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Multicolor connected Ramsey numbers

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Abstract

In this paper we introduce strongly connected t-colorings, which allow us to extend the concept of a connected Ramsey number to the t-color setting, when $t \geq 3$. After proving some general results concerning such colorings, we determine the exact values of the connected Ramsey numbers $r_c^t(K_{1,n})$, $r_c(K_{1,3}, K_{1,4}, K_{1,4})$, $r_c^3(P_4)$, $r_c^3(P_5)$, and $r_c^3(P_6)$, where $K_{1,n}$ denotes a star of order $n+1$, and P_n denotes a path of order n. The paper concludes with some lower bounds, a conjecture, and a discussion concerning future directions of research.

1 Introduction

A *t-coloring* of K_p (the complete graph of order p) is a map

$$f : E(K_p) \longrightarrow \{1, 2, \ldots, t\}.$$

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The *t-color Ramsey number* $r(G_1, G_2, \ldots, G_t)$ is the least $p \in \mathbb{N}$ such that every t-coloring of K_p contains a monochromatic copy of G_i in color i, for some i such that $1 \leq i \leq t$. In the "diagonal" case $G_1 = G_2 = \cdots = G_t$, we shorten the notation to $r^t(G_1)$. A current account of known Ramsey numbers can be found in Radziszowski's dynamic survey [16]. In this paper, we consider a variation of t-color Ramsey numbers by restricting our attention to t-colorings of K_p in which the subgraphs of order p spanned by edges in each color are connected. Before stating our main results, we must present the relevant definitions and background.

A t-coloring of K_p is called a *connected t-coloring* if the subgraphs of order p spanned by edges in each of the t colors are connected. While a t-coloring is not assumed to be surjective, a connected t-coloring is surjective by definition. In 1978, Sumner [17] defined the *(2-color) connected Ramsey number* $r_c(G_1, G_2)$ to be the least $p \in \mathbb{N}$ such that every connected 2-coloring of K_p contains a subgraph isomorphic to G_1 in color 1 or a subgraph isomorphic to G_2 in color 2 (see also [3], [4], [6], [7], [8], and [15]). Our goal in this paper is to develop connected Ramsey theory when $t \geq 3$ colors are used.

A vertex v in a connected t-coloring of K_p is called a *removable vertex* if the coloring induced by $V(K_p) - \{v\}$ is also a connected t-coloring. Theorem 11 of [2] (and the discussion immediately following Theorem 11) implies that when $p \geq 5$, every connected 2-coloring of K_p has a removable vertex. Sumner [17] pointed out that from this result, it follows that if $r_c(G_1, G_2) = p$ and $n \geq p$, then every connected 2-coloring of K_n contains a red copy of G_1 or a blue copy of G_2. As this property is required for proving upper bounds for connected Ramsey numbers, we must start by considering if such a property holds for connected t-colorings when $t \geq 3$. To answer this question, we first show that connected t-colorings exist for sufficiently large complete graphs. Along with many other constructions that we will consider in this paper, this requires the use of certain known factorizations of complete graphs, which we now review.

The following two theorems can be found in König's book [14] (Theorems 2 and 7 of Chapter XI) and Harary's book [9] (Theorems 9.1 and 9.6). Recall that a *1-factor* of a graph G is an independent set of edges that span G. A *Hamiltonian cycle* (*path*) in a graph G is a cycle (path) that spans G.

Theorem 1.1 ([9], [14])**.** *For every $k \in \mathbb{N}$, the complete graph K_{2k} factors into $2k-1$ 1-factors.*

Theorem 1.2 ([9], [14])**.** *For every $k \in \mathbb{N}$, the complete graph K_{2k+1} factors into k Hamiltonian cycles.*

From Theorem 1.2, if we take a factorization of K_{2k+1} into Hamiltonian cycles and remove a single vertex, the resulting K_{2k} is factored into Hamiltonian paths. This leads to the following corollary (see also Theorem 2.3.3 of [12]).

Corollary 1.3 ([12])**.** *For every $k \in \mathbb{N}$, K_{2k} can be factored into k Hamiltonian paths.*

Since a vertex in K_{2k} has degree $2k-1$, it is a leaf in exactly one of the Hamiltonian paths, and an internal vertex in the other $k-1$ Hamiltonian paths. We now use this corollary to show that connected t-colorings exist for complete graphs that are sufficiently large.

Proposition 1.4. *Let $t \geq 2$ and $p > 1$. If $f : E(K_p) \longrightarrow \{1, 2, \ldots, t\}$ is a connected t-coloring, then $p \geq 2t$. Moreover, for every $p \geq 2t$, there exists a connected t-coloring of K_p.*

Proof. Let $t \geq 2$ and $p > 1$. If a connected t-coloring of K_p exists, then there exists a spanning tree spanned by edges in each color class. Such a spanning tree has size $p-1$, from which it follows that $t(p-1) \leq \binom{p}{2}$. This inequality is equivalent to $p \geq 2t$. The second statement follows from the well-known fact that K_{2t} can be decomposed into t Hamiltonian paths (e.g., see Corollary 1.3), and sequentially adding in vertices, joining them to the existing complete graph using at least one edge in each color, produces a connected t-coloring of K_p, for all $p \geq 2t$. □

Unfortunately, when $t \geq 3$ and $p > 2t$, not every connected t-coloring of K_p contains a removable vertex (for example, see Figure 1). In fact, for $t \geq 3$, any

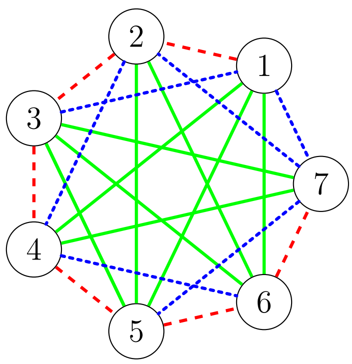


Figure 1: A connected 3-coloring of K_7 that lacks a removable vertex.

connected t-coloring of K_p ($p > 2t$) in which two of the colors span paths of order p which do not have any leaves in common will lack removable vertices. Our solution to this issue is to restrict our attention to connected t-colorings that possess the property we need to prove upper bounds for the corresponding Ramsey numbers.

