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The crossing number of the strong product of a path and a cycle

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Abstract

Let $P_m \boxtimes C_n$ be the strong product on P_m and C_n, where P_m and C_n are the path and the cycle on m and n vertices, respectively. In 2018 Ouyang, Wang, and Huang conjectured that the crossing number of $P_m \boxtimes C_n$ is $(m-1)n$ for $m \geq 3$ and $n \geq 3$. In this paper we confirm the correctness of the conjecture, and we determine the crossing number of a spanning graph of $P_m \boxtimes C_n$.

1 Introduction

One of the concepts characterizing how far a non-planar graph is from being a planar graph is its crossing number. Apart from its graphic interest, crossing number finds applications in other fields, such as VLSI design [1], discrete geometry [4]. However, determining the crossing number of a graph is not easy work. For general graphs, Garey and Johnson [3] have proved that it is NP-complete to determine whether the crossing number of a graph is at most k. For more about the crossing number of graphs, one can refer to [2,8].

Let G_1 and G_2 be two disjoint graphs. The strong product on G_1 and G_2, denoted by $G_1 \boxtimes G_2$, is the graph with vertex set $\{(x,y) : x \in V(G_1) \quad \text{and} \quad y \in V(G_2)\}$ and edge set $\{(x_1,y_1)(x_2,y_2) : x_1 = x_2 \quad \text{and} \quad y_1y_2 \in E(G_2) \text{ or } \quad y_1 = y_2 \quad \text{and} \quad x_1x_2 \in E(G_1) \text{ or } \quad x_1x_2 \in E(G_1) \quad \text{and} \quad y_1y_2 \in E(G_2)\}$.

Let P_s be the path of s vertices, where $s \geq 1$. Let C_t be the cycle of t vertices, where $t \geq 3$. In 2013, Klešč et al. [5] studied the crossing number of the strong product on two paths. They showed that the crossing number of $P_3 \boxtimes P_n$ is $n-3$ for $n \geq 3$, and they conjectured that the crossing number of $P_m \boxtimes P_n$ is $(m-1)(n-1)-4$ for $m, n \geq 4$. The author [6] proved that the conjecture is true except for $m = 4$ and $n = 4$. Ouyang, Wang, and Huang [7] explored the crossing number of $P_m \boxtimes C_n$ in 2018. They obtained the theorem below.

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Theorem 1.1 [7] *$cr(P_2 \boxtimes C_n) = n$ for $n \geq 3$.*

Subsequently, they proposed the following conjecture.

Conjecture 1.2 [7] *$cr(P_m \boxtimes C_n) = (m-1)n$ for $m \geq 3$ and $n \geq 3$.*

This paper confirms the correctness of Conjecture 1.2 in Section 2. In the paper, we always suppose that $P_m = u_1 u_2 \dots u_m$ and $C_n = v_1 v_2 \dots v_n v_1$, where $m \geq 2$ and $n \geq 3$. For $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, the vertex (u_i, v_j) is usually labeled by $z_{i,j}$. If $z_{i,1}z_{i+1,n}$ and $z_{i+1,1}z_{i,n}$ are removed from $P_m \boxtimes C_n$ for $i = 1, 2, \dots, m-1$, then we obtain a spanning subgraph of $P_m \boxtimes C_n$ which is denoted by $F_{m,n}$. We observe that $F_{m,n}$ is also a supergraph of $P_m \boxtimes P_n$. In Section 3, we intend to explore the crossing number of $F_{m,n}$ to understand the gap between $cr(F_{m,n})$ and $cr(P_m \boxtimes C_n)$ as well as the gap between $cr(F_{m,n})$ and $cr(P_m \boxtimes P_n)$.

The rest of this section deals with some terminology for crossing numbers and graph theory.

A *good drawing* of a graph G is a representation of G in the plane which satisfies the following conditions: (1) Each edge is represented by a simple continuous curve which does not pass through any point representing a vertex other than its ends, (2) no two adjacent edges cross each other, (3) no two edges cross each other more than once, (4) each intersection of edges is a crossing rather than tangential, and (5) no three edges cross in a common point. In this paper a drawing is always a good drawing, unless stated otherwise.

The crossing number of the drawing Π of a graph G, denoted by $cr(\Pi)$, is the number of edge crossings in Π. The *crossing number* of G, denoted by $cr(G)$, is the minimum integer k such that G has a drawing with k edge crossings. A drawing Π of G is *optimal* if $cr(\Pi) = cr(G)$.

A graph H is a *subdivision* of G if H is isomorphic to G or H can be obtained from G by inserting vertices of degree two on some edges. Obviously, $cr(H) = cr(G)$ if H is a subdivision of G. A vertex of G is called a *branch vertex* if its degree is at least three in G. For a bipartite graph F, if the vertex set $V(F)$ is written as $X \cup Y$, then our meaning is that both X and Y are two independent sets of F.

2 The crossing number of $P_m \boxtimes C_n$

This section starts with a lemma on an upper bound for $cr(P_m \boxtimes C_n)$.

Lemma 2.1 [7] *$cr(P_m \boxtimes C_n) \leq n(m-1)$ for $m \geq 2$ and $n \geq 3$.*

A (good) drawing of $P_m \boxtimes C_n$ is shown in Figure 1.

In order to obtain a lower bound for $cr(P_m \boxtimes C_n)$, let us start with two lemmas.

Lemma 2.2 *Let H be a subgraph of a connected graph G which is isomorphic to a subdivision of K_5, and let Q be a subgraph of H which is isomorphic to a subdivision of K_4. Then some edge in Q is crossed in any drawing of G.*

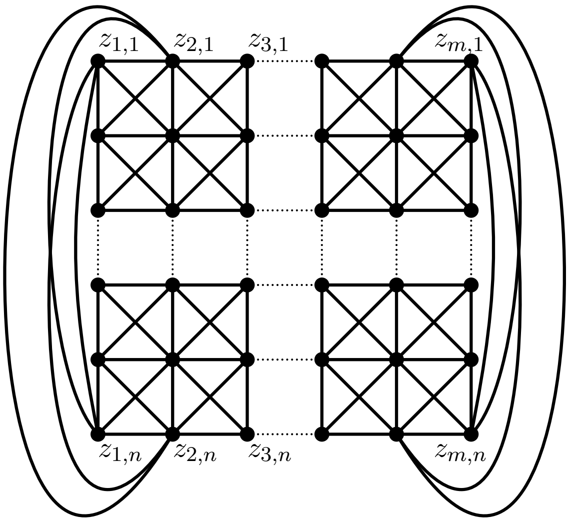

Figure 1: A drawing of $P_m \boxtimes C_n$.

Proof. Let Φ be a drawing of G. Since H is isomorphic to a subdivision of K_5, the drawing of Φ restricted in H has at least one crossing.

Suppose that v_1, v_2, v_3 and v_4 are the four branch vertices of Q. Then those are also the branch vertices of H. If some edge of Q is crossed by another edge of Q, then the lemma holds. Otherwise, the drawing of Φ restricted in Q divides the plane into four regions in which one is unbounded. Suppose that v_5 is the fifth branch vertex of H. No matter which region the vertex v_5 is in, there is at least one edge of Q which is crossed by some edge of $E(H) - E(Q)$ by the Jordan curve theorem. □

By a similar argument to that in the proof of Lemma 2.2, one can show the lemma below.

Lemma 2.3 *Let H be a subgraph of a connected graph G which is isomorphic to a subdivision of $K_{3,3}$, and let Q be a subgraph of H which is isomorphic to a subdivision of $K_{2,3}$. Then some edge in Q is crossed in any drawing of G.*

We intend to obtain a lower bound for $cr(P_m \boxtimes C_n)$. We need the following result.

Lemma 2.4 [4] *The crossing number of the complete graph K_6 is three.*

The lemma below gives a lower bound for $cr(P_m \boxtimes C_n)$.

Lemma 2.5 $cr(P_m \boxtimes C_n) \geq (m-1)n$ *for* $m \geq 2$ *and* $n \geq 3$.

