

Revised ascent sequences avoiding a pattern of length 4

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Abstract

Revised ascent sequences share a similar construction with modified ascent sequences, because both can be obtained from ascent sequences through specific hat maps. These sequences can also be characterized as Cayley permutations where each entry is a leftmost occurrence if and only if it serves as an ascent bottom. In this paper, we systematically investigate revised ascent sequences avoiding a pattern of length 4, deriving several precise enumeration results and establishing corresponding Wilf-equivalence classifications. Building upon these findings, we further propose a number of conjectures for future verification.

1 Introduction

Let n be a positive integer and let $[n] = \{1, 2, \dots, n\}$. An *endofunction* of size n is a mapping $x : [n] \mapsto [n]$, which can be represented as the word $x = x_1x_2 \cdots x_n$, where $x_i = x(i)$ for each $i \in [n]$. For an endofunction $x = x_1x_2 \cdots x_n$, define

$$\begin{aligned} \text{Asctop}(x) &:= \{1\} \cup \{i \mid 2 \leq i \leq n \text{ and } x_{i-1} < x_i\}, \\ \text{Ascbot}(x) &:= \{1\} \cup \{i \mid 1 \leq i \leq n-1 \text{ and } x_i < x_{i+1}\}, \end{aligned}$$

as the sets of *ascent tops* and *ascent bottoms* of x , respectively. Depending on the context, these definitions may also refer to the corresponding values rather than the indices, when no confusion arises. For example, if $x_i < x_{i+1}$, we also call x_i an ascent bottom and x_{i+1} an ascent top. Let $\text{asctop}(x)$ and $\text{ascbot}(x)$ be the cardinality of $\text{Asctop}(x)$ and $\text{Ascbot}(x)$, respectively.

Call an endofunction x an *ascent sequence* if $x_1 = 1$ and $1 \leq x_i \leq \text{asctop}(x_1x_2 \cdots x_{i-1}) + 1$ for all $2 \leq i \leq n$. Ascent sequences were introduced by Bousquet-Mélou, Claesson, Dukes and Kitaev [1] to establish deep connections among three seemingly unrelated

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combinatorial objects: $(2 + 2)$ -free posets [9, 10], a family of permutations avoiding a certain pattern and Stoimenow matchings [16]. All these combinatorial objects have been proved to be enumerated by the *Fishburn numbers*, appearing as sequence A022493 in the OEIS [15] with the following elegant generating function:

$$\sum_{n \geq 0} \prod_{i=1}^n (1 - (1-x)^i) = 1 + x + 2x^2 + 5x^3 + 15x^4 + 53x^5 + 217x^6 + 1014x^7 + \dots$$

The combinatorial objects enumerated by the Fishburn numbers are called *Fishburn structures*.

A *Cayley permutation* is an endofunction x such that x contains each integer between 1 and $\max(x)$. Let Cay_n denote the set of Cayley permutations of length n . *Modified ascent sequences* are originally defined as the bijective images of ascent sequences under a certain hat map [1]. Very recently, Cerbai and Claesson [2] provided an alternative definition of modified ascent sequences as Cayley permutations where each entry is a leftmost occurrence if and only if it is an ascent top. Modified ascent sequences, together with ascent sequences, have played a crucial role in the study of Fishburn structures, giving rise to a substantial body of literature on the subject [3, 4, 6, 7, 11, 12, 13, 14, 17].

Inspired by the definition of modified ascent sequences, Zhou [18] introduced the notion of *revised ascent sequences*. These sequences are defined as Cayley permutations in which each entry is a leftmost occurrence if and only if it serves as an ascent bottom. Let $\text{Nub}(x)$ denote the set of indices of the leftmost occurrences of x . That is,

$$\text{Nub}(x) = \{\min x^{-1}(i) \mid i \in \text{Im}(x)\},$$

where $\text{Im}(x)$ is the set of images of x . Then the set $\hat{\mathcal{B}}_n$ of revised ascent sequences of length n is defined as

$$\hat{\mathcal{B}}_n = \{x \in \text{Cay}_n \mid \text{Ascbot}(x) = \text{Nub}(x)\}.$$

For example, we have

$$\hat{\mathcal{B}}_4 = \{1111, 2212, 2121, 2122, 3123\}.$$

Given two positive integers n, m , let $\hat{\mathcal{B}}_{n,m}$ denote the set of revised ascent sequences in $\hat{\mathcal{B}}_n$ with maximum value m . For example, the sequence $x = 524514352$ is a revised ascent sequence in $\hat{\mathcal{B}}_{9,5}$ because $\text{Ascbot}(x) = \text{Nub}(x) = \{1, 2, 3, 5, 7\}$. By adapting the classic hat map between ascent sequences and modified ascent sequences [1], Zhou [18] established a bijection between ascent sequences of length n and revised ascent sequences of length $n + 1$. As ascent sequences of length n are counted by the n -th Fishburn number, $\hat{\mathcal{B}}_n$ is therefore given by the $(n - 1)$ -th Fishburn number.