When $p > 2t$, we define a *strongly connected t-coloring* of K_p to be a connected t-coloring $f : E(K_p) \longrightarrow \{1, 2, \ldots, t\}$ that contains a sequence of vertices $x_1, x_2, \ldots, x_k$, where $k = p - 2t$, such that the restricted colorings

$$f_\ell : E(K_p - \{x_1, x_2, \ldots, x_\ell\}) \longrightarrow \{1, 2, \ldots, t\},$$

given by $f_\ell = f|_{K_p - \{x_1, x_2, \ldots, x_\ell\}}$, are connected for all $1 \leq \ell \leq k$. With such a coloring, x_1 is a removable vertex for f and for $\ell \geq 1$, x_ℓ is a removable vertex for the (strongly) connected t-coloring $f_{\ell-1}$. By convention, we say that a connected t-coloring of K_{2t} is a strongly connected t-coloring. Of course, every connected 2-coloring is a strongly connected 2-coloring, but this statement is not true when $t \geq 3$. Now we show that strongly connected t-colorings exist for complete graphs that are sufficiently large.

Proposition 1.5. *For every $p \geq 2t$, there exists a strongly connected t-coloring of K_p.*

Proof. We have already noted that K_{2t} can be decomposed into t Hamiltonian paths, from which it follows that there is a strongly connected t-coloring of K_{2t}. To obtain a strongly connected t-coloring of K_p when $p > 2t$, start with a strongly connected t-coloring of K_{2t} and sequentially add $p - 2t$ vertices, joining each of them to the existing complete graph using at least one edge in each color. Reversing this sequence of vertices produces a sequence of removable vertices, showing that the resulting coloring is strongly connected. □

When $t \geq 3$, define the *t-color connected Ramsey number* $r_c(G_1, G_2, \ldots, G_t)$ to be the least $p \in \mathbb{N}$ such that every strongly connected t-coloring of K_p contains a monochromatic copy of G_i in color i for some i such that $1 \leq i \leq t$. In the diagonal case $G_1 = G_2 = \cdots = G_t$, we shorten the notation to $r_c^t(G_1)$. Since every strongly connected t-coloring of K_p is a t-coloring of K_p, it follows that for any graphs $G_1, G_2, \ldots, G_t$,

$$r_c(G_1, G_2, \ldots, G_t) \leq r(G_1, G_2, \ldots, G_t). \tag{1}$$

When equality holds in (1), we say that $(G_1, G_2, \ldots, G_t)$ *is Ramsey-connected.*

Denote by $K_{1,n}$, P_n, and C_n, the star of order $n+1$, the path of order n, and the cycle of order n, respectively. In Section 2, we prove that

$$r_c^t(K_{1,n}) = \begin{cases} t(n-1)+1 & \text{if } t \text{ and } n \text{ are both even} \\ t(n-1)+2 & \text{otherwise,} \end{cases}$$

when $t \geq 3$ and $n \geq 3$, extending the $t = 2$ case that was determined in [4]. In the non-diagonal case, we also show that $r_c(K_{1,3}, K_{1,4}, K_{1,4}) = 9$. In Section 3, we turn our attention to paths, proving that $r_c^3(P_4) = 6$, $r_c^3(P_5) = 7$, and $r_c^3(P_6) = 8$. We conclude with Section 4, in which we provide strongly connected 3-colorings that imply the lower bounds $r_c^3(P_7) \geq 10$, $r_c^3(P_8) \geq 11$, and $r_c^3(C_4) \geq 10$.

2 Multicolor Connected Ramsey Numbers for Stars

In 1973, Burr and Roberts [5] determined the multicolor Ramsey number for stars, generalizing the 2-color result of Harary [10]. They showed that if $t \geq 2$ and $n_1, n_2, \ldots, n_t \in \mathbb{N}$, exactly k of which are even, then

$$r(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t}) = \begin{cases} N-t+1 & \text{if } k \geq 2 \text{ is even} \\ N-t+2 & \text{if } k = 0 \text{ or } k \text{ is odd,} \end{cases} \tag{2}$$

where $N = \sum_{i=1}^{t} n_i$. Monochromatic copies of $K_{1,2}$ are trivially contained in connected t-colorings of K_p when $p \geq 2t$, so we only consider stars that have at least three leaves as the arguments of connected Ramsey numbers.

When $t = 2$, it was shown in [4] that (K_{1,n_1}, K_{1,n_2}) is Ramsey-connected. We now turn our attention to the evaluation of $r_c^t(K_{1,n})$ when $t \geq 3$.

Theorem 2.1. *If $t \geq 2$ and $n \geq 3$, then*

$$r_c^t(K_{1,n}) = \begin{cases} t(n-1)+1 & \text{if } t \text{ and } n \text{ are both even} \\ t(n-1)+2 & \text{otherwise.} \end{cases}$$

Proof. This theorem was proved for $t = 2$ in Theorem 2.1 of [4], so assume that $t \geq 3$. The upper bounds for $r_c^t(K_{1,n})$ follow from Inequality (1) and Equation (2). The theorem will follow from providing strongly connected critical colorings that imply the lower bounds. We break the proof into cases, based on the values of $t \geq 3$ and $n \geq 3$.

Case 1: Let $n = 3$. We will construct a strongly connected t-coloring of K_{2t+1} that avoids a monochromatic copy of $K_{1,3}$. By Theorem 1.2, K_{2t+1} can be factored into t Hamiltonian cycles. Assign a unique color to each Hamiltonian cycle, resulting in every vertex being incident with exactly two edges in each of the t colors. For example, see Figure 2 for the $t = 3$ case. Removing a single vertex results in a

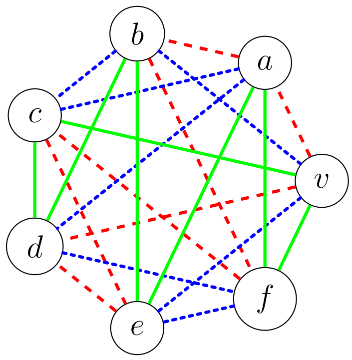


Figure 2: A (strongly) connected 3-coloring of K_7 that lacks a monochromatic $K_{1,3}$.

connected t-coloring of K_{2t}, which is strongly connected by definition. Since we have provided a strongly connected t-coloring of K_{2t+1} that avoids a monochromatic $K_{1,3}$, it follows that $r_c^t(K_{1,3}) \geq 2t + 2$.