Proof. For an arbitrary fixed $n (\geq 3)$, we use the induction on m to show that $cr(P_m \boxtimes C_n) \geq (m-1)n$. If $m = 2$, the inequality holds by Theorem 1.1. Assume that $cr(P_k \boxtimes C_n) \geq (k-1)n$, where $k \geq 2$. Let Φ be an optimal drawing of $P_{k+1} \boxtimes C_n$.

If $n = 3$, let $\bar{G}$ be the subgraph of $P_{k+1} \boxtimes C_n$ induced by all vertices in the set of $\{z_{i,1}, z_{i,2}, z_{i,3} | i = 1, 2\}$. We observe that $\bar{G}$ is isomorphic to the complete graph K_6. By Lemma 2.4, there are three crossings in $\bar{G}$. Let $D' = z_{2,1}z_{2,2}z_{2,3}z_{2,1}$. Then D' is a 3-cycle in which any two edges can not cross each other in Φ. So we can delete three edges in $E(\bar{G}) - E(D')$ to decrease three crossings in Φ. Notice that the obtained graph has a subgraph isomorphic to $P_k \boxtimes C_n$. Then $cr(\Phi) \geq 3 + 3(k-1) = 3k$.

We now suppose that $n \geq 4$. For $i = 1, 2, \ldots, n$, let Q_i be the subgraph of $P_{k+1} \boxtimes C_n$ induced by the four vertices $z_{1,i}$, $z_{1,i+1}$, $z_{2,i}$ and $z_{2,i+1}$, where the addition in the second indices is read modulo n. Then Q_i is isomorphic to K_4 for $i = 1, 2, \ldots, n$.

Let $D_1 = z_{1,1}z_{2,2}z_{1,2}z_{2,1}z_{1,1}$ be a cycle in $P_{k+1} \boxtimes C_n$. Let $R_1 = z_{1,2}z_{1,3} \ldots z_{1,n}z_{1,1}$ and $R_2 = z_{2,2}z_{2,3} \ldots z_{2,n}z_{2,1}$ be two paths in $P_{k+1} \boxtimes C_n$. Then the subgraph H_1 of $P_{k+1} \boxtimes C_n$ induced by all edges in $E(D_1 \cup R_1 \cup R_2) \cup \{z_{1,3}z_{2,3}\}$ is isomorphic to a subdivision of $K_{3,3}$, where the branch vertex set of H_1 is $\{z_{1,1}, z_{1,2}, z_{2,3}\} \cup \{z_{2,1}, z_{2,2}, z_{1,3}\}$. Let H_1' be the subgraph of H_1 induced by all edges in $E(D_1 \cup R_1)$. Then H_1' is isomorphic to a subdivision of $K_{2,3}$. By Lemma 2.3, some edge e_1 in H_1' is crossed in Φ. We consider two cases.

Case 1 The edge e_1 is in D_1.

Since D_1 has four edges, there are four cases to consider.

Case 1.1 The edge e_1 is the edge $z_{1,1}z_{2,1}$.

In this case e_1 is first deleted, and then the obtained drawing is denoted by Φ_1'. Let G_1 be the graph corresponding to Φ_1'. So G_1 contains the two edges $z_{2,1}z_{1,2}$ and $z_{2,1}z_{2,2}$ as well as the two paths $z_{2,1}z_{1,n}z_{1,n-1} \ldots z_{1,3}$ and $z_{2,1}z_{3,1}z_{3,2}z_{3,3}z_{2,3}$. Thus G_1 has a subgraph isomorphic to a subdivision of K_5 with the branch vertex set of $\{z_{2,1}, z_{1,2}, z_{2,2}, z_{1,3}, z_{2,3}\}$ which contains Q_2. By Lemma 2.2, some edge in Q_2 is crossed in Φ_1'.

We now apply the following operations which are denoted by Operation Class I.

(1) If someone of $z_{1,2}z_{2,2}$, $z_{1,2}z_{1,3}$, $z_{1,2}z_{2,3}$ and $z_{2,2}z_{1,3}$ is crossed in Φ_1', then it is deleted, and go to (4). Otherwise, go to (2).

(2) If $z_{2,2}z_{2,3}$ is crossed in Φ_1', then it is redrawn such that it nears to the path $z_{2,2}z_{1,2}z_{2,3}$ in Φ_1' and it crosses at most one edge in S_1, where $S_1 = \{z_{1,2}z_{1,1}, z_{1,2}z_{2,1}, z_{1,2}z_{1,3}\}$, since the degree of $z_{1,2}$ in G_1 is five. If the new drawing of $z_{2,2}z_{2,3}$ have not any crossing, then go to (4). Otherwise, the edge in S_1 crossed by $z_{2,2}z_{2,3}$ is removed and go to (4). If $z_{2,2}z_{2,3}$ has no crossings in Φ_1', then go to (3).

(3) The edge $z_{1,3}z_{2,3}$ must be crossed in Φ_1'. Subsequently, $z_{1,3}z_{2,3}$ is redrawn such that it nears to the path $z_{1,3}z_{1,2}z_{2,3}$ in Φ_1' and it crosses at most one edge in S_2, where $S_2 = \{z_{1,2}z_{1,1}, z_{1,2}z_{2,1}, z_{1,2}z_{2,2}\}$. If the new drawing of $z_{1,3}z_{2,3}$ have not any crossing, then go to (4). If the new drawing of $z_{1,3}z_{2,3}$ has one crossing, then the edge in S_2 crossed by $z_{1,3}z_{2,3}$ is deleted and go to (4).

(4) Let Φ_2' be the obtained drawing, and let G_2 be the graph corresponding to Φ_2'.

Notice that the edges $z_{1,1}z_{2,2}$, $z_{2,1}z_{2,2}$, $z_{2,2}z_{2,3}$, and $z_{1,3}z_{2,3}$ are preserved in G_2. We observe that G_2 must contain one of the edge $z_{2,2}z_{1,3}$ and the path $z_{2,2}z_{1,2}z_{1,3}$, and that G_2 has the path $z_{2,2}z_{3,2}z_{3,3}z_{3,4}z_{2,4}$ and the path $z_{2,2}z_{2,1}z_{1,n}z_{1,n-1} \ldots z_{1,4}$. So G_2 has a subgraph, say B, isomorphic to a subdivision of K_5 with the branch vertex set of $\{z_{2,2}, z_{2,3}, z_{2,4}, z_{1,3}, z_{1,4}\}$ which contains Q_3. For example, if G_2 contains the edge $z_{2,2}z_{1,3}$, then B is shown in Figure 2.

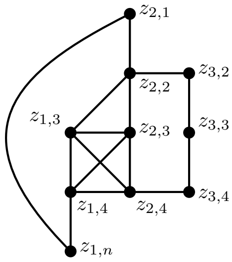

Figure 2: The graph B.

By Lemma 2.2, some edge in Q_3 is crossed in Φ'_2. We then proceed with a similar argument to that for Q_2. Then we obtain the drawing Φ'_3 which corresponds to the graph G_3. If $n = 4$, then we do not proceed with the following argument. Otherwise, G_3 has a subgraph isomorphic to a subdivision of K_5 which contains Q_4 by a similar argument as G_2. Subsequently, Q_4 is discussed as Q_3, and so on. The procedure does not stop until the drawing Φ'_{n-1} is obtained. Let G_{n-1} be the graph corresponding to Φ'_{n-1}.

We observe that G_{n-1} contains the edge $z_{1,1}z_{2,2}$, the cycle $D_3 = z_{1,1}z_{2,n}z_{2,1}z_{1,n}z_{1,1}$ and the path $R_3 = z_{2,2}z_{2,3}\dots z_{2,n-1}$. We also observe that G_{n-1} has a path, say R_4, which is one of the edge $z_{2,n-1}z_{1,n}$ and the path $z_{2,n-1}z_{1,n-1}z_{1,n}$. So G_{n-1} has a subgraph H_2 isomorphic to a subdivision of $K_{3,3}$, where the set of the branch vertices of H_2 is $\{z_{1,1}, z_{2,1}, z_{2,n-1}\} \cup \{z_{2,2}, z_{1,n}, z_{2,n}\}$. See Figure 3.

Let H'_2 be the graph obtained from H_2 by deleting the vertex $z_{2,n}$. Then H'_2 is isomorphic to a subdivision of $K_{2,3}$. By Lemma 2.3, some edge e_2 in H'_2 is crossed in the drawing of Φ'_{n-1} restricted in H'_2.

