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A Fuss-Catalan generalization of the Fine numbers

Paul Drube

Department of Mathematics and Statistics
Valparaiso University
Valparaiso, Indiana, U.S.A.
`paul.drube@valpo.edu`

Abstract

Fine paths are a popular modification of Dyck paths that are not allowed to have “hills”. Enumerating Fine paths of length $2n$ are the Fine numbers $(F_n)_{n\geq 0}$, a well-studied integer sequence that shares many properties with the Catalan numbers. In this paper, we investigate a two-parameter generalization of the Fine numbers known as the (k,r)-Fine numbers $(F_n^{k,r})_{n\geq 0}$, with the k parameter ($k \geq 2$) mirroring the generalization of the Catalan numbers to the Fuss-Catalan numbers, and the r parameter ($1 \leq r \leq k-1$) corresponding to competing definitions of “hills” in k-Dyck paths. Generating functions are derived for all classes of (k,r)-Fine numbers, and these generating functions are used to generalize many pre-existing results involving the Fine numbers. We also use Riordan arrays to provide an additional combinatorial interpretation of all (k,r)-Fine numbers as certain classes of colored, higher-order Motzkin paths. We close the paper with asymptotic results studying the behavior of the (k,r)-Fine numbers as $n \to \infty$.

1 Background: Fine paths and the Fine numbers

A Dyck path of length $2n$ (or semilength n) is an integer lattice path from $(0,0)$ to $(2n,0)$ that uses step set $\{U=(1,1), D=(1,-1)\}$ and stays weakly above the line $y=0$. For any $n \geq 0$, we denote the set of all Dyck paths of length $2n$ by $\mathcal{D}_n$.

It is well-known that Dyck paths of length $2n$ are enumerated by the Catalan numbers $(C_n)_{n\geq 0}$, with $|\mathcal{D}_n| = C_n$ for all $n \geq 0$. These Catalan numbers satisfy $C_n = \frac{1}{2n+1}\binom{2n+1}{n}$ for all $n \geq 0$ and are among the best-studied sequences in all of combinatorics, with hundreds of combinatorial interpretations appearing in Stanley [19], and on OEIS [16] as A000108. It is well-known that the Catalan numbers have ordinary generating function $C(t) = \frac{1-\sqrt{1-4x}}{2x}$, so that $C_n = [t^n]C(t)$.

The literature includes many well-known modifications of Dyck paths. Of specific interest to this paper are Fine paths, which correspond to all Dyck paths that lack

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“hills”. More precisely, a Fine path of length $2n$ is an element of $\mathcal{D}_n$ that lacks a sub-path of the form UD ending at height $y = 0$. We denote the set of all Fine paths of length $2n$ by $\mathcal{F}_n$. Also observe that the existence of a minimal height sub-path UD always coincides with the presence of a peak at height $y = 1$. See Figure 1 for a basic demonstration of these phenomena.

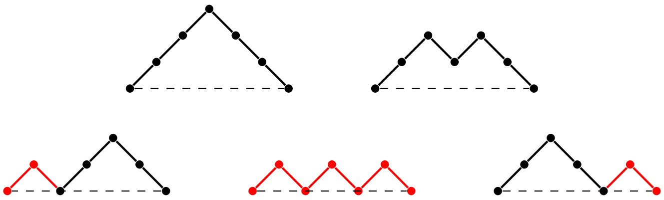

Figure 1: All $C_3 = 5$ Dyck paths in $\mathcal{D}_3$, the top two of which also qualify as Fine paths. Minimal height sub-paths of the form UD (i.e. hills) are highlighted in red.

Fine paths of length $2n$ are enumerated by the Fine numbers $(F_n)_{n\geq 0}$, an integer sequence that begins $1, 0, 1, 2, 6, 18, 57, 186, \ldots$ and appears on OEIS as A000957. Many basic results about Fine numbers were derived as a special case of Dyck path enumerations in Deutsch [8]. Fine numbers later became the subject of in-depth investigations by Deutsch and Shapiro [9] and Cheon, Lee and Shapiro [5]. As many results from those last two papers will be generalized in future sections, we quickly present a number of them below. Here we let $F(t)$ denote the ordinary generating function of the Fine numbers, so that $F_n = [t^n]F(t)$.

$$F(t) \;=\; \frac{1 + 2t - \sqrt{1-4t}}{2t\,(2+t)}, \tag{1}$$

$$F(t) \;=\; \frac{C(t)}{1 + tC(t)} \;=\; \frac{1}{1 - t^2 C(t)^2}, \tag{2}$$

$$1 - (1+2t)F(t) + (t+2t^2)F(t)^2 \;=\; 0, \tag{3}$$

$$F_n \;=\; \sum_{i=0}^{n} (-1)^i \, \frac{i+1}{n+1} \binom{2n-i}{n-i} \quad \text{for all } n \geq 0, \tag{4}$$

$$F_n \;=\; \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{i}{n-i} \binom{2n-2i}{n} \quad \text{for all } n \geq 2, \tag{5}$$

$$F_n \;=\; \sum_{\substack{i_1+\cdots+i_q=n-q,\\ i_1>0,\ldots,i_q>0,\; q>0}} C_{i_1} C_{i_2} \ldots C_{i_q} \quad \text{for all } n \geq 0, \tag{6}$$

$$F_n = (-1)^n \begin{vmatrix} C_0 & 1 & 0 & \dots & 0 \\ C_1 & C_0 & 1 & \dots & 0 \\ C_2 & C_1 & C_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C_n & C_{n-1} & C_{n-2} & \dots & C_0 \end{vmatrix} \quad \text{for all } n \geq 0, \tag{7}$$

$$\begin{vmatrix} F_0 & F_1 & F_2 & \dots & F_n \\ F_1 & F_2 & F_3 & \dots & F_{n+1} \\ F_2 & F_3 & F_4 & \dots & F_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_n & F_{n+1} & F_{n+2} & \dots & F_{2n} \end{vmatrix} = 1, \quad \begin{vmatrix} F_1 & F_2 & F_3 & \dots & F_n \\ F_2 & F_3 & F_4 & \dots & F_{n+1} \\ F_3 & F_4 & F_5 & \dots & F_{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_n & F_{n+1} & F_{n+2} & \dots & F_{2n} \end{vmatrix} = 1-n \text{ for all } n \geq 0, \tag{8}$$

$$\frac{F_n}{C_n} \sim \frac{4}{9} \quad \text{as } n \to \infty, \tag{9}$$

$$F_n \sim \frac{4^{n+1}}{9n\sqrt{n\pi}} \quad \text{as } n \to \infty, \tag{10}$$

For additional, more limited investigations of Fine paths and Fine numbers, see Barcucci et al. [3], Cameron and McLeod [4], DeJager et al. [6], Dershowitz [7], Janjić [14], and Prodinger [17].

Among those papers, of specific interest to what follows are the asymptotic results of Cameron and McLeod [4]. In particular, those authors demonstrate that the expected number of hills in an element of $\mathcal{D}_n$ approaches 1 (with a variance of $\frac{3}{2}$) as $n \to \infty$, and that the probability for an element of $\mathcal{D}_n$ to have an even number of hills approaches $\frac{5}{8}$ as $n \to \infty$. Cameron and McLeod also generalize their asymptotic results to k-Catalan paths in a way that will correspond to our specific case of $r = k - 1$.

1.1 Outline of Results

This paper is structured as follows. Section 2 introduces our Fuss-Catalan generalization of Fine numbers, known as (k, r)-Fine numbers, and then looks to extend all of the non-asymptotic Fine number identities above. This is divided into subsections covering "general" identities, identities that only hold for specific choices of (k, r), and determinant-based identities. Section 3 demonstrates how (k, r)-Fine paths may be recovered via Riordan arrays, allowing such paths to be placed in bijection with certain classes of colored Motzkin paths. In Section 4, we shift our focus from k-Dyck paths that avoid generalized hills to the enumeration of generalized hills in arbitrary k-Dyck paths, once again utilizing proper Riordan arrays to streamline results. Section 5 proceeds with a series of asymptotic results for (k, r)-Fine paths, generalizing the asymptotic results mentioned above. The paper closes with a section highlighting possible directions for future research.

2 (k, r)-Fine paths and the (k, r)-Fine numbers

The primary goal of this paper is to generalize Fine paths from traditional Dyck paths to k-Dyck paths for all $k \geq 2$. Properly done, this will allow us to recover all results from the previous section by setting $k = 2$.

For any $k \geq 2$, a k-Dyck path of length kn (or semilength n) is any integer lattice path from $(0, 0)$ to $(kn, 0)$ that uses step set $\{U = (1, 1), D_{k-1} = (1, 1 - k)\}$ and stays weakly above the line $y = 0$. We denote the set of all k-Dyck paths of length kn by $\mathcal{D}^k_n$. The traditional notion of Dyck paths clearly corresponds to $\mathcal{D}^2_n = \mathcal{D}_n$.

It is well-known that k-Dyck paths of length kn are enumerated by the k-Catalan numbers, which are also known as the Fuss-Catalan numbers. For any $k \geq 2$, we denote the k-Catalan numbers $(C^k_n)_{n\geq 0}$, so that $|\mathcal{D}^k_n| = C^k_n$. The k-Catalan numbers satisfy $C^k_n = \frac{1}{kn+1}\binom{kn+1}{n}$ for all $k \geq 2, n \geq 0$. Fixing $k \geq 2$, we define the generating function of the k-Catalan numbers by $C_k(t)$, so that $C^k_n = [t^n]C_k(t)$. Although a simple closed formula for $C_k(t)$ does not exist when $k \geq 3$, these generating function satisfy $C_k(t) = tC_k(t)^k + 1$ for all $k \geq 2$. For a wealth of information about k-Dyck paths and the k-Catalan numbers, see Hilton and Pedersen [13] or Heubach, Li and Mansour [12].

Before proceeding to our generalization of Fine paths, we require several additional extensions of the k-Catalan numbers. The first of these notions are “partial” k-Dyck paths. A partial k-Dyck path of length $kn + \beta$ and height β is an integer lattice path from $(0, 0)$ to $(kn + \beta, \beta)$ that uses steps $\{U, D_{k-1}\}$ and stays weakly above $y = 0$. We denote the set of all such paths by $\mathcal{D}^k_{n,\beta}$, and define $|\mathcal{D}^k_{n,\beta}| = C^k_{n,\beta}$. For every $k \geq 2, \beta \geq 0$, we then define the generating function $C_{k,\beta}(t)$ by $C^k_{n,\beta} = [t^n]C_{k,\beta}(t)$. Moving forward, when height is specified, we will often supress the word “partial” when discussing partial k-Dyck paths.

In what follows, partial k-Dyck paths will usually appear in conjunction with the following proposition, a more detailed derivation of which may be found in [9].

Proposition 2.1. *For all $\beta \geq 0$, the generating function for k-Dyck paths of height β and length $kn + \beta$ ($n \geq 0$) satisfies*

$$C_{k,\beta}(t) = C_k(t)^{\beta+1}.$$

The other extension of the k-Catalan numbers that will play a role in what follows are the Raney numbers, also known as the two-parameter Fuss-Catalan numbers. As introduced by Hilton and Pedersen [13], for all $k \geq 2, r \geq 1, n \geq 0$ we define the Raney number $R^{k,r}_n = \sum_{i_1+\cdots+i_r=n} C^k_{i_1}C^k_{i_2}\ldots C^k_{i_r}$. This implies $R^{k,r}_n = [t^n]C_k(t)^r$, meaning that the one-parameter k-Catalan numbers may be recovered as $C^k_n = R^{k,1}_n = R^{k,k}_{n-1}$. Also derived in [13] is the closed formula $R^{k,r}_n = \frac{r}{kn+r}\binom{kn+r}{n}$ for all $k \geq 2, r \geq 1, n \geq 0$.

We are now ready for our generalization of Fine paths. The primary challenge here is that the definition of a "hill" is ambiguous for k-Dyck paths with $k > 2$. In light of this difficulty, we offer $k-1$ competing definitions for a k-Catalan analogue of Fine paths, each of which forbids different types of sub-paths that immediately precede a return to $y = 0$. We also immediately generalize to partial Fine paths of any height $\beta \geq 0$.