Case 2: Suppose that $n \geq 5$ is odd. We will construct a strongly connected t-coloring of $K_{(n-1)t+1}$ that avoids a monochromatic $K_{1,n}$. Let $X = \{x_1, x_2, \ldots, x_{2t}\}$ and for each $j \in \{1, 2, \ldots, (n-3)/2\}$, define the set $Y_j := \{y_i^j \mid i \in \{1, 2, \ldots, 2t\}\}$. Let $Y := Y_1 \cup Y_2 \cup \cdots \cup Y_{(n-3)/2}$.

For the edges joining X to Y, color all edges with the forms $x_i y_{i+2c-2}^j$ and $x_i y_{i+2c-3}^j$ (where $j \in \{1, 2, \ldots, (n-3)/2\}$ and the indices in the subscripts are reduced modulo $2t$) with color $c \in \{1, 2, \ldots, t\}$. At this point, each of the vertices in X are incident with exactly $n - 3$ edges in color c and each of the vertices in Y are incident with exactly 2 edges in color c. For example, Figure 3 shows the case where $t = 3$ and $n = 5$.

Introduce a vertex v and consider the subgraph induced by $X \cup \{v\}$, which has order $2t + 1$. By Theorem 1.2, this subgraph factors into t Hamiltonian cycles, each

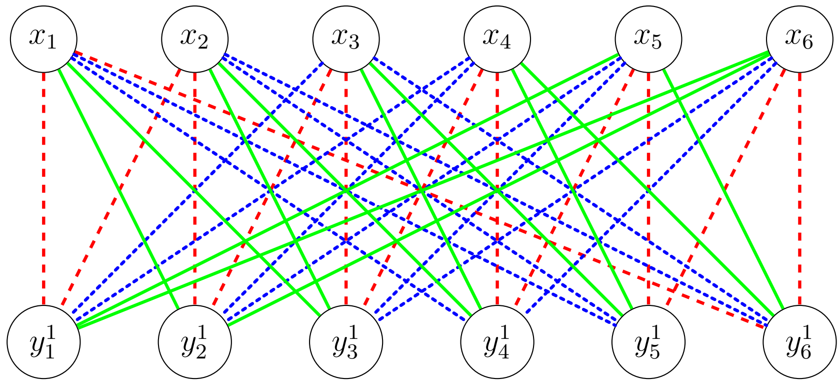


Figure 3: The edges joining X to Y in the case where $t = 3$ and $n = 5$.

of which we color with a unique color. So, each vertex in X is now incident with $n-1$ edges in each color and vertex v is incident with exactly two edges in each color. The subgraph induced by $Y \cup \{v\}$ has order $t(n-3)+1$, which is odd. Applying Theorem 1.2 again, we see that this subgraph can be factored into $\frac{t(n-3)}{2}$ Hamiltonian cycles. Color exactly $\frac{n-3}{2}$ Hamiltonian cycles with each of the t colors. Each vertex in $Y \cup \{v\}$ is now incident with exactly $n-1$ edges in each color.

The t-coloring of $K_{t(n-1)+1}$ with vertex set $X \cup Y \cup \{v\}$ that we have constructed is strongly connected. This is because we can first remove vertex v, then one-by-one, we can remove the vertices in Y, with the resulting t-coloring being connected at each stage. It follows that $r_c^t(K_{1,n}) \geq (n-1)t+2$.

Case 3: Suppose that $n \geq 4$ is even and t is even. We will construct a strongly connected t-coloring of $K_{(n-1)t}$ that avoids a monochromatic $K_{1,n}$. For each $j \in \{1,2,\dots,t\}$, define $X_j = \{x_1^j, x_2^j\}$ and $Y_j = \{y_i^j \mid i \in \{1,2,\dots,n-3\}\}$. Let $X = X_1 \cup X_2 \cup, \cdots \cup X_t$ and $Y = Y_1 \cup Y_2 \cup, \cdots \cup Y_t$.

For the edges that join X to Y, color all edges of the forms $x_1^j y_i^{j+c-1}$ and $x_2^j y_i^{j+c-1}$ with color c (where $c \in \{1,2,\dots,t\}$ and the superscripts are reduced modulo t). At this point every vertex in X is incident with exactly $n-3$ edges in each color, and each vertex in Y is incident with exactly 2 vertices in each color. For example, Figure 4 shows the case where $t = 4$ and $n = 6$.

Now consider the subgraph induced by X. Since $|X| = 2t$, Corollary 1.3 implies that X factors into t Hamiltonian paths. Coloring each of these paths in color $c \in \{1,2,\dots,t\}$ results in every vertex in X being incident with $n-1$ edges in $t-1$ of the colors and $n-2$ edges in one of the colors.

Next, consider the subgraph induced by Y. Since $t(n-3)$ is even, Theorem 1.1 implies that Y factors into $t(n-3)-1$ 1-factors. Coloring $n-3$ 1-factors in each of the colors $1,2,\dots,t-1$, and $n-4$ 1-factors in color t results in every vertex in Y being incident with $n-1$ edges in colors $1,2,\cdots,t-1$ and $n-2$ in color t.

The resulting t-coloring of $K_{t(n-1)}$ with vertex set $X \cup Y$ is strongly connected as we can sequentially remove each of the vertices in Y, and at each stage in the removal process our t-coloring remains connected. Since the coloring lacks a monochromatic

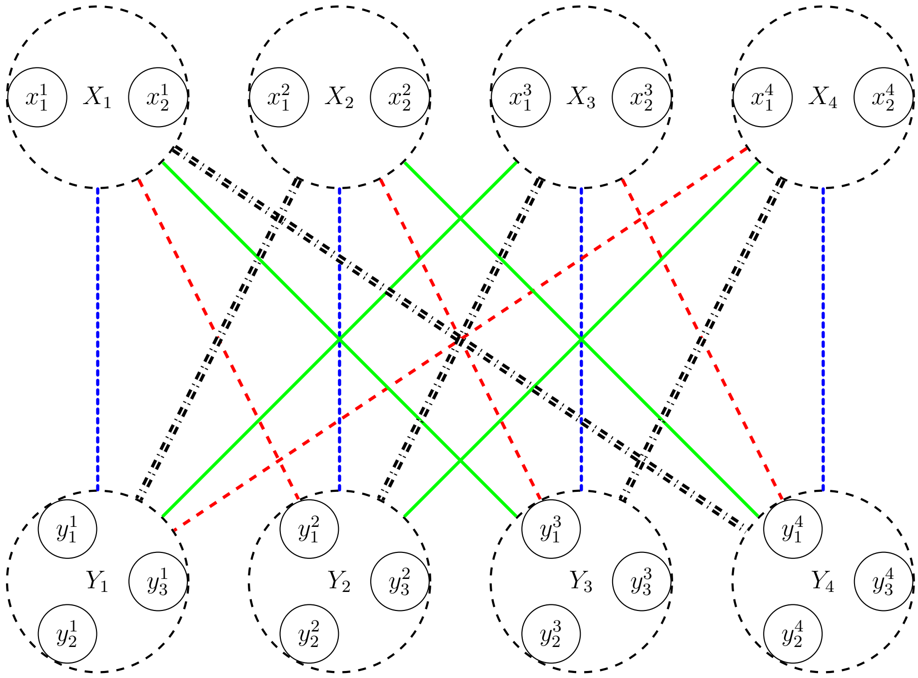


Figure 4: The edges joining X to Y in the case where $t = 4$ and $n = 6$.

