semi-complete, which implies that it contains a Hamiltonian path. Suppose that $u_1 \to u_2 \to \cdots \to u_t$ and $v_1 \to \cdots \to v_s$ are the Hamiltonian paths in T_i and T_{i+1}, respectively.

We aim to construct a path from u_t to v_1.

By Lemma 3.6, the set of uniquely dominated vertices $U_\Gamma(u_t)$ is non-empty. If there is a vertex $y \in U_\Gamma(u_t)$ that lies in a different part than v_1, then $(y, v_1) \in A(D)$, completing the transition from T_i to T_{i+1} (see Figure 7).

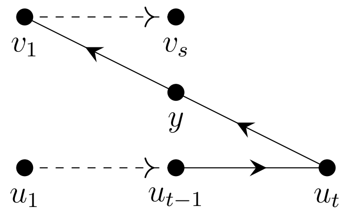

Figure 7: A transition from T_i to T_{i+1}.

Now consider the case when all $y \in U_\Gamma(u_t)$ lie in the same part as v_1. Since D has no sources, there is a vertex $x \in V(D)$ with $(x, v_1) \in A(D)$. If at the same time, $x \in \Gamma$, then for all $y \in U_\Gamma(u_t)$ we have $(y, x) \in A(D)$ (see Figure 8). In this case, we have successfully constructed the path from T_i to T_{i+1}.

Otherwise, if $x \notin \Gamma$ and there exists a vertex $y \in U_\Gamma(u_t)$ such that $(y, x) \in A(D)$, then we again obtain a path from T_i to T_{i+1} (see Figure 8).

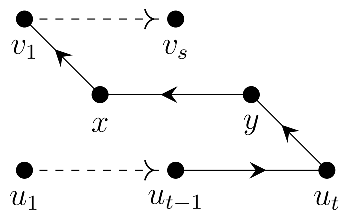

Figure 8: A transition from T_i to T_{i+1}.

Now, assume that $x \notin \Gamma$ and for all $y \in U_\Gamma(u_t)$, it holds that $(y, x) \notin A(D)$. Since x and each $y \in U_\Gamma(u_t)$ lie in different parts, we have $(x, y) \in A(D)$. Thus, $U_\Gamma(u_t) \subseteq N_D^+(x)$, and we can modify the minimum dominating set:

$$\Gamma' = (\Gamma \setminus \{u_t\}) \cup \{x\}.$$

For the new minimum dominating set Γ', the set $U_{\Gamma'}(u_{t-1})$ is non-empty (see Figure 9). Thus, we can select a vertex $z \in U_{\Gamma'}(u_{t-1})$.

Since $x \notin \Gamma$ and $(u_t, x) \notin A(D)$, there exists a vertex $w \in \Gamma \cap \Gamma'$ that dominates x, meaning $(w, x) \in A(D)$. Given that both w and x are in the new minimum dominating set Γ', there must be at least one of the following arcs: (z, w) or (z, x). Therefore, we have successfully constructed a transition from T_i to T_{i+1} (see Figure 10).

Thus, we have demonstrated the existence of a path from a dominating subdigraph with at least 2 vertices to any other dominating subdigraph.

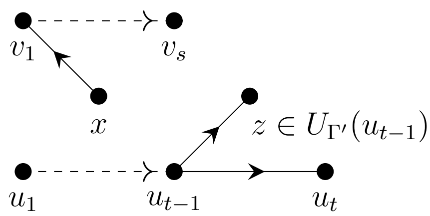

Figure 9: The construction of the new minimum dominating set Γ'.

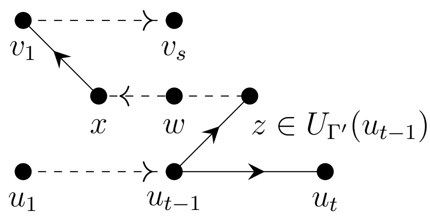

Figure 10: A path from T_i to T_{i+1}.

Next, let us construct the transition between two singleton dominating subdigraphs. Assume $V(T_i) = \{u\}$ and $V(T_{i+1}) = \{v\}$. By construction, u and v lie in the same part. Hence, if there exists a vertex $y \in U_\Gamma(u) \setminus \{u\}$, then $(y, v) \in A(D)$. In this case, we have successfully constructed a path from T_i to T_{i+1} (see Figure 11).

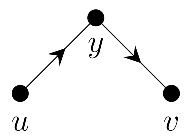

Figure 11: A path from T_i to T_{i+1}.

Now consider the case when $U_\Gamma(u) = \{u\}$. This means that there is no arc of the form (a, u) for any $a \in \Gamma$ as the existence of such an arc would imply that $\Gamma \setminus \{u\}$ is a smaller dominating set.

Since D has no sources, there is a vertex $x \in V(D)$ with $(x, v) \in A(D)$. Recalling that u and v lie in a common part, we can conclude that x and u are adjacent. If $(u, x) \in A(D)$, then we have constructed the desired path from T_i to T_{i+1}. Thus let, $(x, u) \in A(D)$. Then $x \notin \Gamma$. This means that there is $w \in \Gamma$ with $(w, x) \in A(D)$ (see Figure 12). If u and w belong to different parts, then $(u, w) \in A(D)$. This ensures that we have constructed the path from T_i to T_{i+1}.