So fix $k \geq 2$ and take any $1 \leq r \leq k-1$. A (k,r)-Fine path of length $kn+\beta$ and height β is any element of $\mathcal{D}^k_{n,\beta}$ that lacks a sub-path of the form $U^r D_{k-1}$ ending at height 0. We designate the set of all such paths by $\mathcal{F}^{k,r}_{n,\beta}$, and let $|\mathcal{F}^{k,r}_{n,\beta}| = F^{k,r}_{n,\beta}$. For fixed k, r, β, we then define generating functions $F_{k,r,\beta}(t)$ whereby $F^{k,r}_{n,\beta} = [t^n]F_{k,r,\beta}(t)$. When $\beta = 0$, we routinely suppress β in our expressions, and refer to the integer sequence $(F^{k,r}_{n,0})_{n\geq 0} = (F^{k,r}_n)_{n\geq 0}$ as the (k,r)-Fine numbers.

For a basic example of (k,r)-Fine paths with $k = 3$ and various $1 \leq r \leq k-1$, see Figure 2. It is immediate that $(2,1)$-Fine paths correspond to the traditional notion of Fine paths of Section 1, so that $F^{2,1}_{n,0} = F^{2,1}_n = F_n$ for all $n \geq 0$. For fixed $k \geq 2$, we also clearly have $\mathcal{F}^{k,r_1}_{n,\beta} \subseteq \mathcal{F}^{k,r_2}_{n,\beta}$ whenever $r_1 < r_2$, providing for the string of inequalities $F^{k,1}_{n,\beta} \leq F^{k,2}_{n,\beta} \leq \cdots \leq F^{k,k-1}_{n,\beta} \leq C^k_{n,\beta}$.

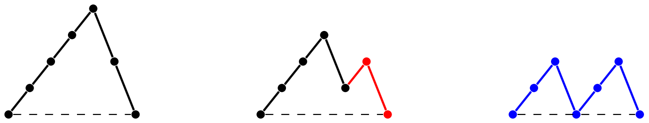

Figure 2: The three elements of $\mathcal{D}^3_2$. The first path is both $(3,2)$-Fine and $(3,1)$-Fine, the second path is $(3,2)$-Fine yet not $(3,1)$-Fine, and the third path is neither $(3,2)$-Fine nor $(3,1)$-Fine. All relevant hills are highlighted.

Although all classes of (k,r)-Fine paths share interesting properties, we will soon see that $(k,1)$-Fine paths and $(k,k-1)$-Fine paths admit a particularly large number of results. In fact, many of the Fine number properties of Section 1 will only generalize to (k,r)-Fine paths when either $r = 1$ or $r = k-1$. Observe that $(k,1)$-Fine paths correspond to k-Dyck paths that lack a "minimal height" peak (at $y = k-1$), whereas $(k,k-1)$-Fine paths correspond to k-Dyck paths that have at least two D_{k-1} steps between every return to $y = 0$.

Before moving on, it should be noted that several generalization of Fine paths and Fine numbers already appear in the literature. One such generalization was introduced by Cameron and McLeod [4] and corresponds to our specific case of $(k,k-1)$-Fine paths. That being said, even in the $r = k-1$ case, we present many results that are well outside the scope of [4]. Another generalization is introduced by Prodinger [17], whose examination of hills in "ternary paths" corresponds to $(3,2)$-Fine Paths. A number of our theorems from Sections 4 and 5 reduce to Prodinger's results in the case of $(k,r) = (3,2)$. More directly related to this paper is the cursory examination of (k,r)-Fine paths provided by DeJager et al. [6]. Although the author's involvement in that paper directly inspired this work, the only results from [6] reiterated here are several basic generating function results (Subsection 2.1) and a handful of results involving higher-order Motzkin paths (Section 3).

2.1 Basic results about (k, r)-Fine paths and the (k, r)-Fine numbers

We begin our investigation of (k, r)-Fine paths with foundational results that hold for all $k \geq 2$ and $1 \leq r \leq k-1$. Our first such result allows us to rewrite the generating function for partial Fine paths in terms of simpler generating functions, similarly to Proposition 2.1:

Proposition 2.2. *Fix $k \geq 2$ and $1 \leq r \leq k-1$. For every $\beta \geq 0$, the generating function for (k, r)-Fine paths of height β and length $kn+\beta$ satisfies*

$$F_{k,r,\beta}(t) \;=\; C_k(t)^{\beta} F_{k,r}(t).$$

Proof. Follows from the decomposition of Figure 3. Note that all but the first block in the decomposition may contain minimal-height sub-paths $U^r D_{k-1}$, as only the first block touches $y = 0$. □

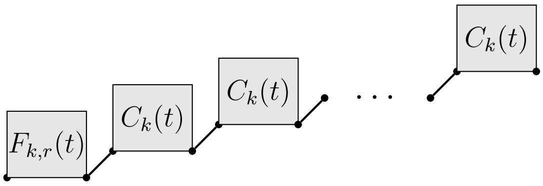

Figure 3: Decomposing a (k, r)-Fine path of height β, for use in Proposition 2.2.

Now looking to the Fine number properties from Section 1, we start with a generating function relationship that directly generalizes the first half of (2).

Proposition 2.3. *Fix $k \geq 2$ and $1 \leq r \leq k-1$. Then the generating functions for the (k, r)-Fine numbers and k-Catalan numbers are related via*

$$F_{k,r}(t) \;=\; \frac{C_k(t)}{1+tC_k(t)^{k-r}}.$$

Proof. Any path $P \in \mathcal{D}_n^k$ is either an element of $\mathcal{F}_n^{k,r}$ or may be decomposed as in Figure 4, where the two blocks are separated by the rightmost instance of a minimal-height sub-path $U^r D_{k-1}$. Notice that the first block in that decomposition may contain additional forbidden sub-paths $U^r D_{k-1}$. Using $C_{k,k-1-r}(t) = C_k(t)^{k-r}$ from Proposition 2.1, our two options for P combine to give

$$C_k(t) \;=\; F_{k,r}(t) + tF_{k,r}(t)\,C_k(t)^{k-r}.$$

□

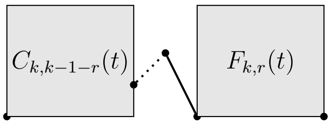

Figure 4: Decomposition of a k-Dyck path in $\mathcal{D}_n^k - \mathcal{F}_n^{k,r}$ by the rightmost instance of a minimal-height sub-path $U^r D_{k-1}$, as used in Proposition 2.3.

Proposition 2.3 may be applied to derive our first closed formula for the (k, r)-Fine numbers. This alternating sum formula is a direct generalization of (4).

Corollary 2.4. *Fix $k \geq 2$ and $1 \leq r \leq k-1$. Then for all $n \geq 0$ we have*

$$F_n^{k,r} \;=\; \sum_{i=0}^{n} (-1)^i \frac{(k-r)i+1}{kn-ri+1} \binom{kn-ri+1}{n-i}.$$

Proof. From Proposition 2.3 we have the generating function identity

$$F_{k,r}(t) \;=\; \sum_{i=0}^{\infty} (-1)^i t^i C_k(t)^{(k-r)i+1}.$$

Comparing coefficients, and using the Raney number property $R_n^{k,r} = [t^n] C_k(t)^r$ yields

$$F_n^{k,r} \;=\; [t^n] F_{k,r}(t) \;=\; \sum_{i=0}^{\infty} (-1)^i [t^n] t^i C_k(t)^{(k-r)i+1} \;=\; \sum_{i=0}^{n} (-1)^i [t^{n-i}] C_k(t)^{(k-r)i+1}$$

$$= \sum_{i=0}^{n} (-1)^i R_{n-i}^{k,(k-r)i+1} = \sum_{i=0}^{n} (-1)^i \frac{(k-r)i+1}{k(n-i)+(k-r)i+1} \binom{k(n-i)+(k-r)i+1}{n-i}.$$

□

Returning to the underlying generating functions, a "first-return decomposition" of (k, r)-Fine paths prompts a generalization of the final equality from (2):

Proposition 2.5. *Fix $k \geq 2$ and $1 \leq r \leq k-1$. Then the generating functions for the (k, r)-Fine numbers and the k-Catalan numbers are related via*

$$\begin{aligned} F_{k,r}(t) \;&=\; \frac{1}{1 - t\left(C_k(t)^{k-1} - C_k(t)^{k-1-r}\right)} \\ &=\; \frac{1}{1 - t^2 C_k(t)^{2k-r-1}\left(1 + C_k(t) + \cdots + C_k(t)^{r-1}\right)}. \end{aligned}$$

Proof. Any non-trivial path $P \in \mathcal{F}_n^k$ has a first-return decomposition of the form shown in Figure 5. The first block in that decomposition is a k-Dyck path of height $k-2$ that cannot terminate in r consecutive U steps. Enumerating such options for the first block is best accomplished by enumerating the complement, consisting of k-Dyck paths of length $kn+\beta$ (for some $\beta > 0$) that terminate in at least r consecutive U steps. This complement is clearly in bijection with all k-Dyck paths of length $kn+\beta-r$ and height $k-2-r$. Citing Proposition 2.1 then allows us to characterize paths in the first block of our decomposition:

$$C^*_{k,k-2}(t) \;=\; C_{k,k-2}(t) - C_{k,k-2-r}(t) \;=\; C_k(t)^{k-1} - C_k(t)^{k-1-r}.$$

Returning to our first-return decomposition then gives the functional equation

$$F_{k,r}(t) \;=\; 1 + tC^*_{k,k-2}(t)\,F_{k,r}(t) \;=\; 1 + t\left(C_k(t)^{k-1} - C_k(t)^{k-1-r}\right)F_{k,r}(t).$$

This relationship directly leads to the first expression for $F_{k,r}(t)$ in the theorem statement. As for our second expression, a bit of algebra and the k-Catalan identity $C_k(t) = tC_k(t)^k + 1$ gives

$$\begin{aligned} t\left(C_k(t)^{k-1} - C_k(t)^{k-1-r}\right) &= tC_k(t)^{k-1-r}\left(C_k(t)^r - 1\right) \\ &= tC_k(t)^{k-1-r}\left(C_k(t) - 1\right)\left(1 + C_k(t) + \cdots + C_k(t)^{r-1}\right) \\ &= tC_k(t)^{k-1-r}\,tC_k(t)^k\left(1 + C_k(t) + \cdots + C_k(t)^{r-1}\right) \\ &= t^2C_k(t)^{2k-r-1}\left(1 + C_k(t) + \cdots + C_k(t)^{r-1}\right). \end{aligned}$$

□

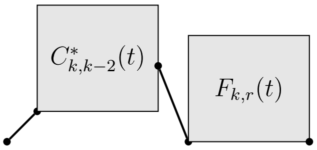

Figure 5: The first-return decomposition of a (k,r)-Fine path, as used in Proposition 2.5. Note that the first block cannot terminate in r consecutive U steps.

2.2 Additional Results for $(k,1)$-Fine Paths

Excepting the determinant identities and asymptotic results, which we delay to future sections, none of the remaining equations from Section 1 directly generalize to (k,r)-Fine paths for all $1 \leq r \leq k-1$. Luckily, some of those equations still accept generalizations for specific choices of r.

The sub-category of (k,r)-Fine paths that satisfy the largest number of additional properties are $(k,1)$-Fine paths. We begin by setting $r=1$ in the relations of Propositions 2.3 and 2.5:

$$F_{k,1}(t) \;=\; \frac{C_k(t)}{1 + tC_k(t)^{k-1}}, \tag{11}$$

$$F_{k,1}(t) \; = \; \frac{1}{1-(tC_k(t)^{k-1})^2} \; = \; \sum_{i=0}(tC_k(t)^{k-1})^{2i}. \tag{12}$$

The simplification of (12) prompts our first result that is specific to the $r = 1$ case: a non-alternating closed formula for the $(k,1)$-Fine numbers that generalizes (5).

Proposition 2.6. *Fix $k \geq 2$. For all $n \geq 2$,*

$$F_n^{k,1} \; = \; \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{2i(k-1)}{kn-2i} \binom{kn-2i}{(k-1)n}.$$

Proof. The formula follows from comparing coefficients in (12). Here we again make use of the Raney number identity $R_n^{k,r} = [t^n]C_k(t)^r$:

$$F_n^{k,1} \; = \; \sum_{i=0}^{\infty}[t^n]t^{2i}C_k(t)^{2i(k-1)} \; = \; \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}[t^{n-2i}]C_k(t)^{2i(k-1)}$$

$$= \; \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} R_{n-2i}^{k,2i(k-1)} \; = \; \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2i(k-1)}{k(n-2i)+2i(k-1)} \binom{k(n-2i)+2i(k-1)}{n-2i}.$$

□

All of our remaining (non-determinant) results for the $r = 1$ case take the form of bijections between $\mathcal{F}_n^{k,1}$ and other sub-categories of k-Dyck paths. We begin by recalling the trivial observation that $\mathcal{F}_n^{k,1}$ corresponds to all k-Dyck paths without a minimal height peak.