$K_{1,n}$, it follows that $r_c^t(K_{1,n}) \geq t(n-1)+1$.

Case 4: Assume that $n \geq 4$ is even and t is odd. We will construct a strongly connected t-coloring of $K_{(n-1)t+1}$ that avoids a monochromatic $K_{1,n}$. As with the previous case, for each $j \in \{1, 2, \ldots, t\}$, define $X_j = \{x_1^j, x_2^j\}$ and $Y_j = \{y_i^j \mid i \in \{1, 2, \ldots, n-3\}\}$. Let $X = X_1 \cup X_2 \cup, \cdots \cup X_t$ and $Y = Y_1 \cup Y_2 \cup, \cdots \cup Y_t$.

For the edges that join X to Y, color all edges of the forms $x_1^j y_i^{j+c-1}$ and $x_2^j y_i^{j+c-1}$ with color c (where $c \in \{1, 2, \ldots, t\}$ and the superscripts are reduced modulo t). At this point every vertex in X is incident with exactly $n-3$ edges in each color, and each vertex in Y is incident with exactly 2 vertices in each color. For example, Figure 5 shows the case where $t = 3$ and $n = 6$.

Now introduce a vertex v and consider the subgraph induced by $X \cup \{v\}$, which has order $2t+1$. By Theorem 1.2, this subgraph factors into t Hamiltonian cycles, each of which we color with a unique color. So, each vertex in X is now incident with $n-1$ edges in each color and vertex v is incident with exactly two edges in each color.

The subgraph induced by $Y \cup \{v\}$ has order $t(n-3)+1$, which is even. By Theorem 1.1, $Y \cup \{v\}$ factors into $t(n-3)$ 1-factors. Coloring $n-3$ 1-factors with each of the colors $1, 2, \ldots, t$ results in v being incident with exactly $n-1$ edges in each color and every vertices in Y being incident with exactly $n-1$ vertices in each color.

The resulting t-coloring of $K_{t(n-1)+1}$ with vertex set $X \cup Y \cup \{v\}$ is strongly connected. This is because we can first remove the vertex v, then one-by-one, each of the vertices in Y, with the resulting t-coloring being connected at each stage. Since

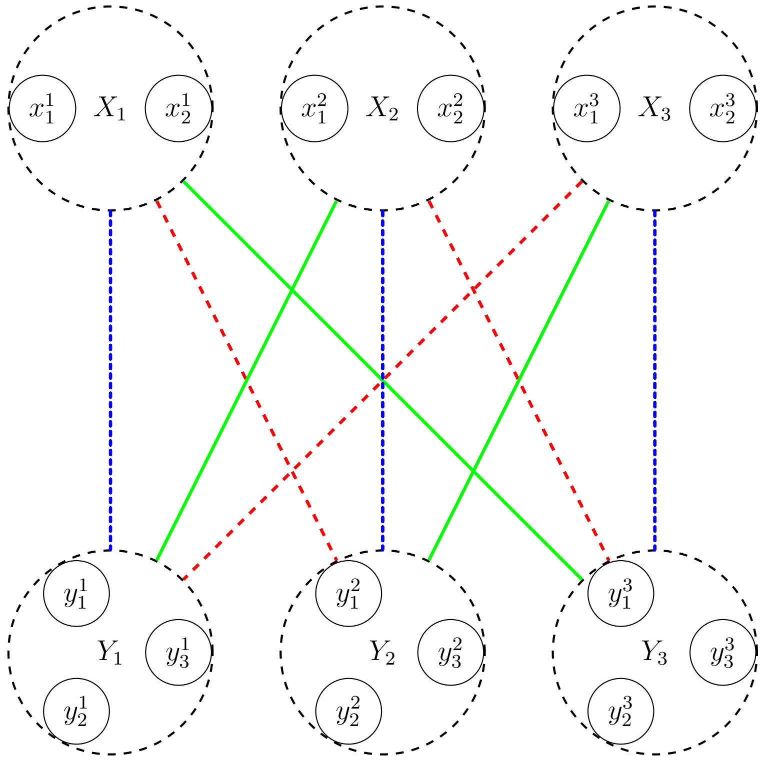


Figure 5: The edges joining X to Y in the case where $t = 3$ and $n = 6$.

the coloring lacks a monochromatic $K_{1,n}$, it follows that $r_c^t(K_{1,n}) \geq (n-1)t + 2$. □

While the general connected Ramsey number $r_c(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t})$ is not determined here, we do provide one non-diagonal case with the next theorem.

Theorem 2.2. *$r_c(K_{1,3}, K_{1,4}, K_{1,4}) = 9$.*

Proof. Since $r(K_{1,3}, K_{1,4}, K_{1,4}) = 9$, by Inequality (1) and Equation (2), it suffices to provide a strongly connected 3-coloring of K_8 that avoids a $K_{1,3}$ in color 1 and a $K_{1,4}$ in colors 2 and 3. Begin with a K_6, which can be factored into 3 Hamiltonian paths by Corollary 1.3 (see Image (A) in Figure 6). Each vertex is a leaf in exactly one of the Hamiltonian paths. Color the Hamiltonian paths distinct colors, letting vertices a and b be the leaves in the Hamiltonian path in color 1 (red), vertices c and d be the leaves in the Hamiltonian paths in color 2 (blue), and vertices e and f be the leaves in the Hamiltonian path in color 3 (green). Introduce vertices u and v. Color edges au and bv with color 1, color edges bu, cu, du, av, dv, and cv with color 2, and color edges eu, fu, ev, fv, and uv with color 3. The result is a connected 3-coloring of K_8 in which vertices u and v can be sequentially removed, resulting in connected 3-colorings at each stage (see Image (B) in Figure 6). Thus, we have produced a strongly connected 3-coloring of K_8 that avoids a $K_{1,3}$ in color 1 and a $K_{1,4}$ in colors 2 and 3. It follows that $r_c(K_{1,3}, K_{1,4}, K_{1,4}) \geq 9$. □