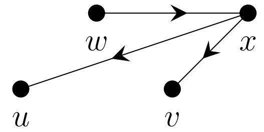

Figure 12: The end of a path from T_i to T_{i+1}.

Otherwise, assume u and w lie in the same part. In this case, we can rebuild the minimum dominating set as follows:

$$\Gamma' = (\Gamma \setminus \{u\}) \cup \{x\}.$$

If T_i was the first dominating subdigraph, then we have constructed a path from x to v that dominates the set $U_\Gamma(u) = \{u\}$. Otherwise, consider the last dominating vertex t from which we had previously constructed a path to T_i (see Figure 13).

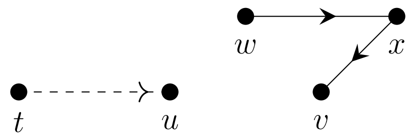

Figure 13: A path from the last dominating vertex t to T_i.

In the new minimum dominating set Γ', the set $U_{\Gamma'}(t)$ is non-empty. Thus, there exists a vertex $z \in U_{\Gamma'}(t)$ such that either $(z, w) \in A(D)$ or $(z, x) \in A(D)$. Consequently, we have constructed an "oblique" path from T_{i-1} to T_{i+1}, while preserving all uniquely dominated vertices of T_i in Γ (see Figure 14).

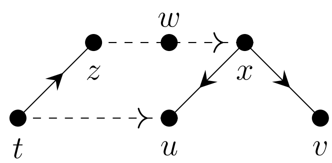

Figure 14: An "oblique" path from T_{i-1} to T_{i+1}.

Therefore, we have shown the existence of a path between two singleton dominating subdigraphs. The next task is to construct a transition from a singleton dominating subdigraph to any other dominating subdigraph. Let

$$V(T_i) = \{u\},$$
$$V(T_{i+1}) = \{v_1, ..., v_s\},$$

where $s = |V(T_{i+1})| \geq 2$. If u, and v_1 lie in the same part, then the construction of the path from T_i to T_{i+1} is similar to the case of two singleton subdigraphs.

Thus, we consider the situation when u and v_1 belong to different parts.

If $u \in U_\Gamma(u)$, then $(u, v_1) \in A(D)$, and the construction is finished. Now let us consider the case when $u \notin U_\Gamma(u)$. If there exists a vertex $y \in U_\Gamma(u)$ such that y and v_1 belong to different parts, then $(y, v_1) \in A(D)$, and the construction of a transition is finished.

Now suppose that for all $y \in U_\Gamma(u)$, the vertices y and v_1 lie in the same part. Since D has no sources, there is a vertex $x \in V(D)$ that has an outgoing arc to v_1. If $x \in \Gamma$, then $(y, x) \in A(D)$ for any $y \in U_\Gamma(u)$. This allows to construct the path from T_i to T_{i+1}.

Otherwise, if $x \notin \Gamma$ and there is $y \in U_\Gamma(u)$ with $(y, x) \in A(D)$, then we have completed the desired path. Further, if for all $y \in U_\Gamma(u)$ we have $(x, y) \in A(D)$, then we can again rebuild the minimum dominating set:

$$\Gamma' = (\Gamma \setminus \{u\}) \cup \{x\}.$$

Since $x \notin \Gamma$ and $(u, x) \notin A(D)$, there exists a vertex $w \in \Gamma \cap \Gamma'$ with $(w, x) \in A(D)$ (see Figure 15).

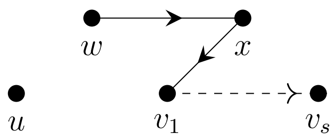

Figure 15: The end of a path from T_i to T_{i+1}.

Let us now consider the last dominating vertex t, from which we have constructed a path to u. Since x and w belong to different parts, there exists a vertex $z \in U_{\Gamma'}(t)$ such that $(z, w) \in A(D)$ or $(z, x) \in A(D)$. This finishes the "oblique" path from T_{i-1} to T_{i+1} while preserving all uniquely dominated vertices of T_i in Γ (see Figure 16).

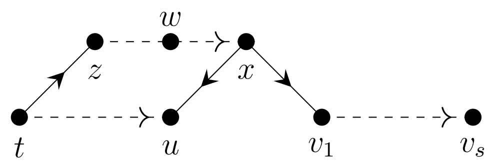

Figure 16: An "oblique" path from T_{i-1} to T_{i+1}.

Thus, we have demonstrated all possible constructions of paths between dominating subdigraphs. To determine the upper bound on the watchman's number on D, we observe that there are m transitions between dominating subdigraphs. The sum of the lengths of their Hamiltonian paths is equal to $\gamma(D) - m$. It is straightforward to check that the largest length of a path (or an "oblique" path) between any two dominating subdigraphs is 3. Consequently, the total length of a watchman's walk in D is at most $\gamma(D) - m + 3m = \gamma(D) + 2m$. Hence,

$$w(D) \leq \gamma(D) + 2m = \gamma(D) + 2\alpha(D[\Gamma]) \leq \gamma(D) + 2\alpha(D).$$

□

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