The first of our non-trivial combinatorial interpretations places $\mathcal{F}_n^{k,1}$ in bijection with paths from $\mathcal{D}_n^k$ that terminate with an even number of consecutive down steps. For a specialization of the result below to $k = 2$, see Deutsch and Shapiro [9].

Proposition 2.7. *For every $k \geq 2$, the generating function for the number of k-Dyck paths that end with an even number of down steps satisfies*

$$C_k(t)|_{even} = F_{k,1}(t).$$

Conversely, for every $k \geq 2$, the generating function for the number of k-Dyck paths that terminate with an odd number of down steps satisfies

$$C_k(t)|_{odd} = tF_{k,1}(t)C_k(t)^{k-1}.$$

Proof. Note that any k-Dyck path of length kn that ends in precisely α down steps must conclude with the sub-path $U(D_{k-1})^{\alpha}$. Deleting this terminal sub-path places the collection of all such paths in bijection with k-Dyck paths of length $kn-\alpha-1 = k(n-\alpha)+(k-1)\alpha-1$ and height $(k-1)\alpha-1$. If we let $C_k(t)|_{\alpha}$ denote the generating function for all k-Dyck paths of length kn that end with precisely α down steps, applying Proposition 2.1 gives

$$C_k(t)|_\alpha = t^\alpha C_{k,(k-1)\alpha-1}(t) = \left(tC_k(t)^{k-1}\right)^\alpha .$$

Focusing upon the summation from (12), we then see that the $\left(tC_k(t)^{k-1}\right)^{2i}$ term enumerates all k-Dyck paths of length kn that terminate in precisely $2i$ consecutive down steps. Ranging over all terms $i \geq 0$ results in the full summation of (12), proving our first result.

As for the second result, observe that (11) is equivalent to $C_k(t) = F_{k,1}(t) + tF_{k,1}(t)C_k(t)^{k-1}$. This demands that $tF_{k,1}(t)C_k(t)^{k-1}$ enumerate the complementary collection of k-Dyck paths. ☐

Although it was not used in the proof of Proposition 2.7, it's relatively straightforward to set up an explicit bijection between k-Dyck paths that terminate with an even number of down steps and $\mathcal{F}_n^{k,1}$. See Figure 6 for one such map.

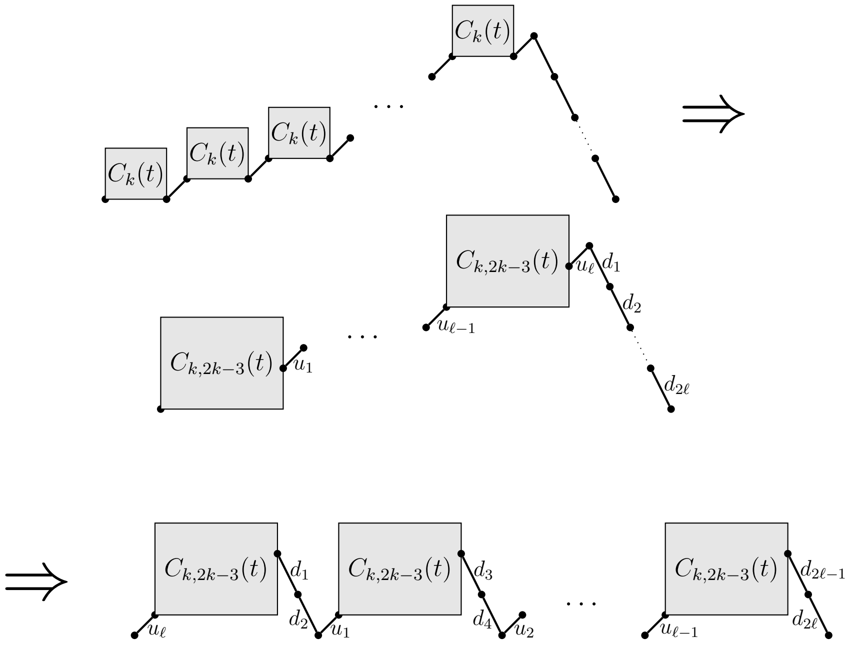

Figure 6: An explicit bijection for the first identity of Proposition 2.7. We begin with a path in $\mathcal{D}_n^k$ that ends with precisely 2ℓ down steps, meaning that the first figure above has $2\ell(k-1)$ boxed sub-paths of height 0. Then group those sub-paths into ℓ sets of $2(k-1)$, so that each new grouping has height $2(k-1)-1 = 2k-3$. Rearranging the labeled steps as shown yields an element of $\mathcal{F}_n^{k,1}$.

2.3 Additional Results for $(k, k-1)$-Fine Paths

The other sub-category of (k, r)-Fine paths that satisfy a number of additional properties are $(k, k-1)$-Fine paths, which coincide with $(k, 1)$-Fine Paths only when

$k = 2$. We once again begin by specializing the generating function identities of Propositions 2.3 and 2.5:

$$F_{k,k-1} = \frac{C_k(t)}{1 + tC_k(t)} \iff C_k(t) = \frac{F_{k,k-1}(t)}{1 - tF_{k,k-1}(t)}, \tag{13}$$

$$F_{k,k-1}(t) = \frac{1}{1 - (t(C_k(t)^{k-1} - 1))} \iff F_{k,k-1}(t) - tF_{k,k-1}(t)\,C_k(t)^{k-1} + tF_{k,k-1}(t) = 1. \tag{14}$$

The first significant result to derive from (13) and (14) is a generating function identity that directly generalizes (3). See Theorem 2.1 of Cheon, lee and Shapiro [5] for a more detailed proof in the case of $k = 2$.

Proposition 2.8. *Take any $k \geq 2$. Then the generating function $F_{k,k-1}(t)$ satisfies*

$$tF_{k,k-1}(t)^k + (1 - tF_{k,k-1}(t))^k - F_{k,k-1}(t)\,(1 - tF_{k,k-1}(t))^{k-1} = 0.$$

Proof. Combining our rearranged versions of (13) and (14) gives the following equation, from which the relationship of the proposition may be derived.

$$F_{k,k-1}(t) - tF_{k,k-1}(t) \left(\frac{F_{k,k-1}(t)}{1 - tF_{k,k-1}(t)} \right)^{k-1} + tF_{k,k-1}(t) = 1.$$

□

Since the left side of the equation from Proposition 2.8 is a degree-k polynomial equation with respect to $y = F_{k,k-1}(t)$, we immediately have the following characterization of $F_{k,k-1}(t)$:

Corollary 2.9. *For every $k \geq 2$, the generating function $F_{k,k-1}(t)$ is the solution to a degree-k polynomial equation involving the variable t.*

For example, when $k = 2$ the equation of Proposition 2.8 simplifies to the degree-2 polynomial equation $1 - (1 + 2t)F_{2,1}(t) + (t + 2t^2)F_{2,1}(t)^2 = 0$. Applying the quadratic formula then yields the closed formula for $F_{2,1}(t)$ given by (1).

The other Fine number property from Section 2 that has a straightforward generalization to the $r = k - 1$ case is (6):

Proposition 2.10. *For every $k \geq 2$ and $n \geq 0$, the $(k, k-1)$-Fine numbers satisfy*

$$F_n^{k,k-1} = \sum_{\substack{i_1 + \cdots + i_q = n - q, \\ i_1 > 0, \ldots, i_q > 0,\ q > 0}} R_{i_1}^{k,k-1} R_{i_2}^{k,k-1} \ldots R_{i_q}^{k,k-1}.$$

Proof. From the first equality of (14) we have

$$F_{k,k-1}(t) = \frac{1}{1 - (t(C_k(t)^{k-1} - 1))} = \sum_{q \geq 0} \left((C_k(t)^{k-1} - 1)^q t^q \right).$$

Extracting the t^n coefficient from the summation of the right gives:

$$\begin{aligned} F_n^{k,k-1} &= [t^n]F_{k,k-1}(t) \;=\; [t^n]\sum_{q\geq 0}\left((C_k(t)^{k-1}-1)^q t^q\right) \;=\; \sum_{q\geq 0}[t^{n-q}](C_k(t)^{k-1}-1)^q \\ &= \sum_{\substack{i_1+\dots+i_q=n-q,\\ q\geq 0}} \left([t^{i_1}](C_k(t)^{k-1}-1)\right)\dots\left([t^{i_q}](C_k(t)^{k-1}-1)\right) \\ &= \sum_{\substack{i_1+\dots+i_q=n-q,\\ i_1>0,\dots,i_q>0,\ q>0}} \left([t^{i_1}]C_k(t)^{k-1}\right)\dots\left([t^{i_q}]C_k(t)^{k-1}\right). \end{aligned}$$

Applying the Raney number identity $R_n^{k,r}=[t^n]C_k(t)^r$ then reduces our expression to the summation from the theorem. □

When $k=2$, notice how the Raney number identity $R_n^{k,1}=C_n^k$ directly reduces the summation of Proposition 2.10 to the identity from (6).

2.4 Determinant Formulas

Up to this point, we have ignored the determinant identities of (7) and (8). In this subsection, we address whether those identities generalize to (k,r)-Fine numbers.

In the case of (7), we may fully generalize the determinant identity to all $k\geq 2$ and $1\leq r\leq k-1$. Sadly, this comes at the cost of "on-diagonal symmetry":

Theorem 2.11. *Fix $k\geq 2$ and $1\leq r\leq k-1$. Then for all $n\geq 0$ we have*

$$F_n^{k,r} \;=\; (-1)^n \begin{vmatrix} C_0^k & 1 & 0 & 0 & \dots & 0 \\ C_1^k & R_0^{k,k-r} & 1 & 0 & \dots & 0 \\ C_2^k & R_1^{k,k-r} & R_0^{k,k-r} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{n-1}^k & R_{n-2}^{k,k-r} & R_{n-3}^{k,k-r} & R_{n-4}^{k,k-r} & \dots & 1 \\ C_n^k & R_{n-1}^{k,k-r} & R_{n-2}^{k,k-r} & R_{n-3}^{k,k-r} & \dots & R_0^{k,k-r} \end{vmatrix}.$$

Proof. From the generating function relation of Proposition 2.3, for all $n\geq 0$ we have

$$C_n^k \;=\; [t^n]C_k(t) \;=\; [t^n]\left(F_{k,r}(t)+t\,F_{k,r}(t)C_k(t)^{k-r}\right).$$

When $n=0$ this expression clearly simplifies to $C_0^k=F_0^{k,r}$. When $n\geq 1$, we can rewrite as below. Here we use the Raney number property $R_n^{k,r}=[t^n]C_k(t)^r$.

$$C_n^k \;=\; \sum_{i=0}^{n}\left([t^i]F_{k,r}(t)\cdot[t^{n-i}](1+t\,C_k(t)^{k-r}\right)$$

$$= F_n^{k,r} + \sum_{i=0}^{n-1} \left([t^i]F_{k,r}(t) \cdot [t^{n-i-1}]C_k(t)^{k-r}\right) = F_n^{k,r} + \sum_{i=0}^{n-1} \left(F_i^{k,r} R_{n-i-1}^{k,k-r}\right).$$

The resulting set of expressions for the C_n^k and $F_n^{k,r}$ may be arranged into the matrix equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ R_0^{k,k-r} & 1 & 0 & 0 & \dots \\ R_1^{k,k-r} & R_0^{k,k-r} & 1 & 0 & \dots \\ R_2^{k,k-r} & R_1^{k,k-r} & R_0^{k,k-r} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} F_0^{k,r} \\ F_1^{k,r} \\ F_2^{k,r} \\ F_3^{k,r} \\ \vdots \end{bmatrix} = \begin{bmatrix} C_0^k \\ C_1^k \\ C_2^k \\ C_3^k \\ \vdots \end{bmatrix}.$$

Fixing $n \geq 0$, we truncate this matrix equation after $n+1$ rows. Focusing on the final unknown $F_n^{k,r}$, Cramer's Rule then guarantees $F_n^{k,r} = \frac{|A_{n+1}|}{|A|}$. Here A is the truncated, $(n+1) \times (n+1)$ version of our matrix above, for which we clearly have $|A| = 1$, whereas A_{n+1} is the matrix below.

$$A_{n+1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & C_0^k \\ R_0^{k,k-r} & 1 & 0 & \dots & 0 & C_1^k \\ R_1^{k,k-r} & R_0^{k,k-r} & 1 & \dots & 0 & C_2^k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ R_{n-2}^{k,k-r} & R_{n-3}^{k,k-r} & R_{n-4}^{k,k-r} & \dots & 1 & C_{n-1}^k \\ R_{n-1}^{k,k-r} & R_{n-2}^{k,k-r} & R_{n-3}^{k,k-r} & \dots & R_0^{k,k-r} & C_n^k \end{bmatrix}.$$

Thus $F_n^{k,r} = |A_{n+1}|$, and the required formula follows from reordering the columns of A_{n+1}. □

In the specific case of $r = k-1$, we may apply the Raney number identity $R_n^{k,1} = C_n^k$ and recover a more elegant formula that directly generalizes (7):

Corollary 2.12. *Fix $k \geq 2$. Then for all $n \geq 0$ we have*

$$F_n^{k,k-1} = (-1)^n \begin{vmatrix} C_0^k & 1 & 0 & 0 & \dots & 0 \\ C_1^k & C_0^k & 1 & 0 & \dots & 0 \\ C_2^k & C_1^k & C_0^k & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{n-1}^k & C_{n-2}^k & C_{n-3}^k & C_{n-4}^k & \dots & 1 \\ C_n^k & C_{n-1}^k & C_{n-2}^k & C_{n-3}^k & \dots & C_0^k \end{vmatrix}.$$

Moving on to (8), we encounter our first Fine number identity that appears to resist any analogue when $k \geq 2$. This lack of a generalization is significant because the proof of (8), as outlined by Deutsch and Shapiro [9], follows directly from the fact that the Fine numbers are a particular case of "Catalan-like numbers" (see Aigner [1]). This suggests that some of the techniques employed by Aigner may not directly generalize from $k = 2$ to $k > 2$.