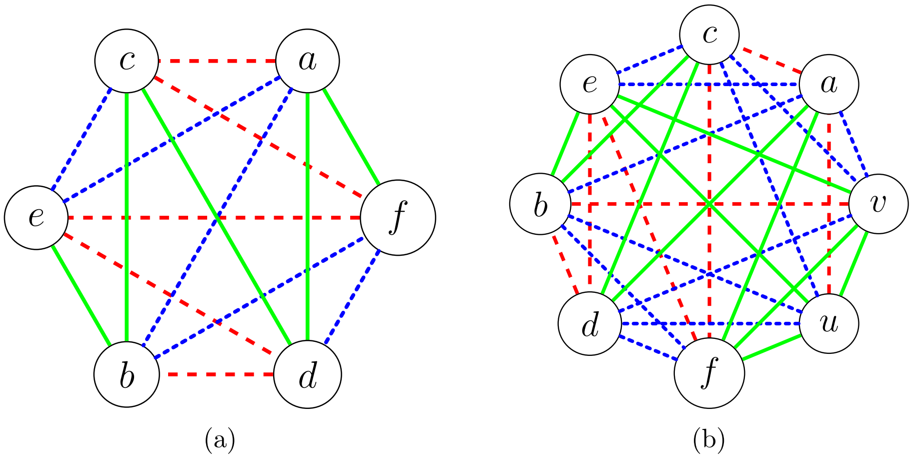


Figure 6: A (strongly) connected 3-coloring of K_6 in which the subgraph spanned by edges in each color is a Hamiltonian path, and a strongly connected 3-coloring of K_8 that lacks a red $K_{1,3}$, a blue $K_{1,4}$, and a green $K_{1,4}$.

3 Diagonal 3-Color Connected Ramsey Numbers for Paths

In 1974, Irving [13] proved that $r^3(P_4) = 6$. From Inequality (1), it follows that $r_c^3(P_4) \leq 6$. By Proposition 1.4, we see that $r_c^3(P_4) \geq 6$, giving us our first multicolor connected Ramsey number for paths:

$$r_c^3(P_4) = 6.$$

As we will only consider strongly connected 3-colorings of K_p for the remainder of this section, we assume that the colors used are red, blue, and green. The subgraphs spanned by the red, blue, and green edges of a strongly connected 3-coloring of K_p will be denoted as G_r, G_b, and G_g respectively. The *maximum degree* of a graph G, is given by

$$\Delta(G) := \max\{\deg_G(x) \mid x \in V(G)\},$$

where $\deg_G(x)$ is the degree of the vertex $x \in V(G)$. We now provide a simple lemma before moving on to the evaluation of other connected Ramsey numbers for paths.

Lemma 3.1. *In every strongly connected* 3*-coloring of* K_p*, the maximum degree of each subgraph* G_r*,* G_b*, and* G_g *is at most* $p-3$*.*

Proof. If a vertex in one of G_r, G_b, or G_g is incident with $p-2$ (or more) edges of the same color, it would become an isolated vertex in one of the other monochromatic subgraphs. This contradicts the assumption that G_r, G_b, and G_g are connected. □

In [18], it was shown that $r^3(P_5) = 9$ and $r^3(P_6) = 10$. We consider the corresponding connected Ramsey numbers in the following two theorems, showing that (P_5, P_5, P_5) and (P_6, P_6, P_6) are not Ramsey-connected.

Theorem 3.2. $r_c^3(P_5) = 7$.

Proof. The lower bound is established from the (strongly) connected 3-coloring of K_6 shown in Figure 7. It follows that $r_c^3(P_5) \geq 7$.

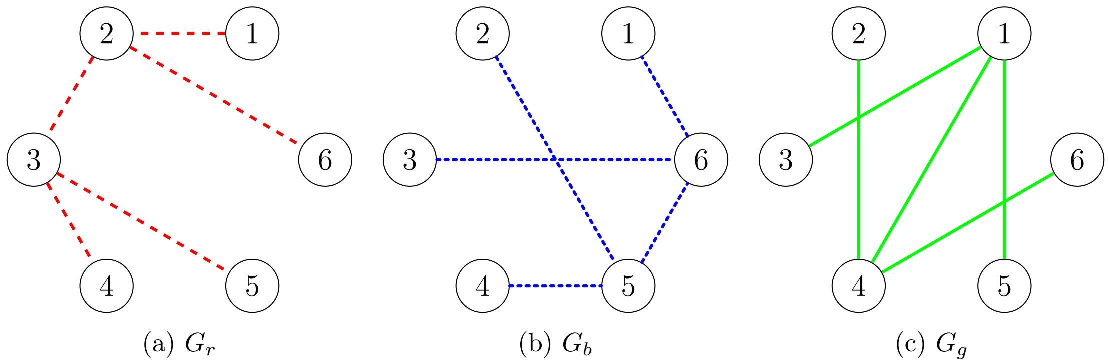


Figure 7: A (strongly) connected 3-coloring of K_6 that avoids a monochromatic P_5.

To establish the reverse inequality, consider a strongly connected 3-coloring of K_7. By the pigeonhole principle, some color must appear on at least 7 edges. Without loss of generality, assume that $|E(G_r)| \geq 7$. Since G_r is connected, but is not a tree, it must contain a cycle C_ℓ. If $\ell \geq 5$, then C_ℓ contains a P_5 as a subgraph. If $\ell = 4$, there must be some additional red edge that joins the C_4 to a vertex not in the C_4 (since G_r is connected) and a red P_5 can be formed (see Image (A) in Figure 8).