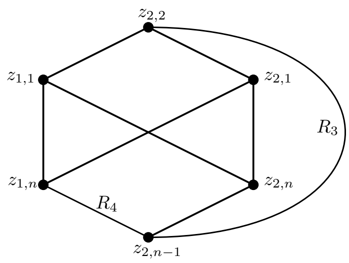

Figure 3: The graph H_2 in which R_3 and R_4 are two paths.

If e_2 is some edge in $\{z_{2,2}z_{1,1},\ z_{2,1}z_{1,n},\ z_{1,1}z_{1,n}\} \cup E(R_4)$, let Φ'_n be the obtained drawing. Let G_n be the graph corresponding to Φ'_n. Then G_n has a subgraph isomorphic to a subdivision of $P_k \boxtimes C_n$. By the assumption, $cr(P_k \boxtimes C_n) \geq n(k-1)$. So $cr(\Phi) \geq cr(\Phi'_n) + n \geq kn$.

If any edge in $\{z_{2,2}z_{1,1},\ z_{2,1}z_{1,n},\ z_{1,1}z_{1,n}\} \cup E(R_4)$ is not crossed, then e_2 is $z_{2,1}z_{2,2}$ or some edge of R_3. If e_2 is $z_{2,1}z_{2,2}$, then it is removed. In this case the obtained graph contains the path $z_{2,2}z_{1,1}z_{1,n}z_{2,1}$, which has a subgraph is isomorphic to a subdivision of $P_k \boxtimes C_n$. Otherwise, e_2 is some edge of R_3 which is crossed by some edge e_3, where e_3 is one of $z_{2,n}z_{1,1}$, $z_{2,n}z_{2,1}$, and $z_{2,n}z_{2,n-1}$. If e_3 is $z_{2,n}z_{1,1}$, then it is removed, which will bring forth a graph isomorphic to a subdivision of $P_k \boxtimes C_n$. If e_3 is $z_{2,n}z_{2,1}$ or $z_{2,n}z_{2,n-1}$, then e_3 is also deleted. Considering that the path $z_{2,n}z_{1,1}z_{1,n}z_{2,1}$ or the path $z_{2,n}z_{1,1}z_{1,n}R_4z_{2,n-1}$ exists, the obtained graph has a subgraph isomorphic to a subdivision of $P_k \boxtimes C_n$. Proceeding with a similar argument as in the previous paragraph, we have that $cr(\Phi) \geq cr(\Phi'_n) + n \geq kn$.

Case 1.2 The edge e_1 is $z_{2,1}z_{1,2}$.

In this case e_1 is first deleted. We also denote Φ'_1 by the obtained drawing, and let G_1 be the graph corresponding to Φ'_1. Then G_1 contains the path $z_{2,1}z_{1,1}z_{1,2}$. Thus G_1 has a subgraph isomorphic to a subdivision of K_5 which contains Q_2 as in Case 1.1. Next, we proceed with a similar argument to Case 1.1. Then $cr(\Phi) \geq cr(\Phi'_n) + n \geq kn$.

Case 1.3 The edge e_1 is $z_{1,1}z_{2,2}$.

Proceeding with a similar argument as in Case 1.1, we obtain the graph G_{n-1} which has the edge $z_{1,1}z_{2,1}$. We consider two cases.

Case 1.3.1 $z_{1,1}z_{1,2}$ is an edge of G_{n-1}.

If G_{n-1} has the edge $z_{1,2}z_{2,2}$, then G_{n-1} has a subgraph isomorphic to a subdivision of H_2. Proceeding with a similar argument as in Case 1.1, we have that $cr(\Phi) \geq kn$. Otherwise, G_{n-1} has the edge $z_{1,2}z_{2,3}$. In this situation, G_{n-1} has the path $z_{1,2}z_{2,3}z_{2,2}$. If $n = 4$, then G_{n-1} has a subgraph isomorphic to a subdivision of K_5 with the branch vertex set of $\{z_{1,1}, z_{2,1}, z_{1,4}, z_{2,4}, z_{2,3}\}$. If $n > 4$, then G_{n-1} has a subgraph isomorphic to a subdivision of $K_{3,3}$ with the branch vertex set of $\{z_{1,1}, z_{2,1}, z_{2,n-1}\} \cup \{z_{1,n}, z_{2,n}, z_{2,3}\}$. Subsequently, we proceed with a similar argument to that in Case 1.1. Then $cr(\Phi) \geq kn$.

Case 1.3.2 $z_{1,1}z_{1,2}$ is not an edge of G_{n-1}.

In this case G_{n-1} has the edge $z_{2,1}z_{1,2}$. Let F_n be the subgraph of G_{n-1} induced by $2n$ vertices $z_{1,1}, z_{1,2}, \ldots, z_{1,n}, z_{2,1}, z_{2,2}, \ldots, z_{2,n}$. Then F_n has the cycle $z_{2,1}z_{2,2}\ldots z_{2,n}z_{2,1}$. For $i = 1, 2, \ldots, n-1$, let Q'_i be the graph obtained from Q_i by deleting the edges $z_{1,i+1}z_{2,i+1}$ and $z_{2,i}z_{2,i+1}$.

In the process of obtaining G_{n-1}, we apply Operation Class I or similar operations to Operation Class I $n-1$ times. For the convenience of argument, those $n-1$ operations are denoted by $OP_1, OP_2, \ldots, OP_{n-1}$. We have the following observations.

Observation 1. For $i = 1, 2, \ldots, n-1$, there is at most one edge which is deleted after Op_i has been applied.

Observation 2. For $i = 1, 2, \ldots, n-1$, there are at most two edges in Q'_i which are removed.

Observation 3. For $i = 1, 2, \ldots, n-2$, there is at most one edge in Q'_{i+1} which is deleted if two edges in Q'_i are removed.

Observation 4. For $i = 1, 2, \ldots, n-2$, if two edges in Q'_i are removed, then the deletion of one edge is caused by applying Op_{i+1}.

For $i = 1, 2, \ldots, n-1$, let l_i be the number of all edges in Q'_i which are removed in the process of obtaining G_{n-1}. Clearly, $l_1 = 2$ and $l_{n-1} \leq 1$. By Observations 1 and 2, we have the following claim.

Claim 1. For $i = 1, 2, \ldots, n-1$, $0 \leq l_i \leq 2$. Moreover, $\sum_{i=1}^{n-1} l_i \leq n-1$.

For the graph F_n, we have the claim below.

Claim 2. F_n has some subgraph isomorphic to a subdivision of $K_{3,3}$.

Suppose on the contrary that the above claim does not hold. We first consider l_{n-1}. Since $l_{n-1} = 0$ or 1, we consider two cases.

Case 1.3.2.1. $l_{n-1} = 0$.

If $l_{n-2} \leq 1$, then F_n has the edge $z_{2,n-2}z_{1,n-1}$ or the path $z_{2,n-2}z_{1,n-2}z_{1,n-1}$. So F_n has a subgraph isomorphic to a subdivision of $K_{3,3}$ with the branch vertex set of $\{z_{2,n-2}, z_{1,n}, z_{2,n}\} \cup \{z_{1,n-1}, z_{2,n-1}, z_{2,1}\}$, a contradiction.

If $l_{n-2} = 2$, and if F_n has one of $z_{2,n-2}z_{1,n-1}$ and $z_{2,n-2}z_{1,n-2}z_{1,n-1}$, then there is a contradiction by a similar argument as the above paragraph.

We now suppose that $l_{n-2} = 2$, and that F_n has neither the edge $z_{2,n-2}z_{1,n-1}$ nor the path $z_{2,n-2}z_{1,n-2}z_{1,n-1}$. In this situation, F_n has the edge $z_{1,n-2}z_{2,n-1}$. Considering that $l_1 = 2$, we have that $n \neq 4$. Otherwise, $\sum_{i=1}^{3} l_i \geq 4$, which violates Claim 1.