To see that (8) lacks a clear generalization, we begin by defining two sequences of matrices, $\{H_n(k,r)\}_{n\geq 0}$ and $\{H'_n(k,r)\}_{n\geq 0}$, as shown below. These are the Hankel matrices corresponding to the sequences $\{F_0^{k,r}\}_{n\geq 0}$ and $\{F_0^{k,r}\}_{n\geq 1}$, respectively.

$$H_n(k,r)=\begin{bmatrix} F_0^{k,r} & F_1^{k,r} & F_2^{k,r} & \dots & F_n^{k,r} \\ F_1^{k,r} & F_2^{k,r} & F_3^{k,r} & \dots & F_{n+1}^{k,r} \\ F_2^{k,r} & F_3^{k,r} & F_4^{k,r} & \dots & F_{n+2}^{k,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_n^{k,r} & F_{n+1}^{k,r} & F_{n+2}^{k,r} & \dots & F_{2n}^{k,r} \end{bmatrix}, H'_n(k,r)=\begin{bmatrix} F_1^{k,r} & F_2^{k,r} & F_3^{k,r} & \dots & F_n^{k,r} \\ F_2^{k,r} & F_3^{k,r} & F_4^{k,r} & \dots & F_{n+1}^{k,r} \\ F_3^{k,r} & F_4^{k,r} & F_5^{k,r} & \dots & F_{n+2}^{k,r} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ F_n^{k,r} & F_{n+1}^{k,r} & F_{n+2}^{k,r} & \dots & F_{2n}^{k,r} \end{bmatrix}.$$

The resulting sequences of determinants, $\{|H_n(k,r)|\}_{n\geq 0}$ and $\{|H'_n(k,r)|\}_{n\geq 0}$, represent the Hankel transforms of the sequences $\{F_n^{k,r}\}_{n\geq 0}$ and $\{F_n^{k,r}\}_{n\geq 1}$, respectively. As such, the original Fine number identity (8) is a statement about the Hankel transforms of the Fine numbers $\{F_n\}_{n\geq 0}$ and the shifted Fine numbers $\{F_n\}_{n\geq 1}$, meaning that any generalization would yield a result about the Hankel transforms of the (k,r)-Fine numbers.

Sadly, no such generalization appears to exist. Manually verifying the first handful of determinants $|H_n(k,r)|$ and $|H'_n(k,r)|$ fails to reveal an observable pattern, even in the case of $k = 3$. Shown below are the start of all relevant Hankel transforms in the case of $k = 3$. None of these sequences correspond to an entry on OEIS.

$$\{|H_n(3,1)|\}_{n\geq 0} = \{1, 1, 2, 6, -95, -17250, \dots\},$$

$$\{|H_n(3,2)|\}_{n\geq 0} = \{1, 1, 11, 170, 7429, 920460, \dots\},$$

$$\{|H'_n(3,1)|\}_{n\geq 0} = \{0, -1, -8, -119, -3680, 2489094827100, \dots\},$$

$$\{|H'_n(3,2)|\}_{n\geq 0} = \{0, -4, -75, -2975, -296746, -78639300 \dots\}.$$

3 Riordan Arrays and (k,r)-Fine Paths as Generalized Motzkin Paths

In this section, we show how the (k,r)-Fine numbers may be recovered via the usage of proper Riordan arrays. This reveals the (k,r)-Fine numbers as a specific class of "higher-order Catalan-like numbers", which generalize the "Catalan-like numbers" of Aigner [1, 2]. It also allows us to provide a general combinatorial interpretation for the (k,r)-Fine numbers as so-called colored Motzkin paths of higher-order.

It should be noted that many of the results from this section already appeared in Section 3.2 of DeJager, Naquin, Seidl and Drube [6]. We recap those results not only for completeness, but to provide a framework for the Riordan array techniques of Sections 4 and 5. For a broader discussion of higher-order Catalan-like numbers, see [6].

We begin with for the necessary background involving Riordan arrays, a more involved treatment of which may be found in Rogers [18] or Merlini et al. [15]. So take a pair of formal power series $d(t), h(t)$ such that $d(0) \neq 0$, $h(0) = 0$, and $h'(0) \neq 0$. Then the proper Riordan array $\mathcal{R}(d(t), h(t))$ is the infinite, lower-triangular array whose (i, j)-entry (for any $0 \leq j \leq i$) is $d_{i,j} = [t^i]d(t)h(t)^j$. Notice that the coefficients of the series $d(t)$ appears as the zeroth column of $\mathcal{R}(d(t), h(t))$.

Essential to our upcoming results is the fact that every proper Riordan array may be generated by a pair of recurrences of the form below:

$$d_{i,j} = \begin{cases} a_0 d_{i-1,j-1} + a_1 d_{i-1,j} + a_2 d_{i-1,j+1} + \cdots, & \text{for all } j \geq 1; \\ z_0 d_{i-1,0} + z_1 d_{i-1,1} + z_2 d_{i-1,2} + \cdots, & \text{for } j = 0. \end{cases} \tag{15}$$

The coefficients in these recurrences may be arranged into a pair of power series $A(t) = \sum_{i=0}^{\infty} a_i t^i$ and $Z(t) = \sum_{i=0}^{\infty} z_i t^i$ that are referred to as the A-sequence and Z-sequence of $\mathcal{R}(d(t), h(t))$, respectively. It may be shown that the A- and Z-sequences of a Riordan array are related to $d(t)$ and $h(t)$ via

$$h(t) = tA(h(t)), \qquad d(t) = \frac{d(0)}{1 - tZ(h(t))}. \tag{16}$$

Also central to much of what follows is the Fundamental Theorem of Riordan Arrays, which describes the result of "infinite matrix multiplication" of $\mathcal{R}(d(t), h(t))$ by the infinite column vector constructed from the coefficients of $f(t) = \sum_{i=0}^{\infty} f_{i,1} t^i$:

$$\mathcal{R}(d(t), h(t)) \cdot f(t) = d(t)h(f(t)). \tag{17}$$

We are now ready to apply the language of proper Riordan arrays to the (k, r)-Fine numbers. For any $k \geq 2$ and $1 \leq r \leq k - 1$, we define $M^{k,r}$ to be the infinite, lower triangular array whose (n, m)-entry corresponds to the number of (k, r)-Fine paths of length kn and height km. Namely, $m_{n,m} = F^{k,r}_{n,km}$ for every $0 \leq m \leq n$. This array does, in fact, represent a proper Riordan array whose leftmost column corresponds to the (k, r)-Fine numbers:

Proposition 3.1. *Take any $k \geq 2$ and $1 \leq r \leq k - 1$. Then $M^{k,r}$ is the proper Riordan array $\mathcal{R}(d(t), h(t))$ with $d(t) = F_{k,r}(t)$ and $h(t) = tC_k(t)^k$.*

Proof. By Proposition 2.2, the m^{th} column of $M^{k,r}$ has generating function

$$t^m F_{k,r}(t) C_k(t)^{km} = F_{k,r}(t) \left(tC_k(t)^k\right)^m.$$

□

Our proper Riordan arrays also admit relatively nice A- and Z-sequences. For a distinct derivation of these sequences, see DeJager et al. [6].

Proposition 3.2. *Take any $k \geq 2$ and $1 \leq r \leq k-1$. Then $M^{k,r}$ is the proper Riordan array with A- and Z-sequences*

$$A(t) = (1+t)^k, \qquad Z(t) = (1+t)^{k-1} - (1+t)^{k-r-1}.$$

In particular, for any $k \geq 2$ and $1 \leq r \leq k-1$, partial (k,r)-Fine paths satisfy the recurrences

$$F^{k,r}_{n,km} = \begin{cases} \displaystyle\sum_{i=0}^{k} \binom{k}{i} F^{k,r}_{n-1,k(m-1+i)}, & \text{if } m \geq 1; \\ \displaystyle\sum_{i=0}^{k-1} \left(\binom{k-1}{i} - \binom{k-r-1}{i} \right) F^{k,r}_{n-1,ki}, & \text{if } m = 0. \end{cases}$$

Proof. In light of Proposition 3.1, we merely need to show that the given sequences $A(t), Z(t)$ satisfy both $h(t) = tA(h(t))$ and $d(t) = \frac{d(0)}{1-tZ(h(t))}$, assuming $d(t) = F_{k,r}(t)$ and $h(t) = tC_k(t)^k$. The first of those equations follows from the k-Catalan relation $C_k(t) = tC_k(t)^k + 1$:

$$tA(h(t)) = t\left(1 + tC_k(t)^k\right)^k = tC_k(t)^k = h(t).$$

Verifying the second of the required equations requires both the relation $C_k(t) = tC_k(t)^k + 1$ and the equality of Proposition 2.5:

$$\begin{aligned} \frac{d(0)}{1-tZ(h(t))} &= \frac{1}{1-t\left((1+tC_k(t)^k)^{k-1} - (1+tC_k(t)^k)^{k-r-1}\right)} \\ &= \frac{1}{1-t\left(C_k(t)^{k-1} - C_k(t)^{k-r-1}\right)} = F_{k,r}(t) = d(t). \end{aligned}$$

□

The proper Riordan arrays of Proposition 3.1 may be used to reverse-engineer a combinatorial interpretation for all classes of (k,r)-Fine numbers as colored Motzkin paths of higher-order. For a more thorough treatment of this topic, see DeJager et al. [6].

For any $\ell \geq 1$, an order-ℓ Motzkin path of length n and height m is an integer lattice path from $(0,0)$ and (n,m) that uses steps $\{U = (1,1), D_0 = (1,0), D_1 = (1,-1), \ldots, D_\ell = (1,-\ell)\}$ and which stays weakly above $y = 0$. Observe that the traditional notion of Motzkin paths corresponds to the $\ell = 1, m = 0$ case. We denote the set of all such paths by $\mathcal{M}^\ell_{n,m}$, and set $|\mathcal{M}^\ell_{n,m}| = M^\ell_{n,m}$.

Once again fixing $\ell \geq 1$, take any pair $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{\ell-1}), \vec{\beta} = (\beta_0, \ldots, \beta_{\ell-1})$ of ℓ-tuples of non-negative integers. An $(\vec{\alpha}, \vec{\beta})$-colored Motzkin path of order ℓ, length n, and height m is a path from $\mathcal{M}^\ell_{n,m}$ where, for every $0 \leq i \leq \ell-1$, D_i steps ending at height $y = 0$ carry one of α_i colors and D_i steps ending at height $y > 0$ carry one of β_i colors. We denote the set of all such paths by $\mathcal{M}^{\ell}_{n,m}(\vec{\alpha}, \vec{\beta})$, and set $|\mathcal{M}^{\ell}_{n,m}(\vec{\alpha}, \vec{\beta})| = M^{\ell}_{n,m}(\vec{\alpha}, \vec{\beta})$. See Figure 7 for a basic example of colored Motzkin paths of higher order.