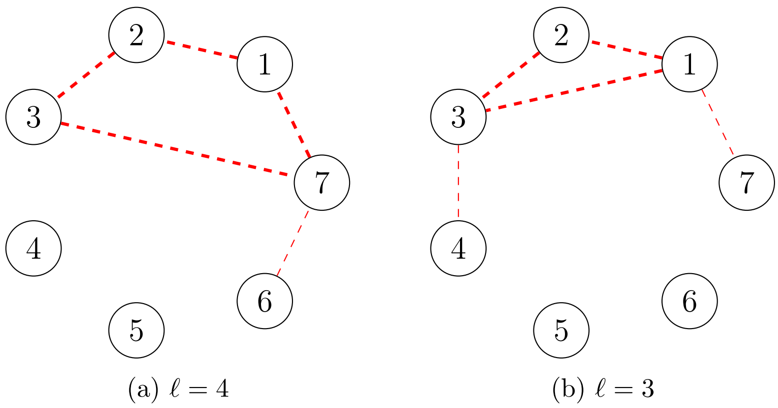
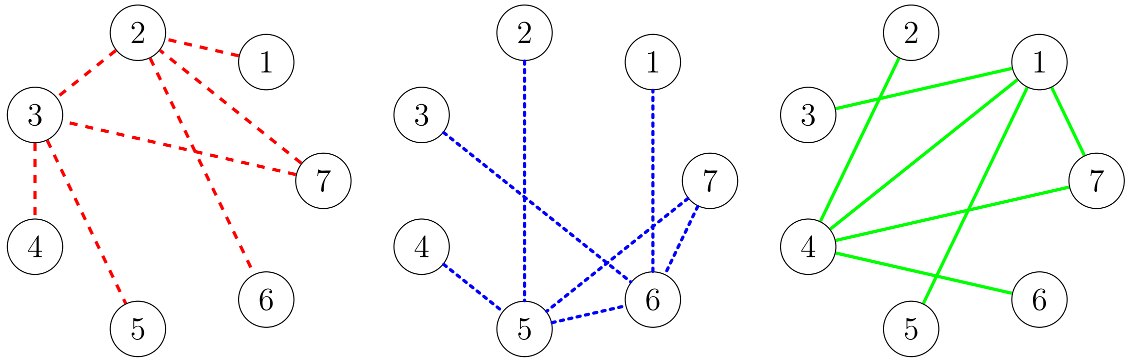


Figure 8: The subgraph G_r contains a cycle C_ℓ with $\ell = 4$ or $\ell = 3$.

Finally, assume that $\ell = 3$, denote the vertices in the red C_3 by x_1, x_2, and x_3, and denote the remaining vertices by y_1, y_2, y_3, and y_4. If there exists a red path of the form $x_i y_j y_k$ (where $j \neq k$), then this path, along with the other two vertices in the C_3, form a red P_5. If no such path exists, then each y_j must join to the C_3 with a red edge. By Lemma 3.1, $\Delta(G_r) \leq 4$, from which it follows that there exists x_i, x_m, y_j, and y_k such that $x_i y_j$ and $x_m y_k$ are red, with $i \neq m$ and $j \neq k$ (for example, see Image (B) in Figure 8). These vertices, along with the other vertex in the red C_3, form a red P_5. It follows that $r_c^3(P_5) \leq 7$. □

Theorem 3.3. $r_c^3(P_6) = 8$.

Proof. The lower bound is established from the strongly connected 3-coloring of K_7 shown in Figure 9 (observe that vertex 7 is a removable vertex). Therefore, $r_c^3(P_6) \geq 8$.



(a) G_r (b) G_b (c) G_g

Figure 9: A strongly connected 3-coloring of K_7 that avoids a monochromatic P_6

To establish the reverse inequality, consider a strongly connected 3-coloring of K_8. Since $|E(K_8)| = 28$, one of G_r, G_b, and G_g must have size at least 10 by the pigeonhole principle. Without loss of generality, assume that $|E(G_r)| \geq 10$. Since the size of G_r is greater than 7, it must contain a cycle. Let C_ℓ be such a cycle. If $\ell \geq 6$, then G_r contains a red C_6, which contains a red P_6. We break the rest of the proof into cases based on the size of ℓ.

Case 1: Suppose that $\ell = 5$. Then the red C_5 must have a red pendant edge since the coloring is connected. Such an edge, along with the vertices in the C_5, form a red P_6.

Case 2: Suppose that $\ell = 4$. Let the red C_4 be given by $x_0x_1x_2x_3x_0$ and label the other vertices by y_0, y_1, y_2, and y_3. If any red path of the form $x_iy_jy_k$ $(j \neq k)$ exists, then $y_ky_jx_ix_{i+1}x_{i+2}x_{i+3}$ is a red P_6 (here, the indices are reduced modulo 4). If no such red path exists, then the subgraph induced by $\{y_0, y_1, y_2, y_3\}$ does not contain any red edges, and each y_j must join via a red edge to some x_i. If for any $j \in \{0, 1, 2, 3\}$, y_jx_i and y_jx_{i+1} are red, then $y_jx_{i+1}x_{i+2}x_{i+3}x_iy_j$ is a red C_5, reducing the argument back to that of Case 1. If for $j \neq k$, y_jx_i and y_kx_{i+1} are red, then $y_kx_{i+1}x_{i+2}x_{i+3}x_iy_j$ is a red P_6. It follows that the only red edges joining $\{y_0, y_1, y_2, y_3\}$ to $\{x_0, x_1, x_2, x_3\}$ join to vertices that are not adjacent in the C_4. Without loss of generality, suppose that all such edges only join to the vertices x_1 and x_3. Consider two subcases.

Subcase 2.1: Suppose that x_1x_3 is red. By Lemma 3.1, $\Delta(G_r) \leq 5$, from which it follows that at most two red edges join $\{y_0, y_1, y_2, y_3\}$ to each of x_1 and x_3. Without loss of generality, suppose that y_0x_3, y_1x_3, y_2x_1, and y_3x_1 are red (see Figure 10). Since no additional red edges can join $\{y_0, y_1, y_2, y_3\}$ to $\{x_1, x_3\}$ and $|E(G_r)| \geq 10$, it follows that x_0x_2 must be red. Then $y_0x_3x_0x_2x_1y_2$ is a red P_6.