We now suppose that $n \geq 5$. Since $\sum_{i=1}^{n-1} l_i \leq n-1$, there is some j such that $l_j = 0$, where $2 \leq j \leq n-3$. Let k be the largest number such that $l_k = 0$, where $k \leq n-3$. If $n = 5$, then $k = 2$. In this situation, G_{n-1} has a subgraph isomorphic to a subdivision of $K_{3,3}$ with the branch vertex set of $\{z_{2,1}, z_{1,3}, z_{2,3}\} \cup \{z_{1,2}, z_{2,2}, z_{2,4}\}$, a contradiction. If $n > 5$, then $l_j \geq 1$ for $j = k+1, \ldots, n-3$. If $l_{k+1} = 2$, then $l_{k+2} = 1$ by Observation 4. Notice that the removed edge in Q'_{k+2} is caused by Op_{k+3}, and that $l_{k+3} \geq 1$. So $l_{k+3} = 1$, and so on. At last, $l_{n-2} \leq 1$, a contradiction.

If $l_{k+1} = 1$, then the deleted edge in Q'_{k+1} is caused by Op_{k+1}, since $l_k = 0$. If $l_{k+2} = 2$, then we proceed a similar argument as the case that $l_{k+1} = 2$, which follows that $l_{n-2} = 1$, a contradiction. So $l_{k+2} = 1$. Furthermore, $l_j = 1$ for $j = k+3, \ldots, n-3$. Thus $\sum_{i=k}^{n-1} l_i = n-1-k$, which implies that $\sum_{i=1}^{k-1} l_i \leq k$.

If $l_{k-1} \leq 1$, then F_n has one of $z_{2,k-1}z_{1,k}$ and $z_{2,k-1}z_{1,k-1}z_{1,k}$. If F_n has the edge $z_{1,k+1}z_{2,k+2}$, then F_n has a subgraph, say T_1, which is isomorphic to a subdivision of $K_{3,3}$ with the branch vertex set of $\{z_{2,k-1}, z_{1,k+1}, z_{2,k+1}\} \cup \{z_{1,k}, z_{2,k}, z_{2,k+2}\}$, a contradiction. Otherwise, F_n has the edge $z_{1,k+1}z_{1,k+2}$, since $l_{k+1} = 1$. Considering that $l_{k+2} = 1$, F_n has one of $z_{1,k+2}z_{2,k+2}$ and $z_{1,k+2}z_{2,k+3}$. Then F_n has a subgraph isomorphic to a subdivision of T_1, a contradiction. Refer to Figure 4. Hence $l_{k-1} = 2$. In this situation, if F_n has one of $z_{2,k-1}z_{1,k}$ and $z_{2,k-1}z_{1,k-1}z_{1,k}$, then F_n has a subgraph isomorphic to a subdivision of $K_{3,3}$ by a similar argument as the above. We now suppose that F_n has neither $z_{2,k-1}z_{1,k}$ nor $z_{2,k-1}z_{1,k-1}z_{1,k}$. So F_n contains $z_{1,k-1}z_{2,k}$.

Since $l_1 = 2$ and $l_{k-1} = 2$, we have that there is some c such that $l_c = 0$, where $2 \leq c \leq k-2$. Furthermore, we can suppose that c is the largest number. Proceeding

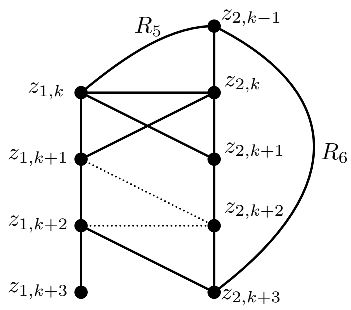

Figure 4: R_5 is one of $z_{2,k-1}z_{1,k}$ and $z_{2,k-1}z_{1,k-1}z_{1,k}$ and $R_6 = z_{2,k-1}z_{2,k-2}\ldots z_{2,k+3}$.

with a similar argument as the case that $l_{n-1} = 0$ and $l_{n-2} = 2$, we have that either F_n has a subgraph isomorphic to a subdivision of $K_{3,3}$ or $l_2 = 0$. If $l_2 = 0$, then $l_3 = 1$ or 2. Next, we proceed with a similar argument to that for l_k. Then F_n either has a subgraph isomorphic to a subdivision of $K_{3,3}$ or $l_{k-1} = 1$, a contradiction.

Case 1.3.2.2. $l_{n-1} = 1$.

In this case, F_n has one of $z_{1,n-1}z_{2,n}$ and $z_{1,n-1}z_{1,n}z_{2,n}$. Considering that $l_1 = 2$ and $\sum_{i=1}^{n-1} l_i \leq n-1$, we have that there is some s such that $l_s = 0$, where $2 \leq s \leq n-2$. Let t be the smallest number such that $l_t = 0$, where $t \geq 2$. If $n = 4$, then $t = 2$. In this situation, F_n has one of $z_{1,3}z_{2,4}$ and $z_{1,3}z_{1,4}z_{2,4}$. So F_n has a subgraph, say T_2, isomorphic to a subdivision of $K_{3,3}$ with the branch vertex set of $\{z_{2,1}, z_{1,3}, z_{2,3}\} \cup \{z_{1,2}, z_{2,2}, z_{2,4}\}$, a contradiction. We now suppose that $n \geq 5$. Then $l_j \geq 1$ for $j = 2, 3, \ldots, t-1$. If $l_{t-1} = 2$, we first consider that $t \geq 3$. In this situation, the two deleted edges are caused by applying Op_{t-1} and Op_t. So $l_{t-2} = 1$. Moreover, the deleted edge in Q'_{t-2} is caused by applying Op_{t-2}. Furthermore, $l_{t-3} = 1$, and so on. At last, $l_1 = 1$, a contradiction.

If $t = 2$, then $l_{t-1} = l_1 = 2$. We now argue l_3. If $l_3 \leq 1$, then F_n has a subgraph isomorphic to a subdivision of T_2, a contradiction. So $l_3 = 2$. In this situation, if F_n has one of $z_{1,3}z_{2,4}$ and $z_{1,3}z_{1,4}z_{2,4}$, then F_n has a subgraph isomorphic to a subdivision of T_2, a contradiction. So F_n has the edge $z_{2,3}z_{1,4}$. Notice that $\sum_{i=1}^{n-1} l_i \leq n-1$. Then there is some t' such that $l_{t'} = 0$, where $4 \leq t' \leq n-2$. We can suppose that t' is the smallest number. If $t' > 4$, then we proceed a similar argument to that in the above paragraph, which implies that $l_3 = 1$, a contradiction. If $t' = 4$, then l_4 is argued as l_2, and so on. Finally, either F_n has a subgraph isomorphic to a subdivision of $K_{3,3}$, or one of $l_{n-1} = 0$ and $l_{n-1} = 2$ holds, a contradiction.

If $l_{t-1} = 1$, then F_n has one of $z_{2,t-1}z_{1,t}$ and $z_{2,t-1}z_{1,t-1}z_{1,t}$. In this situation, l_{t+1} is argued. If $l_{t+1} \leq 1$, then F_n has one of $z_{2,t+2}z_{1,t+1}$ and $z_{2,t+2}z_{1,t+2}z_{1,t+1}$. So F_n has a subgraph isomorphic to a subdivision of $K_{3,3}$ with the branch vertex set of $\{z_{2,t-1}, z_{1,t+1}, z_{2,t+1}\} \cup \{z_{1,t}, z_{2,t}, z_{2,t+2}\}$, a contradiction. So $l_{t+1} = 2$. Since $\sum_{i=1}^{t} l_i \geq t$, we have that $\sum_{i=t+1}^{n-1} l_i \leq n-1-t$. Then there is some q with $l_q = 0$, where $t+2 \leq q \leq n-2$. We can suppose that q is the smallest number. Next, we proceed with a similar argument as l_t, and so on. Then F_n either has a subgraph isomorphic to a subdivision of $K_{3,3}$ or one of $l_{n-1} = 0$ and $l_{n-1} = 2$ holds, a contradiction. So

Claim 2 holds.

Let H_3 be a subgraph of F_n which is isomorphic to a subdivision of $K_{3,3}$. We delete some edge e_3 in H_3 which is crossed in G_{n-1} to obtain the drawing Φ'_n. It is not hard to find that $G_{n-1} - e_3$ has a subgraph isomorphic to a subdivision of $P_k \boxtimes C_n$. Then $cr(\Phi) \geq cr(\Phi'_n) + n \geq kn$.