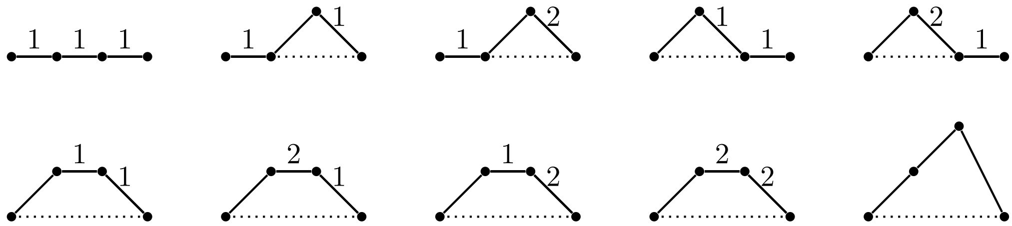

Figure 7: All paths in $\mathcal{M}^2_{3,0}(\vec{\alpha}, \vec{\beta})$ with $\vec{\alpha} = (1, 2)$ and $\vec{\beta} = (2, 2)$. Notice that up steps and the steepest down steps are always uncolored.

Our terminology for higher-order Motzkin paths has been carefully chosen so that such paths are straightforward to enumerate using proper Riordan arrays. To that end, for every $\ell \geq 1$ and pair of appropriate ℓ-tuples $\vec{\alpha}, \vec{\beta}$, we define an infinite array whose (n, m) entry is $M^{\ell}_{n,m}(\vec{\alpha}, \vec{\beta})$. Since $M^{\ell}_{n,m}(\vec{\alpha}, \vec{\beta}) = 0$ unless $0 \leq m \leq n$, this array is lower-triangular, and we refer to any such array as the $(\vec{\alpha}, \vec{\beta})$-colored Motzkin triangle.

Every $(\vec{\alpha}, \vec{\beta})$-colored Motzkin triangle corresponds to a proper Riordan array, as given below. For a proof of this result, see DeJager et al. [6].

Proposition 3.3. *Fix $\ell \geq 1$, and take any pair $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{\ell-1})$, $\vec{\beta} = (\beta_0, \ldots, \beta_{\ell-1})$ of ℓ-tuples of non-negative integers. Then the $(\vec{\alpha}, \vec{\beta})$-Motzkin triangle is a proper Riordan array with A- and Z-sequences given by*

$$A(t) = 1 + \beta_0 t + \cdots + \beta_{\ell-1} t^{\ell} + t^{\ell+1}, \qquad Z(t) = \alpha_0 + \alpha_1 t + \cdots + \alpha_{\ell-1} t^{\ell-1} + t^{\ell}.$$

Since proper Riordan arrays are fully determined by their A- and Z-sequences, comparing Propositions 3.2 and 3.3 shows that the (k, r)-Fine numbers enumerate $(\vec{\alpha}, \vec{\beta})$-colored Motzkin paths of higher order for appropriate choices of $\vec{\alpha}, \vec{\beta}$:

Corollary 3.4. *Fix $k \geq 2$ and $1 \leq r \leq k - 1$. Then $M^{k,r}$ is identical to the (α, β)-colored Motzkin triangle for the $(k-1)$-tuples $\vec{\alpha} = (\alpha_0, \ldots, \alpha_{k-2})$ and $\vec{\beta} = (\beta_0, \ldots, \beta_{k-2})$ with $\alpha_i = \binom{k-1}{i} - \binom{k-r-1}{i}$ and $\beta_i = \binom{k}{i+1}$ for all $0 \leq i \leq k - 2$.*

In particular, the (k, r)-Fine paths of $\mathcal{F}^{k,r}_{n,km}$ are in bijection with the $(\vec{\alpha}, \vec{\beta})$-colored Motzkin paths of $\mathcal{M}^{k-1}_{n,m}(\vec{\alpha}, \vec{\beta})$ when $\alpha_i = \binom{k-1}{i} - \binom{k-r-1}{i}$ and $\beta_i = \binom{k}{i+1}$ for all $0 \leq i \leq k - 2$.

See Table 1 for an illustration of the $(k-1)$-tuples $\vec{\alpha}, \vec{\beta}$ that ensure the bijection of Corollary 3.4, for small values of $k \geq 2$.

$\vec{\alpha}, \vec{\beta}$	$\boldsymbol{r=1}$	$\boldsymbol{r=2}$	$\boldsymbol{r=3}$	$\boldsymbol{r=4}$
$\boldsymbol{k=2}$	$(0), (2)$	-	-	-
$\boldsymbol{k=3}$	$(0,1), (3,3)$	$(0,2), (3,3)$	-	-
$\boldsymbol{k=4}$	$(0,1,2), (4,6,4)$	$(0,2,3), (4,6,4)$	$(0,3,3), (4,6,4)$	-
$\boldsymbol{k=5}$	$(0,1,3,3), (5,10,10,5)$	$(0,2,5,4), (5,10,10,5)$	$(0,3,6,4), (5,10,10,5)$	$(0,4,6,4), (5,10,10,5)$

Table 1: The $(k-1)$-tuples $\vec{\alpha}, \vec{\beta}$ such that $F^{k,r}_{n,km} = M^{k-1}_{n,m}(\vec{\alpha}, \vec{\beta})$, as shown in Corollary 3.4.

4 Enumeration by Number of Forbidden Sub-paths

Up to this point, we have only considered whether a k-Dyck path contains at least one minimal height sub-path of the form $U^r D_{k-1}$. Here we broaden our scope by accounting for the number of minimal height sub-paths $U^r D_{k-1}$ in a given k-Dyck path.

So fix $k \geq 2$, $1 \leq r \leq k-1$, and $m \geq 0$, and define $\mathcal{P}^{k,r}_{n,m}$ to be the set of all paths in $\mathcal{D}^k_n$ with precisely m instances of $U^r D_{k-1}$ ending at height 0. Clearly $\mathcal{P}^{k,r}_{n,0} = F^{k,r}_n$, meaning that this notion represents a direct generalization of (k,r)-Fine paths.

Then let $|\mathcal{P}^{k,r}_{n,m}| = P^{k,r}_{n,m}$. As shown in Proposition 4.1, these cardinalities are best understood via the construction of a proper Riordan array. It is extremely important to observe that the Riordan array $\widetilde{M}^{k,r}$ described below is distinct from the Riordan array $M^{k,r}$ of Section 3. In particular, whereas the the m^{th} column of $M^{k,r}$ corresponded to the number of (k,r)-Fine paths of height km, the m^{th} column of $\widetilde{M}^{k,r}$ corresponds to k-Dyck paths of height 0 and precisely m "forbidden subpaths". It is still the case that, in both Riordan arrays, the leftmost column provides the generating function $F_{k,r}(t)$ for (k,r)-Fine paths of height 0.

Proposition 4.1. *Fix $k \geq 2$ and $1 \leq r \leq k-1$, and consider the integer triangle $\widetilde{M}^{k,r}$ whose (n,m) entry is $P^{k,r}_{n,m}$. Then $\widetilde{M}^{k,r}$ corresponds to the proper Riordan array $\mathcal{R}(F_{k,r}(t), tC_k(t)^{k-1-r}F_{k,r}(t))$.*
In particular, for all $k \geq 2$, $1 \leq r \leq k-1$, and $m \geq 0$,

$$P^{k,r}_{n,m} = [t^n] F_{k,r}(t) \left(tC_k(t)^{k-1-r}F_{k,r}(t)\right)^m .$$

Proof. Any $P \in \mathcal{P}^{k,r}_{n,m}$ may be decomposed as in Figure 8. Here, all of the boxed sub-paths are (k,r)-Fine paths, whereas all but the final sub-path ends at height $(k-1)-r$. By Proposition 2.2, $F_{k,r,k-1-r}(t) = C_k(t)^{k-1-r}F_{k,r}(t)$. □

It should be noted that Prodinger [17] derives a bivariate generating function for all of the $P^{k,r}_{n,m}$ $(n, m \geq 0)$ in the specific case of $(k,r) = (3,2)$, albeit without reference to the underlying Riordan array. It may be shown that Proposition 4.1 reduces to Prodinger's result when $(k,r) = (3,2)$.

In addition to providing an enumeration of the $\mathcal{P}^{k,r}_{n,m}$, the Riordan array of Proposition 4.1 may be used to derive various general statistics involving k-Dyck paths. As

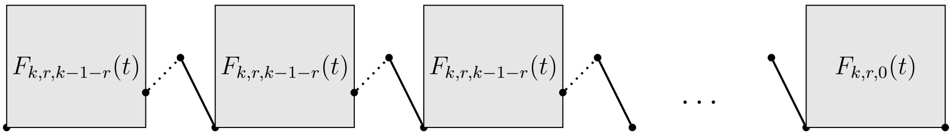

Figure 8: Decomposition of a k-Dyck path via instances of minimal-height sub-paths U^rD_{k-1}, as used in Proposition 4.1.

demonstrated by Theorem 4.2, all this requires is a clever application of the Fundamental Theorem of Riordan Arrays. Note that Theorem 4.2 is a direct generalization of results from Cameron and McLeod [4], where the authors restrict their attention to the $(2,1)$-Fine and $(3,2)$-Fine cases.

Theorem 4.2. *Fix $k \geq 2$ and $1 \leq r \leq k-1$. When ranging over all paths $P \in \mathcal{D}_n^k$, then the total number of instances of U^rD_{k-1} ending at height* 0 *is*

$$[t^n]\, tC_k(t)^{k-r+1} = \frac{k-r+1}{kn-r+1}\binom{kn-r+1}{n-1}.$$

Proof. Via the Fundamental Theorem of Riordan Arrays, a generating function for the numbers of required sub-paths is obtained by multiplying $\widetilde{M}^{k,r}$ by the infinite column vector corresponding to $g(t) = 0 + t + 2t^2 + 3t^3 + \cdots = \frac{t}{(1-t)^2}$:

$$\mathcal{R}(F_{k,r}(t), tC_k(t)^{k-1-r}F_{k,r}(t)) \cdot \left(\frac{t}{(1-t)^2}\right) = \frac{F_{k,r}(t)\, tC_k(t)^{k-1-r}F_{k,r}(t)}{\left(1 - tC_k(t)^{k-1-r}F_{k,r}(t)\right)^2}.$$

Applying the identity $F_{k,r}(t) = \frac{C_k(t)}{1+tC_k(t)^{k-r}}$ of Proposition 2.3 allows the simplification

$$= \frac{tC_k(t)^{k-1-r}\left(\frac{C_k(t)}{1+tC_k(t)^{k-r}}\right)^2}{\left(1 - tC_k(t)^{k-1-r}\left(\frac{C_k(t)}{1+tC_k(t)^{k-r}}\right)\right)^2} = \frac{\left(\frac{tC_k(t)^{k-r+1}}{\left(1+tC_k(t)^{k-r}\right)^2}\right)}{\left(\frac{\left(1+tC_k(t)^{k-r}-tC_k(t)^{k-r}\right)^2}{\left(1+tC_k(t)^{k-r}\right)^2}\right)} = tC_k(t)^{k-r+1}.$$

The closed formula of the theorem follows from the Raney identity $[t^n]C_k(t)^\ell = \frac{\ell}{kn+\ell}\binom{kn+\ell}{n}$. □

The Riordan array $\widetilde{M}^{k,r}$ may also be used to derive formulas for the number of k-Dyck paths with either an even number or odd number of relevant sub-paths U^rD_{k-1}. To that end, we define $\mathcal{P}^{k,r}_{n,even} = \bigcup_{i\geq 0}\mathcal{P}^{k,r}_{n,2i}$ and $\mathcal{P}^{k,r}_{n,odd} = \bigcup_{i\geq 0}\mathcal{P}^{k,r}_{n,2i+1}$. Then let $|\mathcal{P}^{k,r}_{n,even}| = P^{k,r}_{n,even}$ and $|\mathcal{P}^{k,r}_{n,odd}| = P^{k,r}_{n,odd}$. Note that Theorem 4.3 once again represents a direct generalization of Cameron and McLeod [4].