Subcase 2.2: Suppose that x_1x_3 is not red (without loss of generality, assume that it is blue). Since $\Delta(G_r) \leq 5$, it follows that not all of y_0, y_1, y_2, and y_3 can join

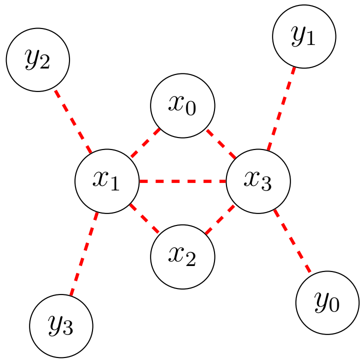


Figure 10: The strongly connected 3-coloring of K_8 that corresponds with Subcase 2.1 of the proof of Theorem 3.3.

to one of x_1 and x_3 via red edges. Without loss of generality, suppose that y_0x_3 and y_3x_1 are red. If x_0x_2 is red, then $y_0x_3x_0x_2x_1y_3$ is a red P_6. So, assume that x_0x_2 is not red. In order for G_r to have size at least 10, two of the vertices from $\{y_0, y_1, y_2, y_3\}$ must join to $\{x_1, x_3\}$ with two red edges and the other two must join with one red edge. Without loss of generality, suppose that y_0x_3, y_1x_3, y_1x_1, y_2x_3, y_2x_1, and y_3x_1 are red (see Figure 11). In order for the 3-coloring to be connected, both y_0x_1 and y_3x_3 must be green. However, this results in the blue edge x_1x_3 being isolated, contradicting the assumption that we are considering a strongly connected 3-coloring.

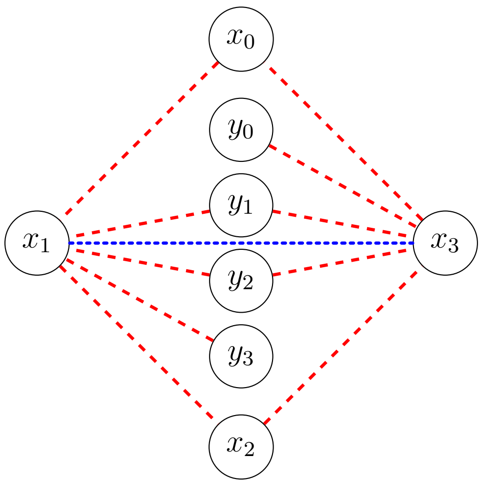
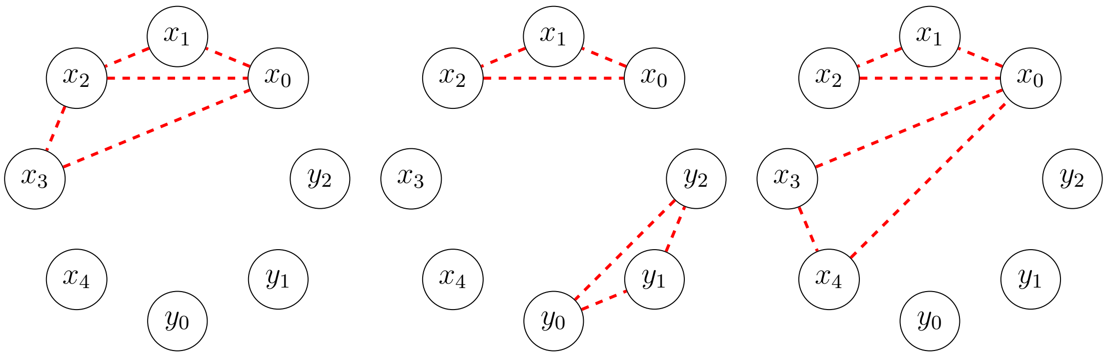


Figure 11: The strongly connected 3-coloring of K_8 that corresponds with Subcase 2.2 of the proof of Theorem 3.3.

Case 3: Suppose that $\ell = 3$ and no red cycle exists with length more than 3. Suppose that a red C_3 is given by $x_0x_1x_2x_0$. If we remove the edges x_0x_1 and x_0x_2, then the resulting red subgraph still has order 8 and size 8 (although it may no longer be connected). As such, it contains another cycle, which must necessarily be a C_3. So, G_r contains at least two C_3-subgraphs. If they have an edge in common (see Image (A) in Figure 12), then together, a red C_4 can be formed, contradicting the assumption in this case. If they are disjoint (see Image (B) in Figure 12), some red path must join them together, from which a red P_6 can be formed. If they have a single vertex in common, say x_0 (see Image (C) in Figure 12), then assume that the other vertices in the second triangle are x_3 and x_4 and the vertices not in the triangles are given by y_0, y_1, and y_2. Now, x_0 can join to at most one of y_0, y_1, and



(a) Common edge (b) Disjoint (c) Common vertex

Figure 12: A strongly connected 3-coloring of K_8 that contains two red C_3-subgraphs.

y_2 with a red edge since $\Delta(G_r) \leq 5$. Also, if the subgraph induced by $\{y_0, y_1, y_2\}$ is a red K_3, then it is disjoint from the subgraph induced by $\{x_0, x_1, x_2\}$, and falls under the subcase we just argued. So, assume that the subgraph induced by $\{y_0, y_1, y_2\}$ contains at most two red edges. Since $|E(G_r)| \geq 10$, at least one red edge must join $\{x_1, x_2, x_3, x_4\}$ to $\{y_0, y_1, y_2\}$ allowing for a red P_6 to be formed.

In all cases, we see that a red P_6 is formed. It follows that $r_c^3(P_6) \leq 8$, completing the proof of the theorem. □

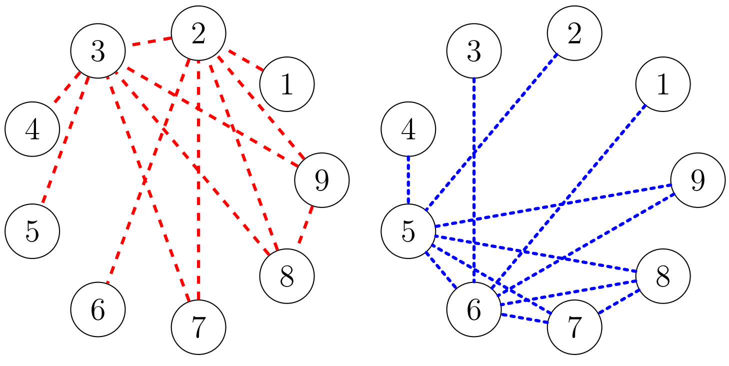
4 Conclusion

We conclude by describing some directions for future work in connected Ramsey theory. With regard to stars, the cases completed so far lead to the following conjecture.