Case 1.4. The edge e_1 is $z_{1,2}z_{2,2}$.

Let G'_1 be the graph obtained from G by removing e_1. Then G'_1 contains two edges $z_{2,2}z_{1,1}$, $z_{2,2}z_{2,1}$ and two paths $z_{2,2}z_{2,3}\dots z_{2,n}z_{2,1}$ and $z_{2,2}z_{1,3}\dots z_{1,n}z_{1,1}$. So G'_1 has a subgraph isomorphic to a subdivision of K_5 with the branch vertex set of $\{z_{2,2}, z_{1,1}, z_{2,1}, z_{1,n}, z_{2,n}\}$ which contains Q_n. Next, we proceed with a similar argument to Case 1.1. The difference is that Q_n, $Q_{n-1}, \dots, Q_2$ will be discussed in this order in the case. Thus $cr(\Phi) \geq cr(\Phi'_n) + n \geq kn$.

Case 2. The edge e_1 is not in D_1.

Then e_1 is some edge in $R_1 = z_{1,2}z_{1,3}\dots z_{1,n}z_{1,1}$. Without loss of generality, suppose that e_1 is $z_{1,i}z_{1,i+1}$, where $2 \leq i \leq n$ and the addition is read modulo n. We also denote Φ'_1 by the drawing obtained from Φ_1 be deleting e_1, and let G_1 be the graph corresponding to Φ'_1. Then G_1 contains the path $z_{1,i+1}z_{2,i}z_{2,i+1}$. So G_1 has a subgraph isomorphic to a subdivision of K_5 with the branch vertex set of $\{z_{2,i}, z_{1,i+1}, z_{2,i+1}, z_{1,i+2}, z_{2,i+2}\}$ which contains Q_{i+1}. Next, we proceed with a similar argument as in Case 1.1. The difference is that Q_{i+1}, $Q_{i+2}, \dots, Q_{i+n-1}$ will be discussed in this order in the case, where the addition is read modulo n.

Therefore, $cr(\Phi) \geq kn$. By the arbitrariness of n, we conclude that $cr(P_m \boxtimes C_n) \geq (m-1)n$ for $m \geq 2$ and $n \geq 3$. □

By Lemmas 2.1 and 2.5, we have the following result.

Theorem 2.6 *$cr(P_m \boxtimes C_n) = (m-1)n$ for $m \geq 2$ and $n \geq 3$.*

3 The crossing number of $F_{m,n}$

It is easy to obtain a drawing of $F_{m,n}$ with $(m-1)(n-1)$ crossings from the drawing shown in Figure 1. For example, a drawing of $F_{4,4}$ is shown in Figure 5. So we have the following lemma.

Lemma 3.1 *$cr(F_{m,n}) \leq (m-1)(n-1)$ for $m \geq 2$ and $n \geq 3$.*

We now study the crossing number of $F_{2,n}$. First, we obtain a lemma below.

Lemma 3.2 *Let G be the graph obtained from $F_{2,n}$ by deleting the edges $z_{1,1}z_{1,n}$ and $z_{2,1}z_{2,n}$ and adding the edges $z_{1,1}z_{2,n}$ and $z_{2,1}z_{1,n}$. Then G is isomorphic to $F_{2,n}$*

Proof. Let $D_1 = z_{1,1}z_{1,2}\dots z_{1,n-1}z_{2,n}z_{1,1}$ and $D_2 = z_{2,1}z_{2,2}\dots z_{2,n-1}z_{1,n}z_{2,1}$ be two cycles in G. Let $D'_1 = z_{1,1}z_{1,2}\dots z_{1,n-1}z_{1,n}z_{1,1}$ and $D'_2 = z_{2,1}z_{2,2}\dots z_{2,n-1}z_{2,n}z_{2,1}$ be two cycles in $F_{2,n}$. Define a mapping φ from $V(D_1 \cup D_2)$ to $V(D'_1 \cup D'_2)$ as follows:

(1) For $i = 1, 2, \dots, n-1$, $\varphi(z_{1,i}) = z_{1,i}$ and $\varphi(z_{2,i}) = z_{2,i}$.

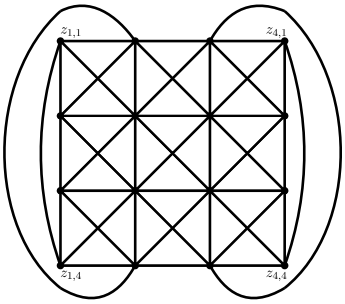

Figure 5: A drawing of $F_{4,4}$.

(2) $\varphi(z_{1,n}) = z_{2,n}$, and $\varphi(z_{2,n}) = z_{1,n}$.

Then φ is a bijective mapping from $V(D_1 \cup D_2)$ to $V(D_1' \cup D_2')$. Notice that the subgraph of $F_{2,n}$ induced by the vertices $z_{1,n-1}, z_{2,n-1}, z_{1,n}$ and $z_{2,n}$ is isomorphic to K_4. Then it is not hard to check that φ is an isomorphism from G to $F_{2,n}$. □

Lemma 3.3 *$cr(F_{2,n}) \geq n - 1$ for $n \geq 4$.*

Proof. We first show that $cr(F_{2,4}) \geq 3$. Let Φ be an optimal drawing of $F_{2,4}$. Let Q_i be the subgraph of $F_{2,4}$ induced by the four vertices $z_{1,i}$, $z_{2,i}$, $z_{1,i+1}$ and $z_{2,i+1}$ for $i = 1, 2, 3$. Suppose that $R_1 = z_{2,3}z_{1,4}z_{1,1}$, $R_2 = z_{2,3}z_{2,4}z_{2,1}$, and $R_3 = z_{1,2}z_{2,3}z_{2,2}$. Then $Q_1 \cup \{R_1, R_2, R_3\}$ is isomorphic to a subdivision of K_5 with branch vertex set of $\{z_{1,1}, z_{2,1}, z_{1,2}, z_{2,2}, z_{2,3}\}$. By Lemma 2.2, some edge e_1 in Q_1 is crossed in Φ. Let Φ_1' be the drawing obtained from Φ by deleting e_1, and let G_1 be the graph corresponding to Φ_1'. Then G_1 contains one of the two paths $z_{1,2}z_{1,1}z_{2,2}$ and $z_{1,2}z_{2,1}z_{2,2}$. We consider two cases.

Case 1 The edge e_1 is not the edge $z_{1,2}z_{2,2}$.

In this case, there are two cases to consider.

Case 1.1 The edge e_1 is neither $z_{1,1}z_{1,2}$ nor $z_{2,1}z_{2,2}$.

Then G_1 has two paths $z_{2,4}z_{1,4}z_{1,1}z_{1,2}$ and $z_{2,4}z_{2,1}z_{2,2}$. In this situation, G_1 has a subgraph isomorphic to a subdivision of K_5 with the branch vertex set of $\{z_{1,2}, z_{2,2}, z_{1,3}, z_{2,3}, z_{2,4}\}$ which contains Q_2. By Lemma 2.2, some edge e_2 in Q_2 is crossed in Φ_1'. Let Φ_2' be the drawing obtained from Φ_1' by removing e_2, which corresponds to the graph G_2. If e_2 is the edge $z_{1,3}z_{2,3}$, then G_2 has the cycle $D_1 = z_{1,2}z_{2,3}z_{2,2}z_{1,3}z_{1,2}$, and G_2 contains one of $z_{1,1}z_{2,2}$ and $z_{2,1}z_{1,2}$. Without loss of generality, suppose that G_2 contains $z_{1,1}z_{2,2}$. So G_2 has a subgraph isomorphic to a subdivision of $K_{3,3}$ with the branch vertex set of $\{z_{1,2}, z_{2,2}, z_{2,4}\} \cup \{z_{1,1}, z_{1,3}, z_{2,3}\}$. Thus $cr(\Phi_2') \geq 1$, which implies that $cr(\Phi) \geq 3$. If e_2 is not the edge $z_{1,3}z_{2,3}$, then G_2 has one of the two paths $z_{1,3}z_{1,2}z_{2,3}$ and $z_{1,3}z_{2,2}z_{2,3}$ as well as Q_3. Since one of $z_{1,1}z_{2,2}$ and $z_{1,2}z_{2,1}$ is in G_2, we have that G_2 has a minor isomorphic to K_5 with the branch vertex set of $\{z_{1,1}, z_{1,2}, z_{1,3}, z_{2,3}, z_{1,4}, z_{2,4}\}$, $\{z_{1,1}, z_{2,2}, z_{1,3}, z_{2,3}, z_{1,4}, z_{2,4}\}$, $\{z_{2,1}, z_{2,2}, z_{1,3}, z_{2,3}, z_{1,4}, z_{2,4}\}$, or $\{z_{2,1}, z_{1,2}, z_{1,3}, z_{2,3}, z_{1,4}, z_{2,4}\}$. Hence $cr(\Phi_2') \geq 1$, which means that $cr(\Phi) \geq 3$.