Theorem 4.3. *Fix $k \geq 2$ and $1 \leq r \leq k-1$. Then the number of k-Dyck paths $P \in \mathcal{D}_n^k$ with an even number of instances of U^rD_{k-1} ending at height* 0 *is*

$$P^{k,r}_{n,even} = [t^n]\left(\frac{C_k(t) + tC_k(t)^{k-r+1}}{1 + 2tC_k(t)^{k-r}}\right).$$

Alternatively, the number of k-Dyck paths $P \in \mathcal{D}_n^k$ with an odd number of instances of $U^r D_{k-1}$ ending at height 0 *is*

$$P_{n,odd}^{k,r} = [t^n]\left(\frac{tC_k(t)^{k-r+1}}{1+2tC_k(t)^{k-r}}\right).$$

Proof. A generating function for the number of k-Dyck paths with an even number of required sub-paths is obtained by multiplying the Riordan array $\widetilde{M}^{k,r}$ by the infinite column vector corresponding to $g_e(t) = 1 + t^2 + t^4 + \cdots = \frac{1}{1-t^2}$:

$$\mathcal{R}(F_{k,r}(t), tC_k(t)^{k-1-r}F_{k,r}(t)) \cdot \left(\frac{1}{1-t^2}\right) \;=\; \frac{F_{k,r}(t)}{1-(tC_k(t)^{k-1-r}F_{k,r}(t))^2}.$$

Using the identity of Proposition 2.3 then allows the simplification

$$\begin{aligned} = \frac{\left(\frac{C_k(t)}{1+tC_k(t)^{k-r}}\right)}{1-\left(tC_k(t)^{k-1-r}\left(\frac{C_k(t)}{1+tC_k(t)^{k-r}}\right)\right)^2} \;&=\; \frac{\left(\frac{C_k(t)}{1+tC_k(t)^{k-r}}\right)}{\left(\frac{(1+tC_k(t)^{k-r})^2-t^2C_k(t)^{2k-2r}}{(1+tC_k(t)^{k-r})^2}\right)} \\ &=\; \frac{C_k(t)(1+tC_k(t)^{k-r})}{1+2tC_k(t)^{k-r}}. \end{aligned}$$

The enumeration of "odd-instance paths" may be pursued equivalently, multiplying $\widetilde{M}^{k,r}$ by the vector corresponding to $g_o(t) = t + t^3 + t^5 + \cdots = \frac{t}{1-t^2}$. However, it is easier to simply subtract the number of "even-instance paths" from $C_k(t)$. □

5 Asymptotic Results

In this final section, we investigate statistics that hold for (k,r)-Fine paths in the limit of $n \to \infty$. This will result in generalizations of various asymptotic results from Deutsch and Shapiro [9] and Cameron and McLeod [4], two of which we have already presented in Section 1 as (9) and (10).

As our upcoming results make frequent usage of descending factorials, we pause to recap that concept here. So let x, n be non-negative integers. Then the associated descending factorial is the expression $(x)_n = x^{\underline{n}} = x(x-1)(x-2)\dots(x-n+1)$, where we set the result to 1 whenever $n = 0$. Notice that $x^{\underline{n}}$ is always a length-n product.

Descending factorials frequently appear in Taylor series. This is most evident in how repeated differentiation of $f(x) = \frac{1}{1-x}$ yields the well-known identity $\frac{n!}{(1-x)^{n+1}} = \sum_{i=n}^{\infty} i^{\underline{n}} x^{i-n}$. Less commonplace yet central to upcoming proofs is the identity below.

Lemma 5.1. *Fix $n \geq 0$. For all $|x| < 1$,*

$$\sum_{i=n}^{\infty} i\, i^{\underline{n}} x^{i-n} \;=\; \frac{n!(n+x)}{(1-x)^{n+2}}.$$

Proof. Temporarily denote $A_n = \sum_{i=n}^{\infty} i\, i^{\underline{n}} x^{i-n}$. It is clear that A_n converges for all $|x| < 1$. We look to show that $(1-x)A_n = \frac{n!(n+x)}{(1-x)^{n+1}}$. On the left side of this equation we have

$$(1-x)A_n \;=\; (1-x)\sum_{i=n}^{\infty} i\, i^{\underline{n}} x^{i-n} \;=\; \sum_{i=n}^{\infty} \left(i\, i^{\underline{n}} - (i-1)(i-1)^{\underline{n}}\right) x^{i-n}.$$

Applying the aforementioned identity $\frac{n!}{(1-x)^{n+1}} = \sum_{i=n}^{\infty} i^{\underline{n}} x^{i-n}$ then allows us to rewrite the right side of our desired equation as

$$\frac{n!(n+x)}{(1-x)^{n+1}} \;=\; (n+x)\sum_{i=n}^{\infty} i^{\underline{n}} x^{i-n} \;=\; \sum_{i=n}^{\infty} \left(n\, i^{\underline{n}} + (i-1)^{\underline{n}}\right) x^{i-n}.$$

For all $i \geq n$, manipulation of descending factorials then gives

$$\begin{aligned} i\, i^{\underline{n}} - (i-1)(i-1)^{\underline{n}} \;&=\; n\, i^{\underline{n}} + (i-n) i^{\underline{n}} - (i-1)(i-1)^{\underline{n}} \\ &=\; n\, i^{\underline{n}} + i(i-1)^{\underline{n}} - (i-1)(i-1)^{\underline{n}} \;=\; n\, i^{\underline{n}} + (i-1)^{\underline{n}}. \end{aligned}$$

□

We are now ready for our first major asymptotic result, which identifies the fraction of k-Dyck paths that feature precisely m minimal-height sub-paths of the form $U^r D_{k-1}$:

Theorem 5.2. *For all $k \geq 2$, $1 \leq r \leq k-1$, and $m \geq 0$, as $n \to \infty$ we approach*

$$\frac{P^{k,r}_{n,m}}{C^k_n} \;\sim\; \frac{k^r (k-1)^{(r-1)m}}{\left(k^r + (k-1)^{r-1}\right)^{m+2}} \left(k^r - (k-1)^r + r(k-1)^{r-1} + m(k-r)k^r\right).$$

Proof. Proposition 4.1 and the identity $F_{k,r}(t) = \frac{C_k(t)}{1+tC_k(t)^{k-r}}$ from Proposition 2.3 give

$$P^{k,r}_{n,m} \;=\; [t^n] t^m C_k(t)^{(k-1-r)m} \left(\frac{C_k(t)}{1+tC_k(t)^{k-r}}\right)^{m+1} \;=\; [t^n] \frac{t^m C_k(t)^{km-rm+1}}{\left(1+tC_k(t)^{k-r}\right)^{m+1}}.$$

The identity $\dfrac{n!}{(1-x)^{n+1}} = \displaystyle\sum_{i=n}^{\infty} i^{\underline{n}} x^{i-n}$ then allows us to rewrite our formula as an infinite sum:

$$P^{k,r}_{n,m} \;=\; [t^n] t^m C_k(t)^{km-rm+1} \frac{1}{m!} \sum_{i=m}^{\infty} i^{\underline{m}} \left(-tC_k(t)^{k-r}\right)^{i-m}$$

$$= \frac{1}{m!}\sum_{i=m}^{\infty} [t^n] t^i (-1)^{i-m} i^{\underline{m}} C_k(t)^{ki-ri+1} \;=\; \frac{1}{m!}\sum_{i=m}^{\infty} [t^{n-i}] (-1)^{i-m} i^{\underline{m}} C_k(t)^{ki-ri+1}.$$

Using the Fuss-Catalan identity $[t^n]C_k(t)^\ell = \frac{\ell}{kn+\ell}\binom{kn+\ell}{n}$, we derive a ratio of cardinalities:

$$\begin{aligned}
\frac{P^{k,r}_{n,m}}{C^k_n} &= \frac{\dfrac{1}{m!}\displaystyle\sum_{i=m}^{\infty}[t^{n-i}]\,(-1)^{i-m}\,i^{\underline{m}}C_k(t)^{ki-ri+1}}{[t^n]C_k(t)} \\
&= \frac{1}{m!}\sum_{i=m}^{\infty}\left((-1)^{i-m}\,i^{\underline{m}}\frac{ki-ri+1}{kn-ri+1}\binom{kn-ri+1}{n-i}\right)\Big/\left(\frac{1}{kn+1}\binom{kn+1}{n}\right) \\
&= \frac{1}{m!}\sum_{i=m}^{\infty}\left((-1)^{i-m}\,i^{\underline{m}}(ki-ri+1)\,\frac{(kn-ri)!}{(n-i)!(kn-ri+1-n+i)!}\right)\Big/\left(\frac{(kn)!}{n!(kn-n+1)!}\right) \\
&= \frac{1}{m!}\sum_{i=m}^{\infty}(-1)^{i-m}\,i^{\underline{m}}(ki-ri+1)\,\frac{n^{\underline{i}}(kn-n+1)^{\underline{ri-i}}}{(kn)^{\underline{ri}}}.
\end{aligned}$$

We now wish to take the limit $n \to \infty$ of our summation. By Tannery's Theorem, the equality $\lim_{n\to\infty}\sum_{i=n}^{\infty} a_i(n) = \sum_{i=1}^{\infty}\lim_{n\to\infty} a_i(n) = \sum_{i=1}^{\infty} b_i$ holds if the resulting sequence of the b_i is absolutely convergent. To that end, notice

$$\begin{aligned}
&\lim_{n\to\infty}\left((-1)^{i-m}\,i^{\underline{m}}(ki-ri+1)\,\frac{n^{\underline{i}}(kn-n+1)^{\underline{ri-i}}}{(kn)^{\underline{ri}}}\right) \\
&= (-1)^{i-m}\,i^{\underline{m}}(ki-ri+1)\,\frac{1^i\,(k-1)^{ri-i}}{k^{ri}} \\
&= (-1)^{-m}\,i^{\underline{m}}((k-r)i+1)\left(\frac{-(k-1)^{r-1}}{k^r}\right)^i = b_i.
\end{aligned}$$

As $i^{\underline{m}}((k-r)i+1)$ is a fixed-degree polynomial with respect to i, and since $|\frac{-(k-1)^{r-1}}{k^r}| < 1$ for all $k \geq 2$ and $1 \leq r \leq k-1$, the Ratio Test guarantees that the sequence of the b_i is absolutely convergent for all relevant k, r. Thus

$$\lim_{n\to\infty}\frac{P^{k,r}_{n,m}}{C^k_n} = \frac{1}{m!}\sum_{i=m}^{\infty}(-1)^{-m}\,i^{\underline{m}}((k-r)i+1)\left(\frac{-(k-1)^{r-1}}{k^r}\right)^i.$$

All that remains is to evaluate this series. We expand into a sum of two series by splitting the term $(k-r)i+1$, and consider the two series separately.

The second series from the resulting expression merely requires another application of the identity $\sum_{i=n}^{\infty} i^{\underline{n}}x^{i-n} = \frac{n!}{(1-x)^{n+1}}$. Noting $|x| < 1$ for all k, r, when $x = \frac{-(k-1)^{r-1}}{k^r}$ we have

$$\begin{aligned}
&\frac{1}{m!}\sum_{i=m}^{\infty}(-1)^{-m}\,i^{\underline{m}}\left(\frac{-(k-1)^{r-1}}{k^r}\right)^i \\
&= \frac{(-1)^{-m}}{m!}\left(\frac{-(k-1)^{r-1}}{k^r}\right)^m\sum_{i=m}^{\infty} i^{\underline{m}}\left(\frac{-(k-1)^{r-1}}{k^r}\right)^{i-m}
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{m!}\left(\frac{(k-1)^{r-1}}{k^r}\right)^m \left(\frac{m!}{\left(1+\frac{(k-1)^{r-1}}{k^r}\right)^{m+1}}\right) \\ &= \frac{k^r\,(k-1)^{(r-1)m}}{(k^r+(k-1)^{r-1})^{m+1}}. \end{aligned}$$

The first series from our expression can manipulated into the form from Lemma 5.1. Here we again make use of the fact that $|x| = |\frac{-(k-1)^{r-1}}{k^r}| < 1$ for all k, r:

$$\begin{aligned} &\frac{1}{m!}\sum_{i=m}^{\infty}(-1)^{-m}i^{\underline{m}}(k-r)i\left(\frac{-(k-1)^{r-1}}{k^r}\right)^i \\ &= \frac{((-1)^m(k-r)}{m!}\left(\frac{-(k-1)^{r-1}}{k^r}\right)^m \sum_{i=m}^{\infty} i\,i^{\underline{m}}\left(\frac{-(k-1)^{r-1}}{k^r}\right)^{i-m} \\ &= \frac{(k-r)(k-1)^{(r-1)m}}{m!\,k^{rm}}\left(\frac{m!\left(m-\frac{(k-1)^{r-1}}{k^r}\right)}{\left(1+\frac{(k-1)^{r-1}}{k^r}\right)^{m+2}}\right) \\ &= (k-r)(k-1)^{(r-1)m}k^r\frac{(mk^r-(k-1)^{r-1})}{(k^r+(k-1)^{r-1})^{m+2}}. \end{aligned}$$

Combining our two evaluations yields an expression equivalent to that of the theorem statement:

$$\begin{aligned} \lim_{n\to\infty}\frac{P^{k,r}_{n,m}}{C^k_n} &= \frac{k^r\,(k-1)^{(r-1)m}}{(k^r+(k-1)^{r-1})^{m+1}} + (k-r)(k-1)^{(r-1)m}k^r\frac{(mk^r-(k-1)^{r-1})}{(k^r+(k-1)^{r-1})^{m+2}} \\ &= \frac{k^r(k-1)^{(r-1)m}}{(k^r+(k-1)^{r-1})^{m+2}}\left(k^r+(k-1)^{r-1}+(k-r)(mk^r-(k-1)^{r-1})\right). \end{aligned}$$

□

Setting $m = 0$ in Theorem 5.2 then yields the asymptotic ratio of (k, r)-Fine paths to k-Dyck paths, as given below in Theorem 5.3. Notice that the $(k, r) = (2, 1)$ case of this theorem directly recovers (9), as originally proven by Deutsch and Shapiro [9]. See Table 2 for all relevant specializations of Theorem 5.3 with $k \leq 7$.