Conjecture 4.1. *For $t \geq 2$ and $n_i \geq 3$, where $1 \leq i \leq t$, $(K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t})$ is Ramsey-connected.*

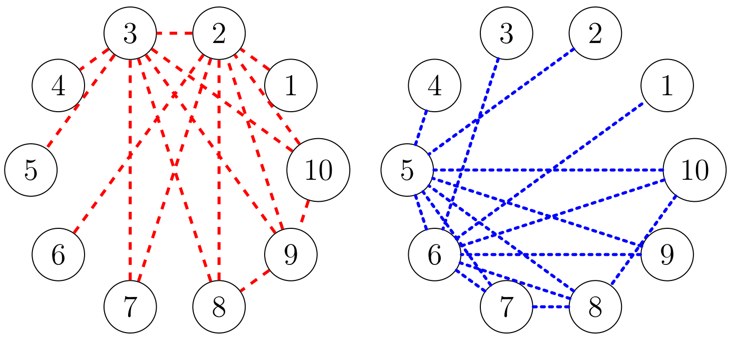
Our work in Section 3 might lead one to conjecture that $r_c^3(P_n) = n + 2$ when $n \geq 4$. However, we now give two strongly connected 3-colorings that disprove this possibility. In Figure 13, a strongly connected 3-coloring of K_9 is given in which a monochromatic P_7 is avoided. Observe that vertices 7, 8, and 9 can be sequentially removed, creating a connected 3-coloring at each step. It follows that $r_c^3(P_7) \geq 10$.

For the path P_8, consider the connected 3-coloring of K_{10} shown in Figure 14. Observe that the vertices 7, 8, 9, and 10 can be sequentially removed, resulting in a



(a) G_r (b) G_b (c) G_g

Figure 13: A strongly connected 3-coloring of K_9 that avoids a monochromatic P_7.



(a) G_r (b) G_b (c) G_g

Figure 14: A strongly connected 3-coloring of K_{10} that avoids a monochromatic P_8

connected 3-coloring at each step. It follows that it is strongly connected, and since this 3-coloring avoids a monochromatic P_8, we obtain the lower bound $r_c^3(P_8) \geq 11$.

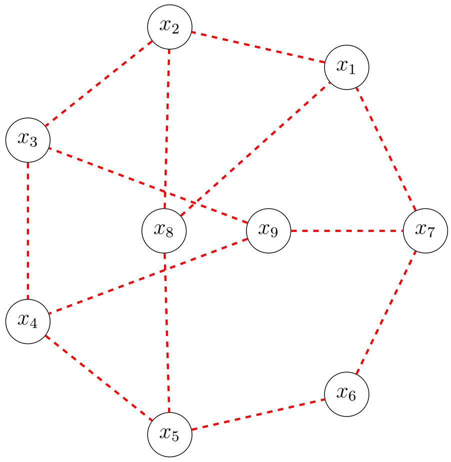
Besides the 3-color strongly connected Ramsey numbers for paths, one could also consider the case of cycles. In 1984, it was shown that $r^3(C_4) = 11$ [1]. In Figure 15, a 3-coloring of K_9 is given that avoids a monochromatic C_4.

Observe that vertices 7, 8 and 9 can be sequentially removed, resulting in a connected coloring at each step. This coloring, along with Inequality 1 implies that

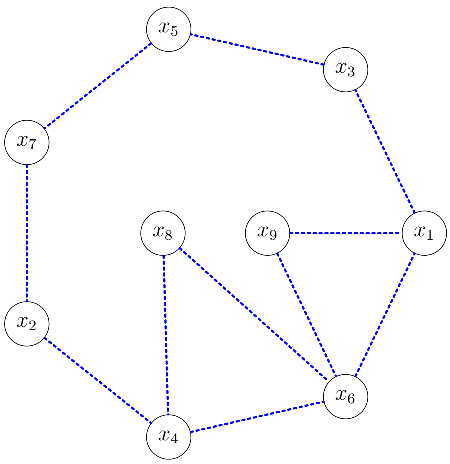
$$10 \leq r_c^3(C_4) \leq 11.$$

Now that connected Ramsey theory has been extended to the multicolor setting, one can consider other standard variations. For example, for $1 \leq s < t$, the *weakened connected Ramsey number* $r_c^{s,t}(G)$ can be defined to be the least $p \in \mathbb{N}$ such that every strongly connected t-coloring of K_p contains a subgraph isomorphic to G that is spanned by edges using at most s colors (cf., [11]). When $s = 1$, this definition agrees with that of the usual connected Ramsey number and these numbers exist since

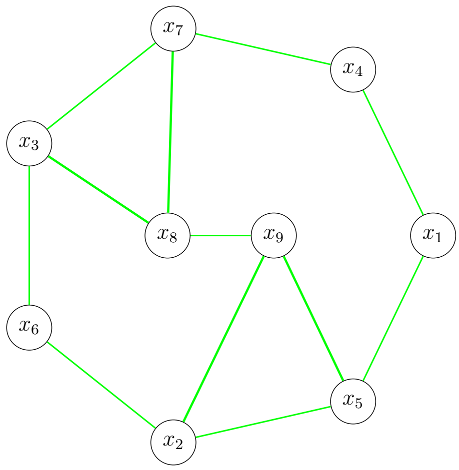
$$r_c^{s,t}(G) \leq r_c^t(G),$$



(a) G_r



(b) G_b



(c) G_g

Figure 15: A strongly connected 3-coloring of K_9 that avoids a monochromatic C_4.

for all s such that $1 \leq s < t$.

In [15], star-critical connected Ramsey numbers were considered when $t = 2$. To define them when $t \geq 3$, denote by $K_n \sqcup K_{1,k}$ the graph formed by taking K_n and a single vertex, and joining them with exactly k edges. The *star-critical connected Ramsey number* $r_c^*(G_1, G_2, \ldots, G_t)$ is then defined to be the least k such that $K_{r_c(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,k}$ contains a monochromatic copy of G_i in color i, for some i such that $1 \leq i \leq t$. These numbers always exist since

$$K_{r_c(G_1,G_2,\ldots,G_t)-1} \sqcup K_{1,r_c(G_1,G_2,\ldots,G_t)-1} = K_{r_c(G_1,G_2,\ldots,G_t)}.$$

A good starting point for determining the values of some multicolor star-critical connected Ramsey numbers would be to consider the cases of the connected Ramsey numbers determined in this paper.

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