Case 1.2 The edge e_1 is one of $z_{1,1}z_{1,2}$ and $z_{2,1}z_{2,2}$.

Without loss of generality, suppose that e_1 is $z_{2,1}z_{2,2}$. Then G_1 contains the paths $z_{1,1}z_{1,2}z_{1,3}$, $z_{1,1}z_{2,2}z_{2,3}$, and $z_{1,1}z_{2,1}z_{2,4}$. So G_1 has a subgraph isomorphic to a subdivision of K_5 with the branch vertex set of $\{z_{1,1}, z_{1,3}, z_{2,3}, z_{1,4}, z_{2,4}\}$ which contains Q_3. By Lemma 2.2, some edge e_3 in Q_3 is crossed in Φ'_1. We also denote by Φ'_2 the drawing obtained from Φ'_1 by deleting e_3. Let G_2 be the graph corresponds to Φ'_2. Then G_2 contains one of the two paths $z_{1,3}z_{2,4}z_{2,3}$ and $z_{1,3}z_{1,4}z_{2,3}$. So G_2 has a subgraph isomorphic to a subdivision of $K_{3,3}$ with the branch vertex set of $\{z_{1,2}, z_{2,2}, z_{1,4}\} \cup \{z_{1,1}, z_{1,3}, z_{2,3}\}$ or $\{z_{1,2}, z_{2,2}, z_{2,4}\} \cup \{z_{1,1}, z_{1,3}, z_{2,3}\}$. Thus $cr(\Phi'_2) \geq 1$, which implies that $cr(\Phi) \geq 3$.

Case 2 The edge e_1 is the edge $z_{1,2}z_{2,2}$.

In this case we also denote by G_1 the graph obtained from G by deleting e_1. Obviously, G_1 has three paths $z_{1,1}z_{1,2}z_{1,3}$, $z_{1,1}z_{2,2}z_{2,3}$, and $z_{1,1}z_{2,1}z_{2,4}$. Proceeding with a similar argument as in Case 1.1, we have that $cr(\Phi) \geq 3$.

We now use induction on n to show that $cr(F_{2,n}) \geq n-1$. If $n = 4$, then the proposition holds by the former paragraphs. Assume that the proposition is true if $n = k$, where $k \geq 4$. We now consider the case that $n = k+1$. Suppose that Π is an optimal drawing of $F_{2,k+1}$. The graph Q_1 is defined as in the first paragraph in the proof. By the similar argument to that in the first paragraph in the proof, some edge in Q_1 is crossed in Π.

If some edge in $E(Q_1) - \{z_{1,2}z_{2,2}\}$ is crossed in Π, then it is deleted. Let Π' be the obtained drawing, and let H be the graph corresponding to Π'. Then H contains the two edges $z_{1,1}z_{1,2}$ and $z_{2,1}z_{2,2}$ or two edges $z_{1,1}z_{2,2}$ and $z_{2,1}z_{1,2}$. If the former occurs, then H has a subgraph isomorphic to a subdivision of $F_{2,k}$. If the latter occurs, then H also has a subgraph isomorphic to a subdivision of $F_{2,k}$ by Lemma 3.2. By the induction assumption, $cr(H) \geq k-1$, we have that $cr(\Pi) \geq k$. If any edge in $E(Q_1) - \{z_{1,2}z_{2,2}\}$ is not crossed in Π, then $z_{1,2}z_{2,2}$ must be crossed in Π. Next, the edge $z_{1,2}z_{2,2}$ is redrawn such that it nears to the path $z_{1,2}z_{1,1}z_{2,2}$ in Π which crosses exactly $z_{1,1}z_{2,1}$ or have not any crossing, since the degree of $z_{1,1}$ in H is four. If the former occurs, then $z_{1,1}z_{2,1}$ is deleted. Otherwise, there is nothing to do. Let Π'' be the obtained drawing, and let H' be the graph corresponding to Π''. Then H' has a subgraph isomorphic to a subdivision of $F_{2,k}$. By the induction assumption, $cr(H') \geq k-1$. So $cr(\Pi) \geq k$. Hence $cr(F_{2,n}) \geq n-1$ for $n \geq 4$. □

Lemma 3.4 *$cr(F_{m,n}) \geq (m-1)(n-1)$ for $m \geq 2$ and $n \geq 4$.*

Proof. For an arbitrary fixed $n (\geq 4)$, we use the induction on m. The base case is that $m = 2$. By Lemma 3.3, $cr(F_{2,n}) \geq n-1$. Assume that $cr(F_{k,n}) \geq (k-1)(n-1)$, where $k \geq 2$. Suppose that Π is an optimal drawing of $F_{k+1,n}$.

For $i = 1, 2, \ldots, n-1$, let Q_i be the subgraph of $F_{k+1,n}$ induced by the four vertices $z_{1,i}$, $z_{2,i}$, $z_{1,i+1}$ and $z_{2,i+1}$. Clearly, Q_i is isomorphic to K_4 for $i = 1, 2, \ldots, n-1$. As in the proof of Lemma 3.3, it can be found that $F_{k+1,n}$ has a subgraph isomorphic to a subdivision of K_5 which contains Q_1. So some edge in Q_1 is crossed in Π by Lemma 2.2. We apply the following operations which are denoted by Operation Class II.

(1) If one of $z_{1,1}z_{2,1}$, $z_{1,1}z_{1,2}$, $z_{1,1}z_{2,2}$ and $z_{2,1}z_{1,2}$ is crossed in Π, then it is deleted, and go to (4). Otherwise, go to (2).
(2) If $z_{1,2}z_{2,2}$ is crossed in Π, then it is redrawn such that it nears to the path $z_{1,2}z_{1,1}z_{2,2}$ in Π. If the new drawing of $z_{1,2}z_{2,2}$ has no crossings, then go to (4). Otherwise, $z_{1,2}z_{2,2}$ can be redrawn such that it exactly crosses $z_{1,1}z_{2,1}$, since the degree of $z_{1,1}$ in $F_{k+1,n}$ is four. Subsequently, $z_{1,1}z_{2,1}$ is deleted, and go to (4). If $z_{1,2}z_{2,2}$ has no crossings in Π, then go to (3).
(3) The edge $z_{2,1}z_{2,2}$ must be crossed in Π by Lemma 2.2. Subsequently, it is redrawn such that it nears to the path $z_{2,1}z_{1,1}z_{2,2}$. If the new drawing of $z_{2,1}z_{2,2}$ have not any crossing, then go to (4). Otherwise, $z_{2,1}z_{2,2}$ can be redrawn such that it exactly crosses $z_{1,1}z_{1,2}$. Next, the edge $z_{1,1}z_{1,2}$ is deleted and go to (4).
(4) Let Π'_1 be the obtained drawing, and let G_1 be the graph corresponding to Π'_1.