Theorem 5.3. *For all $k \geq 2$ and $1 \leq r \leq k-1$, as $n \to \infty$ we approach*

$$\frac{F^{k,r}_n}{C^k_n} \sim \frac{k^r\,(k^r-(k-1)^r+r\,(k-1)^{r-1})}{(k^r+(k-1)^{r-1})^2}.$$

With help from Stirling's approximation, Theorem 5.3 gives a closed formula approximation of $F^{k,r}_n$ as $n \to \infty$. Notice that the $(k, r) = (2, 1)$ case of this formula corresponds to (10), an independent proof of which may be found in Deutsch and Shapiro [9].

	r = 1	**r = 2**	**r = 3**	**r = 4**	**r = 5**	**r = 6**
k = 2	$\frac{4}{9} \approx 0.4444$					
k = 3	$\frac{3}{8} = 0.3750$	$\frac{81}{121} \approx 0.6694$				
k = 4	$\frac{8}{25} = 0.3200$	$\frac{208}{361} \approx 0.5762$	$\frac{4096}{5329} \approx 0.7686$			
k = 5	$\frac{5}{18} \approx 0.2778$	$\frac{425}{841} \approx 0.5054$	$\frac{13625}{19881} \approx 0.6853$	$\frac{390625}{474721} \approx 0.8229$		
k = 6	$\frac{12}{49} \approx 0.2449$	$\frac{756}{1681} \approx 0.4497$	$\frac{35856}{58081} \approx 0.6173$	$\frac{1517616}{2019241} \approx 0.7516$	$\frac{60466176}{70576801} \approx 0.8567$	
k = 7	$\frac{7}{32} \approx 0.2186$	$\frac{49}{121} \approx 0.4050$	$\frac{80605}{143641} \approx 0.5612$	$\frac{4727569}{6848689} \approx 0.6903$	$\frac{260693377}{327718609} \approx 0.7955$	$\frac{13841287201}{15731430625} \approx 0.8799$

Table 2: Asymptotic ratios $\dfrac{F_n^{k,r}}{C_n^k}$, as given by Theorem 5.3.

Corollary 5.4. *For all $k \geq 2$ and $1 \leq r \leq k-1$, as $n \to \infty$ we approach*

$$F_n^{k,r} \;\sim\; \frac{k^r\left(k^r-(k-1)^r+r(k-1)^{r-1}\right)}{\left(k^r+(k-1)^{r-1}\right)^2\sqrt{2\pi k(k-1)n^3}}\left(\frac{k^k}{(k-1)^{k-1}}\right)^n.$$

Proof. Using $C_n^k = \frac{1}{kn+1}\binom{kn+1}{n}$ and Stirling's approximation $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, it is straightforward to derive the following formula for C_n^k as $n \to \infty$. The result then follows from Theorem 5.3.

$$C_n^k \;\sim\; \frac{1}{\sqrt{2\pi k(k-1)n^3}}\left(\frac{k^k}{(k-1)^{k-1}}\right)^n.$$

□

Also following from Theorem 5.3 is a formula for the ratio for consecutive (k,r)-Fine numbers as $n \to \infty$. Interestingly, this result depends only upon k.

Corollary 5.5. *For all $k \geq 2$ and $1 \leq r \leq k-1$, as $n \to \infty$ we approach*

$$\frac{F_n^{k,r}}{F_{n+1}^{k,r}} \;\sim\; \frac{(k-1)^{k-1}}{k^k}.$$

Proof. Follows directly from Corollary 5.4, with $\frac{n}{n+1} \to 1$ as $n \to 1$. □

We now move on to a distinct asymptotic result that relates to our enumeration of "even instance" and "odd instance" paths in Theorem 4.3. Note that the $(k,r) = (2,1)$ and $(k,r) = (3,2)$ specializations of Theorem 5.6 recover one of the core results from Cameron and McLeod [4], and that our proof is modeled after the methodologies of that paper.

Theorem 5.6. *Fix $k \geq 2$ and $1 \leq r \leq k-1$. As $n \to \infty$, we approach*

$$\frac{P_{n,even}^{k,r}}{C_n^k} \;\sim\; \frac{2(k-1)^{2r-2}+(r+3)k^r(k-1)^{r-1}-k^{r+1}(k-1)^{r-1}+k^{2r}}{\left(k^r+2(k-1)^{r-1}\right)^2}.$$

Proof. In light of Theorem 5.2, we merely need to simplify the summation below:

$$\frac{P^{k,r}_{n,even}}{C^k_n} \sim \sum_{m\geq 0\ even} \frac{k^r\,(k-1)^{(r-1)m}}{(k^r+(k-1)^{r-1})^{m+2}} \left(k^r-(k-1)^r+r\,(k-1)^{r-1}+m\,(k-r)\,k^r\right).$$

To accomplish this, we split our summation in two:

$$= \sum_{m\geq 0\ even} \frac{k^r\,(k^r-(k-1)^r+r\,(k-1)^{r-1})}{(k^r+(k-1)^{r-1})^2}\left(\frac{(k-1)^{r-1}}{k^r+(k-1)^{r-1}}\right)^m$$

$$+ \sum_{m\geq 0\ even} \frac{k^{2r}\,(k-r)m}{(k^r+(k-1)^{r-1})^2}\left(\frac{(k-1)^{r-1}}{k^r+(k-1)^{r-1}}\right)^m \;=\; \mathcal{S}_1+\mathcal{S}_2.$$

Begin by noting $(k-1)^{r-1} < (k-1)^{r-1}+k^r$ for all $k\geq 2, r\geq 1$, implying $|x| = |\frac{(k-1)^{r-1}}{k^r+(k-1)^{r-1}}| < 1$ for all $k\geq 2, r\geq 1$. This guarantees that both series above actually converge.

For the first series $\mathcal{S}_1$, we use the standard identity $\displaystyle\sum_{m\geq 0\ even} x^m = \frac{1}{1-x^2}$ to give

$$\mathcal{S}_1 \;=\; \frac{k^r\,(k^r-(k-1)^r+r\,(k-1)^{r-1})}{(k^r+(k-1)^{r-1})^2}\left(\frac{1}{1-\left(\frac{(k-1)^{r-1}}{k^r+(k-1)^{r-1}}\right)^2}\right)$$

$$= \frac{k^r\,(k^r+(1-k+r)(k-1)^{r-1})}{(k^r+(k-1)^{r-1})^2-(k-1)^{2(r-1)}} \;=\; \frac{k^r-(k-r-1)(k-1)^{r-1}}{k^r+2(k-1)^{r-1}}.$$

On to the series $\mathcal{S}_2$, the identity $\displaystyle\sum_{m\geq 0} m x^m = \frac{x}{(1-x)^2}$ gives $\displaystyle\sum_{m\geq 0\ even} m x^m = \frac{2x^2}{(1-x^2)^2}$ and

$$\mathcal{S}_2 \;=\; \frac{k^{2r}\,(k-r)}{(k^r+(k-1)^{r-1})^2}\left(\frac{2\left(\frac{(k-1)^{r-1}}{k^r+(k-1)^{r-1}}\right)^2}{\left(1-\left(\frac{(k-1)^{r-1}}{k^r+(k-1)^{r-1}}\right)^2\right)^2}\right)$$

$$= \frac{2\,k^{2r}\,(k-r)(k-1)^{2(r-1)}}{\left((k^r+(k-1)^{r-1})^2-(k-1)^{2(r-1)}\right)^2} \;=\; \frac{2\,(k-r)(k-1)^{2r-2}}{\left(k^r+2\,(k-1)^{r-1}\right)^2}.$$

Summing $\mathcal{S}_1+\mathcal{S}_2$ may then be reduced to the required expression:

$$\mathcal{S}_1+\mathcal{S}_2 \;=\; \frac{k^r-(k-r-1)(k-1)^{r-1}}{k^r+2(k-1)^{r-1}} + \frac{2\,(k-r)(k-1)^{2r-2}}{\left(k^r+2\,(k-1)^{r-1}\right)^2}$$

$$= \frac{(k^r-(k-r-1)(k-1)^{r-1})\,(k^r+2\,(k-1)^{r-1})+2\,(k-r)(k-1)^{2r-2}}{\left(k^r+2\,(k-1)^{r-1}\right)^2}.$$

□

For most choices of $1 \leq r \leq k-1$, the ratio of Theorem 5.6 does not significantly simplify, but this is not the case for $r = 1$. The $r = k-1$ case also admits a slight simplification. Since $r = 1$ and $r = k-1$ were the two sub-classes of (k,r)-Fine paths that received additional treatment in Section 2, we present those specializations below. We also include Table 3 for an explicit calculation of all asymptotic ratios with $k \leq 7$.

Corollary 5.7. *Fix $k \geq 2$. As $n \to \infty$, the proportion of k-Dyck paths in $\mathcal{D}_n^k$ with an even number of minimal height sub-paths of the form $U^1 D_{k-1}$ approaches*

$$\frac{P_{n,even}^{k,1}}{C_n^k} \sim \frac{4k+2}{(k+2)^2}.$$

Similarly, as $n \to \infty$, the proportion of k-Dyck paths in $\mathcal{D}_n^k$ with an even number of minimal height sub-paths of the form $U^{k-1} D_{k-1}$ approaches

$$\frac{P_{n,even}^{k,k-1}}{C_n^k} \sim \frac{k^{2k-2} + 2k^{k-1}(k-1)^{k-2} + 2(k-1)^{2k-4}}{\left(k^{k-1} + 2(k-1)^{k-2}\right)^2}.$$

	r = 1	**r = 2**	**r = 3**	**r = 4**	**r = 5**	**r = 6**
k = 2	$\frac{5}{8} = 0.6250$					
k = 3	$\frac{14}{25} = 0.5600$	$\frac{125}{169} \approx 0.7396$				
k = 4	$\frac{1}{2} = 0.5000$	$\frac{161}{242} \approx 0.6653$	$\frac{2705}{3362} \approx 0.8046$			
k = 5	$\frac{22}{49} \approx 0.4490$	$\frac{73}{121} \approx 0.6033$	$\frac{18137}{24649} \approx 0.7358$	$\frac{478817}{567009} \approx 0.8445$		
k = 6	$\frac{13}{32} \approx 0.4063$	$\frac{583}{1058} \approx 0.5510$	$\frac{23953}{35378} \approx 0.6771$	$\frac{936433}{1195058} \approx 0.7836$	$\frac{35483713}{40734338} \approx 0.8711$	
k = 7	$\frac{10}{27} \approx 0.3704$	$\frac{1885}{3721} \approx 0.5066$	$\frac{107893}{172225} \approx 0.6265$	$\frac{5858113}{8025889} \approx 0.7299$	$\frac{307616353}{376321201} \approx 0.8174$	$\frac{15791896801}{17742506401} \approx 0.8901$

Table 3: Asymptotic ratios $\frac{P_{n,even}^{k,r}}{C_n^k}$, as given by Theorem 5.6.

For our final set of asymptotic results, we derive formulas for the expected number of minimal height sub-paths of the form $U^r D_{k-1}$ in a k-Dyck path as $n \to \infty$, along with the asymptotic variance of that statistic. Here we once again follow the methodologies of Cameron and McLeod [4], whose core result corresponds to the $(k,r) = (3,2)$ case of Theorem 5.8. The $(k,r) = (3,2)$ specialization of Theorem 5.8 similarly yields the primary result from Section 3 of Prodinger [17].

After the proof of Theorem 5.8, we include Table 4 for an explicit calculation of expected values for all (k,r) with $k \leq 7$.

Theorem 5.8. *Fix $k \geq 2$ and $1 \leq r \leq k-1$. For any $P \in \mathcal{D}_n^k$, let X denote the number of sub-paths in P of the form $U^r D_{k-1}$ that end at height 0. Ranging over all $P \in \mathcal{D}_n^k$, the mean and variance of X approach the following values as $n \to \infty$:*

$$E(X) \sim \frac{k-r+1}{k}\left(\frac{k-1}{k}\right)^{r-1},$$

$$Var(X) \sim \left(\frac{(k-r+1)k^{r-1} + (2k-2r+1-k^2+2kr-r^2)(k-1)^{r-1}}{k^{r+1}}\right)\left(\frac{k-1}{k}\right)^{r-1}.$$

Proof. For any finite $n \geq 0$, let $E_n(X)$ denote the expected number of relevant sub-paths across all $P \in \mathcal{D}_n^k$. This value $E_n(X)$ is the sum total of all instances of X in $\mathcal{D}_n^k$, divided by the cardinality of $\mathcal{D}_n^k$. Utilizing Theorem 4.2 gives

$$E_n(X) \;=\; \frac{[t^n]\,tC_k(t)^{k-r+1}}{[t^n]C_k(t)} \;=\; \frac{[t^{n-1}]\,C_k(t)^{k-r+1}}{[t^n]C_k(t)}.$$

Applying the Raney number identity $[t^n]C_k(t)^\ell = \frac{\ell}{kn+\ell}\binom{kn+\ell}{n}$, this ratio becomes

$$\begin{aligned}
E_n(X) \;&=\; \frac{\frac{k-r+1}{k(n-1)+k-r+1}\binom{k(n-1)+k-r+1}{n-1}}{\frac{1}{kn+1}\binom{kn+1}{n}} \\
&=\; \frac{(k-r+1)(kn+1)\,n!\,(kn-r+1)!\,(kn-n+1)!}{(kn-r+1)\,(n-1)!\,(kn+1)!\,(kn-n-r+2)!} \\
&=\; \frac{(k-r+1)\,n\,(kn-r)!\,(kn-n+1)!}{(kn)!\,(kn-n-r+2)!} \\
&=\; (k-r+1)\,\frac{n}{kn}\,\frac{(kn-r)!}{(kn-1)!}\,\frac{(kn-n+1)!}{(kn-n-r+2)!} \\
&=\; \frac{k-r+1}{k}\,\frac{(kn-n+1)^{\underline{r-1}}}{(kn-1)^{\underline{r-1}}}.
\end{aligned}$$

Taking the limit $n \to \infty$ then gives our first asymptotic result:

$$E_n(X) \;\sim\; \lim_{n\to\infty}\left(\frac{k-r+1}{k}\,\frac{((k-1)n+1)^{\underline{r-1}}}{(kn-1)^{\underline{r-1}}}\right) \;=\; \frac{k-r+1}{k}\left(\frac{k-1}{k}\right)^{r-1}.$$

On to the asymptotic variance, we begin by recalling that $\mathrm{Var}(X) = E(X)^2 - E(X^2)$. Thus we merely require an asymptotic formula for $E(X^2)$.

To that end, we wish to derive a generating function where k-Dyck paths are weighted according to the square of the number of relevant sub-paths U^rD_{k-1}. This may be accomplished via multiplication of the Riordan array $\widetilde{M}^{k,r}$ by $g(t) = 0 + 1^2t + 2^2t^2 + 3^2t^3 + \cdots = \frac{t+t^2}{(1-t)^3}$. The Fundamental Theorem of Riordan Arrays then gives

$$\begin{aligned}
&\mathcal{R}(F_{k,r}(t), tC_k(t)^{k-1-r}F_{k,r}(t)) \cdot \left(\frac{t+t^2}{(1-t)^3}\right) \\
&\quad= \frac{F_{k,r}(t)\,tC_k(t)^{k-1-r}\,F_{k,r}(t)\,\left(1 + tC_k(t)^{k-1-r}\,F_{k,r}(t)\right)}{(1 - tC_k(t)^{k-1-r}\,F_{k,r}(t))^3}.
\end{aligned}$$

Using the identity from Proposition 2.3 allows us to simplify:

$$= \quad \frac{tC_k(t)^{k-1-r}\left(\frac{C_k(t)}{1+tC_k(t)^{k-r}}\right)^2\left(1 + tC_k(t)^{k-1-r}\left(\frac{C_k(t)}{1+tC_k(t)^{k-r}}\right)\right)}{\left(1 - tC_k(t)^{k-1-r}\left(\frac{C_k(t)}{1+tC_k(t)^{k-r}}\right)\right)^3}$$

$$= \frac{\frac{tC_k(t)^{k-r+1}}{\left(1+tC_k(t)^{k-r}\right)^2}\left(\frac{1+tC_k(t)^{k-r}+tC_k(t)^{k-r}}{1+tC_k(t)^{k-r}}\right)}{\left(\frac{1+tC_k(t)^{k-r}-tC_k(t)^{k-r}}{1+tC_k(t)^{k-r}}\right)^3} = tC_k(t)^{k-r+1} + 2t^2C_k(t)^{2(k-r)+1}.$$

If we define $E_n(X^2)$ to be the expected value for the square of the number of relevant sub-paths across all $P \in \mathcal{D}_n^k$, the generating function above may be applied to give

$$E_n(X^2) = \frac{[t^n]\left(tC_k(t)^{k-r+1} + 2t^2C_k(t)^{2(k-r)+1}\right)}{[t^n]C_k(t)} = E_n(X) + \frac{[t^{n-2}]2C_k(t)^{2(k-r)+1}}{[t^n]C_k(t)}.$$

Using the identity $[t^n]C_k(t)^\ell = \frac{\ell}{kn+\ell}\binom{kn+\ell}{n}$ and proceeding similarly to calculations for $E_n(X)$:

$$\begin{aligned}
E_n(X^2) &= E_n(X) + \frac{\frac{2(2(k-r)+1)}{k(n-2)+2(k-r)+1}\binom{k(n-2)+2(k-r)+1}{n-2}}{\frac{1}{kn+1}\binom{kn+1}{n}} \\
&= E_n(X) + \frac{2(2k-2r+1)(kn+1)(kn-2r+1)!\,n!\,(kn-n+1)!}{(kn-2r+1)(n-2)!\,(kn-n-2r+3)!\,(kn+1)!} \\
&= E_n(X) + \frac{(4k-4r+2)(n)(n-1)}{(kn)(kn-1)}\,\frac{(kn-2r)!}{(kn-2)!}\,\frac{(kn-n+1)!}{(kn-n-2r+3)!} \\
&= E_n(X) + \frac{(4k-4r+2)(n^2-n)}{(k^2n^2-kn)}\,\frac{((k-1)n+1)^{\underline{2r-2}}}{(kn-2)^{\underline{2r-2}}}.
\end{aligned}$$

Letting $n \to \infty$ and inserting our earlier result for $E(X)$:

$$\begin{aligned}
E(X^2) &\sim E(X) + \lim_{n\to\infty}\left(\frac{(4k-4r+2)(n^2-n)}{(k^2n^2-kn)}\,\frac{((k-1)n+1)^{\underline{2r-2}}}{(kn-2)^{\underline{2r-2}}}\right) \\
&= \frac{k-r+1}{k}\left(\frac{k-1}{k}\right)^{r-1} + \frac{4k-4r+2}{k^2}\left(\frac{k-1}{k}\right)^{2r-2}.
\end{aligned}$$

□

6 Future Investigations

As seen with the determinant formulas of Subsection 2.4, there exist fundamental results about Fine numbers that resist easy generalization to (k,r)-Fine numbers. Below is a more detailed account of several such results, presented alongside other avenues for future investigation.

- Deutsch and Shapiro [9] derive the "internal recurrence" below, which holds for all $n \geq 2$:

	r = 1	**r = 2**	**r = 3**	**r = 4**	**r = 5**	**r = 6**
k = 2	1					
k = 3	1	$\frac{4}{9} \approx 0.4444$				
k = 4	1	$\frac{9}{16} = 0.5625$	$\frac{9}{32} \approx 0.2813$			
k = 5	1	$\frac{16}{25} = 0.6400$	$\frac{48}{125} = 0.3840$	$\frac{128}{625} = 0.2048$		
k = 6	1	$\frac{25}{36} \approx 0.6944$	$\frac{25}{54} \approx 0.4630$	$\frac{125}{432} \approx 0.2894$	$\frac{625}{3888} \approx 0.1608$	
k = 7	1	$\frac{36}{49} \approx 0.7347$	$\frac{180}{343} \approx 0.5248$	$\frac{864}{2401} \approx 0.3599$	$\frac{3888}{16807} \approx 0.2313$	$\frac{15552}{117649} \approx 0.1322$

Table 4: Average number $E(X)$ of minimal height sub-paths $U^r D_{k-1}$ in a k-Dyck path $P \in \mathcal{D}_n^k$ as $n \to \infty$, as given by Theorem 5.8.

$$2(n+1)F_n = (7n-5)F_{n-1} + 2(2n-1)F_{n-2}.$$

This recurrence is closely related to the formula $2F_n + F_{n-1} = C_n$, which itself may be derived via manipulation of basic generating function relations (2). Do there exist internal recurrences of this type for (k,r)-Fine numbers when $k > 2$? Note that the methodology leading to the specific recurrence above appears to require $(k,r) = (2,1)$.

- On a related note, Deutsch and Shapiro [9] introduce a “hill-killer involution” for $\mathcal{D}_n^2$. This map is used to derive the equality $2F_n + F_{n-1} = C_n$ mentioned above. Do there exist “generalized hill-killer involutions” that can be used to identify (k,r)-Fine paths when $k > 2$?

- In Subsection 2.4, we saw that the Fine number determinant identities of (8) failed to generalize to (k,r)-Fine numbers. That being said, it may still be possible to derive semi-succinct formulas for the determinants of the Hankel matrices $H_n(k,r)$ and $H'_n(k,r)$ when $k > 2$. Such formulas would need to specialize to the seemingly random, non-OEIS sequences provided at the end of Subsection 2.4. More generally, do there exist elegant formulas for the Hankel transforms of $\{F_n^{k,r}\}_{n \geq i}$, for at least some choices of (k,r) with $k > 2$, and at least some choices of $0 \leq i \leq k-1$?

- Is there an approachable generalization of the results in this paper to generalized k-Dyck paths that both begin and end at a non-zero height? See Drube [11] for a possible starting point in this investigation.

References

[1] M. Aigner, Catalan-like numbers and determinants, *J. Combin. Theory Ser. A* **87** (1999), 33–51.

[2] M. Aigner, Enumeration via ballot numbers, *Discrete Math.* **308** (2008), 2544–2563.

[3] E. Barcucci, E. Pergola, R. Pinzani and R. Rinaldi, ECO method and hill-free generalized Motzkin paths, *Sém. Lothar. Combin.* **46** (2001/2), Article B46b.

[4] N. Cameron and J. McLeod, Returns and hills on generalized Dyck paths, *J. Integer Sequences* **19** (2016), Article 16.6.1.

[5] G. Cheon, S. Lee and L. Shapiro, The Fine numbers refined, *European J. Combin.* **31** (2010), 120–128.

[6] I. DeJager, M. Naquin, F. Seidl and P. Drube, Colored Motzkin Paths of Higher Order, *J. Integer Sequences* **24** (2021), Article 21.4.6.

[7] N. Dershowitz, Nonleaf patterns in trees: protected nodes and Fine numbers, *J. Integer Sequences* **23** (2020), Article 20.7.7.

[8] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999), 167–202.

[9] E. Deutsch and L. W. Shapiro, A survey of the Fine numbers, *Discrete Math.* **241** (2001), 241–265.

[10] P. Drube, Generalized path pairs and Fuss-Catalan triangles, *Australas. J. Combin.* **80** (1) (2021), 143–156.

[11] P. Drube, Raised k-Dyck paths, *J. Integer Sequences* **26** (2023), Article 23.6.7.

[12] S. Heubach, N. Y. Li and T. Mansour, Staircase tilings and k-Catalan structures, *Discrete Math.* **308** (2008), 5954–5964.

[13] P. Hilton and J. Pedersen, Catalan numbers, their generalizations, and their uses, *Math. Intelligencer* **13** (1991), 64–75.

[14] M. Janjić, On enumeration of Dyck paths with colored hills, *J. Integer Sequences* **21** (2018), Article 18.9.7.

[15] D. Merlini, D. G. Rogers, R. Sprugnoli and M. C. Verri, On some alternative characterizations of Riordan arrays, *Canad. J. Math.* **49** (2) (1997), 301–320.

[16] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, `https://oeis.org` (2019).

[17] H. Prodinger, Returns, hills, and t-ary trees, *J. Integer Sequences* **19** (2016), Article 16.7.2.

[18] D. G. Rogers, Pascal triangles, Catalan numbers, and renewal arrays, *Discrete Math.* **22** (1978), 301–310.

[19] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, 2015.

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