Then $cr(\Pi'_1) \leq cr(\Pi) - 1$, and G_1 contains the path $z_{2,1}z_{2,2}z_{1,2}$ and one of $z_{2,1}z_{1,2}$ and $z_{2,1}z_{1,1}z_{1,2}$. In addition, G_1 contains the two paths $z_{2,1}z_{2,n}z_{1,n-1}\dots z_{1,3}$ and $z_{2,1}z_{3,1}z_{3,2}z_{3,3}z_{2,3}$. So G_1 has a subgraph isomorphic to a subdivision of K_5 with the branch vertex set $\{z_{2,1}, z_{1,2}, z_{2,2}, z_{1,3}, z_{2,3}\}$ which contains Q_2. By Lemma 2.2, some edge in Q_2 is crossed in Π'_1. Next, we proceed with similar operations to Operation Class I. Then we obtain drawing Π'_2 with $cr(\Pi'_2) \leq cr(\Pi'_1) - 1$. Let G_2 be the graph corresponding to Π'_2. If $n = 4$, then we do not proceed with the following argument. Otherwise, we observe that G_2 contains the paths $z_{2,2}z_{2,3}z_{1,3}$, $z_{2,2}z_{2,1}z_{2,n}z_{1,n}z_{1,n-1}\dots z_{1,4}$, and $z_{2,2}z_{3,2}z_{3,3}z_{3,4}z_{2,4}$. So G_2 has a subgraph isomorphic to a subdivision of K_5 with the branch vertex set $\{z_{2,2}, z_{1,3}, z_{2,3}, z_{1,4}, z_{2,4}\}$ which contains Q_3. Subsequently, Q_3 is argued as Q_2, and so on. After Q_{n-2} has been dealt with, let Π'_{n-2} be the obtained drawing, and let G_{n-2} be the graph corresponding to Π'_{n-2}.

We now consider G_{n-2}. We observe that G_{n-2} has the path $z_{2,n-2}z_{2,n-3}\dots z_{2,1}$, and one of $z_{2,2}z_{1,1}$ and $z_{2,2}z_{2,1}z_{1,1}$. Without loss of generality, suppose that G_{n-2} has the edge $z_{2,2}z_{1,1}$. Since G_{n-2} has the paths $z_{2,n-2}z_{2,n-3}\dots z_{2,2}z_{1,1}z_{1,n}$ and $z_{2,n-2}z_{3,n-2}z_{3,n-1}z_{3,n}z_{2,n}$, we have that G_{n-2} has a subgraph isomorphic to a subdivision of K_5 with the branch vertex set $\{z_{2,n-2}, z_{1,n-1}, z_{1,n}, z_{2,n-1}, z_{2,n}\}$ which contains Q_{n-1}. So some edge e in Q_{n-1} is crossed in Π'_{n-2} by Lemma 2.2. Subsequently, e is deleted. Let Π'_{n-1} be the obtained drawing, and let G_{n-1} be the graph corresponding to Π'_{n-1}. If e is not the edge $z_{2,n-1}z_{2,n}$, then G_{n-1} has a subgraph isomorphic to a subdivision of $F_{k,n}$. If e is the edge $z_{2,n-1}z_{2,n}$, then G_{n-1} has the path the path $z_{2,n-1}z_{1,n}z_{2,n}$. So G_{n-1} also has a subgraph isomorphic to a subdivision of $F_{k,n}$. Then $cr(\Pi'_{n-1}) \leq cr(\Pi'_{n-2}) - 1 \leq cr(\Pi) - (n-1)$. Hence $cr(\Pi) \geq cr(\Pi'_{n-1}) + (n-1)$. Since $cr(\Pi'_{n-1}) \geq (k-1)(n-1)$, we have that $cr(\Pi) \geq k(n-1)$. By the arbitrariness of n, we have that $cr(F_{m,n}) \geq (m-1)(n-1)$ for $m \geq 2$ and $n \geq 4$. □

The theorem below follows from Lemmas 3.1 and 3.4 directly.

Theorem 3.5 *$cr(F_{m,n}) = (m-1)(n-1)$ for $m \geq 2$ and $n \geq 4$.*

The only missing case is $F_{m,3}$. We now explore it.

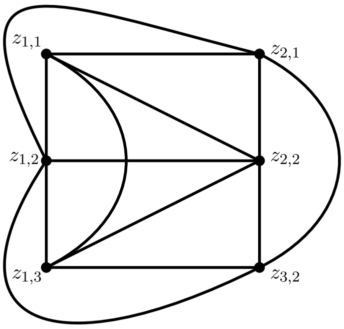

Figure 6: A drawing of $F_{2,3}$.

Lemma 3.6 *$cr(F_{m,3}) \leq m-1$ for $m \geq 2$.*

Proof Two (good) drawings of $F_{2,3}$ and $F_{3,3}$ are shown in Figures 6 and 7, respectively. So $cr(F_{m,3}) \leq m-1$ if $m = 2$ or 3.

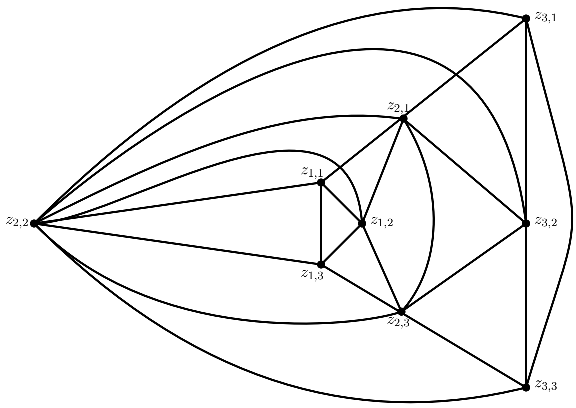

Figure 7: A drawing of $F_{3,3}$.

If $m \geq 4$, then we give a drawing of $F_{m,3}$ which is similar to that of $F_{3,3}$ shown in Figure 7. Let $D_1 = z_{m-1,2}z_{m,1}z_{m,3}z_{m-1,2}$. We first draw the cycle D_1 in the plane such that each edge is represented by a segment. Thus D_1 can be viewed as a triangle. For the convenience of discussion, let l be the bisector of the angle $\angle z_{m,1}z_{m-1,2}z_{m,3}$.

The vertex $z_{m,2}$ is first placed at the mid-point of $z_{m,1}z_{m,3}$. If $m \equiv 0 (\bmod 2)$ and $m \geq 6$, then $z_{m-3,2}, \ldots, z_{3,2}, z_{1,2}, z_{2,2}, \ldots, z_{m-2,2}$ in this order are put in the line l in the interior of D_1 such that $z_{m-3,2}$ is near to $z_{m-1,2}$. If $m = 4$, then $z_{1,2}$ and $z_{2,2}$ in this order are placed in the line l in the interior of D_1 such that $z_{1,2}$ is near to $z_{3,2}$. If $m \equiv 1 (\bmod 2)$, then $z_{m-3,2}, \ldots, z_{2,2}, z_{1,2}, z_{3,2}, \ldots, z_{m-2,2}$ in this order are put in the line l in the interior of D_1 such that $z_{m-3,2}$ is near to $z_{m-1,2}$. Next, the two segments L_1 and L_2 are drawn in the interior of D_1, where L_1 connects $z_{1,2}$ to $z_{m,1}$ and L_2 connects $z_{1,2}$ to $z_{m,3}$. The vertices $z_{1,1}, z_{2,1}, \ldots, z_{m-1,1}$ in this order are placed in the segment L_1 such that $z_{1,1}$ is near to $z_{1,2}$. Simultaneously, $z_{1,3}, z_{2,3}, \ldots, z_{m-1,3}$ in this order are placed in the segment L_2 such that $z_{1,3}$ is near to $z_{1,2}$.

Let D_2 be the cycle $z_{1,2}z_{1,1}z_{2,1}\dots z_{m-1,1}z_{m,1}z_{m,2}z_{m,3}z_{m-1,3}\dots z_{2,3}z_{1,3}z_{1,2}$. If $m \equiv 0 (\text{mod } 2)$, then the edge $z_{i,1}z_{i,3}$ is drawn in the interior of D_2 such that it passes between $z_{i-1,2}$ and $z_{i+1,2}$ for $i = 3, \dots, m-1$, and $z_{i,1}z_{i,3}$ is drawn in the exterior of D_2 such that it passes between $z_{i-1,2}$ and $z_{i+1,2}$ for $i = 2, 4, \dots, m-2$. In addition, the edge $z_{m,1}z_{m,3}$ is drawn in the exterior of D_1, and the edge $z_{1,1}z_{1,3}$ is drawn in the interior of D_2 such that it passes between $z_{1,2}$ and $z_{2,2}$. For $i = 2, 4, \dots, m-2$, the edges $z_{i,2}z_{i,1}$, $z_{i,2}z_{i,3}$, $z_{i,2}z_{i-1,1}$, $z_{i,2}z_{i-1,3}$, $z_{i,2}z_{i+1,1}$ and $z_{i,2}z_{i+1,3}$ are drawn in the interior of D_2. If $m \geq 6$, then the edges $z_{i,2}z_{i,1}$, $z_{i,2}z_{i,3}$, $z_{i,2}z_{i-1,1}$, $z_{i,2}z_{i-1,3}$, $z_{i,2}z_{i+1,1}$ and $z_{i,2}z_{i+1,3}$ are drawn between the interior of D_1 and the exterior of D_2 for $i = 3, 5, \dots, m-3$. Subsequently, the edges $z_{m-1,2}z_{m-1,1}$, $z_{m-1,2}z_{m-1,3}$, $z_{m-1,2}z_{m-2,1}$, and $z_{m-1,2}z_{m-2,3}$ are also drawn in the interior of D_1 and the exterior of D_2. If $m = 4$, then $z_{3,2}z_{3,1}$, $z_{3,2}z_{3,3}$, $z_{3,2}z_{2,1}$, and $z_{3,2}z_{2,3}$ are drawn between the interior of D_1 and the exterior of D_2. In addition, the edges $z_{1,2}z_{2,1}$ and $z_{1,2}z_{2,3}$ are drawn between the interior of D_1 and the exterior of D_2, and the edges $z_{m,2}z_{m-1,1}$ and $z_{m,2}z_{m-1,3}$ are drawn in the interior of D_2. For $i = 3, 5, \dots, m-1$, the edges $z_{i,2}z_{i-1,2}$ and $z_{i,2}z_{i+1,2}$ are drawn such that they cross exactly $z_{i-1,1}z_{i,1}$ and $z_{i,1}z_{i+1,1}$, respectively. In addition, $z_{1,2}z_{2,2}$ is drawn in such a way that it crosses $z_{1,1}z_{1,3}$ only. Thus a drawing of $F_{m,3}$ is completed which has $2 \cdot \frac{m-2}{2} + 1 (= m-1)$ crossings in this case. For example, the drawing of $F_{4,3}$ is shown in Figure 8.

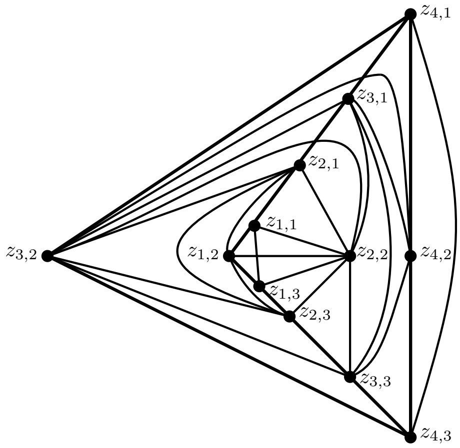

Figure 8: A drawing of $F_{4,3}$.

If $m \equiv 1 (\text{mod } 2)$, then the edge $z_{i,1}z_{i,3}$ is drawn in the interior of D_2 such that it passes between $z_{i-1,2}$ and $z_{i+1,2}$ for $i = 2, 4, \dots, m-1$, and $z_{i,1}z_{i,3}$ is drawn in the exterior of D_2 such that it passes between $z_{i-1,2}$ and $z_{i+1,2}$ for $i = 3, \dots, m-2$. In addition, the edge $z_{1,1}z_{1,3}$ is drawn in the exterior of D_2 such that it passes between $z_{1,2}$ and $z_{2,2}$ and the edge $z_{m,1}z_{m,3}$ is drawn in the exterior of D_1. For $i = 3, 5, \dots, m-2$, the edges $z_{i,2}z_{i,1}$, $z_{i,2}z_{i,3}$, $z_{i,2}z_{i-1,1}$, $z_{i,2}z_{i-1,3}$, $z_{i,2}z_{i+1,1}$ and $z_{i,2}z_{i+1,3}$ are drawn in the interior of D_2. For $i = 2, 4, \dots, m-3$, the edges $z_{i,2}z_{i,1}$, $z_{i,2}z_{i,3}$, $z_{i,2}z_{i-1,1}$, $z_{i,2}z_{i-1,3}$, $z_{i,2}z_{i+1,1}$ and $z_{i,2}z_{i+1,3}$ are drawn between the interior of D_1 and the exterior of D_2. In addition, the edges $z_{m-1,2}z_{m-1,1}$, $z_{m-1,2}z_{m-1,3}$, $z_{m-1,2}z_{m-2,1}$, and $z_{m-1,2}z_{m-2,3}$, are also drawn between the interior of D_1 and the exterior of D_2.

Subsequently, the edges $z_{1,2}z_{2,1}$, $z_{1,2}z_{2,3}$, $z_{m,2}z_{m-1,1}$, and $z_{m,2}z_{m-1,3}$ are drawn in the interior of D_2. For $i = 4, \ldots, m-1$, the edges $z_{i,2}z_{i-1,2}$ and $z_{i,2}z_{i+1,2}$ are drawn such that they cross exactly $z_{i-1,1}z_{i,1}$ and $z_{i,1}z_{i+1,1}$, respectively. In addition, $z_{2,2}z_{1,2}$ is drawn such that it crosses $z_{1,1}z_{1,3}$ only, and $z_{2,2}z_{3,2}$ is drawn such that it exactly crosses $z_{2,1}z_{3,1}$. Thus a drawing of $F_{m,3}$ is completed which has $2 \cdot \frac{m-1}{2} (= m-1)$ crossings in this case. □

Lemma 3.7 *$cr(F_{m,3}) \geq m-1$ for $m \geq 2$.*

Proof We use the induction on m to show that $cr(F_{m,3}) \geq m-1$. The base case is that $m = 2$. Since $F_{2,3}$ has a subgraph isomorphic to a subdivision of K_5, we have that $cr(F_{2,3}) \geq 1$.

Assume that $cr(F_{k,3}) \geq k-1$, where $k \geq 2$. Let Φ be an optimal drawing of $F_{k+1,3}$. Let Q be the subgraph of $F_{k+1,3}$ induced by the four vertices $z_{1,1}, z_{2,1}, z_{1,2}$ and $z_{2,2}$. Then Q is isomorphic to K_4. It is easy to find that $F_{k+1,3}$ has a subgraph H isomorphic to a subdivision of K_5 which contains Q as in the proof of Lemma 3.4. By Lemma 2.2, some edge e in Q has at least one crossing in Φ. Delete e from $F_{k+1,3}$. If e is not the edge $z_{2,1}z_{2,2}$, it is obvious that $F_{k+1,3} - e$ has a subgraph isomorphic to $F_{k,3}$. If e is the edge $z_{2,1}z_{2,2}$, then there is a path $z_{2,1}z_{1,1}z_{2,2}$ in $Q - e$. So $F_{k+1,3} - e$ has a subgraph isomorphic to a subdivision of $F_{k,3}$. Hence, $cr(F_{k+1,3}) \geq cr(F_{k,3}) + 1$. By the inductional assumption, $cr(F_{k,3}) \geq k-1$. So $cr(F_{k+1,3}) \geq k$. Therefore, $cr(F_{m,3}) \geq m-1$ for $m \geq 2$. □

The theorem below follows from Lemmas 3.6 and 3.7 directly.

Theorem 3.8 *$cr(F_{m,3}) = m-1$ for $m \geq 2$.*

By Theorems 3.5 and 3.8, the crossing number of $F_{m,n}$ is determined.

Remark: For any $j \in \{1, 2, \ldots, n-1\}$, let H_j be the graph obtained from $P_m \boxtimes C_n$ by deleting $z_{i,j}z_{i+1,j+1}$ and $z_{i+1,j}z_{i,j+1}$ for $i = 1, 2, \ldots, m-1$, then H_j is isomorphic to $F_{m,n}$. Thus $cr(H_j) = cr(F_{m,n})$. If $m \geq 3$ and $n \geq 4$, the crossing number of $F_{m,n}$ is $m-1$ less than that of $P_m \boxtimes C_n$ and is 4 greater than that of $P_m \boxtimes P_n$.

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