Let y be a sequence of positive integers. The *standardization* of y , denoted by $\text{std}(y)$, is the sequence obtained from y by relabeling the sequence according to their relative order. More specifically, the smallest integers in y are replaced with 1, the second smallest with 2, and so on, while preserving their original order in y . For

example, we have $\text{std}(8382683) = 4241342$. Let $x \in \text{Cay}_n$ and $\sigma \in \text{Cay}_k$ be two Cayley permutations with $k \leq n$. An *occurrence* of σ in x is a subsequence $x_{i_1}x_{i_2}\cdots x_{i_k}$ whose standardization equals σ . We say x *contains* the pattern σ if x contains an occurrence of σ . Otherwise, we say that x *avoids* the pattern σ or x is σ -*avoiding*. For example, the revised ascent sequence 5135245 contains two occurrences of the pattern 213, namely, the subsequences 324 and 325, but it avoids the pattern 112. Let $\hat{\mathcal{B}}_n(\sigma)$ denote the set of revised ascent sequences of length n that avoid the pattern σ and let

$$\hat{\mathcal{B}}_\sigma(t) = \sum_{n \geq 1} |\hat{\mathcal{B}}_n(\sigma)| t^n$$

be the generating function of $\hat{\mathcal{B}}_n(\sigma)$. Let $\hat{\mathcal{B}}_{n,m}(\sigma)$ denote the set of revised ascent sequences in $\hat{\mathcal{B}}_{n,m}$ that avoid the pattern σ .

Given two Cayley permutations σ and τ , they are said to be *Wilf-equivalent* if $|\hat{\mathcal{B}}_n(\sigma)| = |\hat{\mathcal{B}}_n(\tau)|$ for any $n \geq 1$. They are said to be *strong-Wilf-equivalent* if $\hat{\mathcal{B}}_n(\sigma) = \hat{\mathcal{B}}_n(\tau)$ for any $n \geq 1$. We denote the Wilf-equivalence and strong-Wilf-equivalence of σ and τ by $\sigma \sim \tau$ and $\sigma \simeq \tau$, respectively.

Zhou [18] conducted a systematic investigation of revised ascent sequences that avoid a single pattern of length 3, deriving explicit enumeration results for all cases except the pattern 111. The objective of this paper is to study the enumeration of revised ascent sequences avoiding a single pattern of length 4, as well as to classify such patterns according to Wilf-equivalence. Our findings, including precise enumeration results and a number of conjectures, are presented in Table 1. This work parallels the research of Duncan and Steingrímsson on ascent sequences [8] and Cerbai's work on modified ascent sequences [5].

The rest of this paper is organized as follows.

In Section 2, we present some fundamental properties of revised ascent sequences, including a basic result concerning their Wilf-equivalence.

In Section 3, we enumerate revised ascent sequences that avoid certain relatively simple patterns of length 4. For the patterns 2213, 2313, and 3123, we establish strong-Wilf-equivalence between these patterns and others whose enumeration is known. For the patterns 3312, 4213, and 4312, we derive their enumeration results by analyzing the structures of corresponding pattern avoiding revised ascent sequences and applying generating function techniques.

In Section 4, we first derive a recurrence relation for $|\hat{\mathcal{B}}_{n,m}(3212)|$, which leads to a partial differential equation for its bivariate generating function. Solving this equation yields an explicit enumeration formula for $|\hat{\mathcal{B}}_{n,m}(3212)|$. The enumeration of $\hat{\mathcal{B}}_{n,m}(2312)$ then follows directly from Corollary 2.5.

In Section 5 and Section 6, we enumerate the sets $\hat{\mathcal{B}}_n(3122)$ and $\hat{\mathcal{B}}_n(2132)$ by establishing a bijection between $\hat{\mathcal{B}}_n(3122)$ and $\hat{\mathcal{B}}_n(3212)$, and a bijection between $\hat{\mathcal{B}}_n(2132)$ and $\hat{\mathcal{B}}_n(122)$, respectively.

In Section 7, we propose several conjectures concerning the enumeration and Wilf-equivalence of revised ascent sequences, which provide directions for future research in this area.

σ	$ \hat{\mathcal{B}}_n(\sigma) _{n \geq 2}$	Reference
111	1,1,1, ...	[18]
221	1,2,3, ...	[18]
312, 122	2^{n-2}	[18]
231, 321, 3231	$2^{n-1} - n + 1$	[18]
121, 211, 2121	A026898	[18]
132, 3132	A082582	[18]
111	1, 1, 2, 4, 10, 29, 97, ...	open problem
112, 2213, 2132	Bell number	[18] and Sections 3.1, 6
213, 2313	Catalan number	[18] and Section 3.2
123, 3123	A105633	[18] and Section 3.3
3312	A001519	Section 3.4
4213	A181768	Section 3.5
4312, 3412	New	Section 3.6
3212, 2312, 3122	A047970	Sections 4, 5
3121	A368636	Conjecture 7.3
1233	A137550	Conjecture 7.4

Table 1: Enumeration of revised ascent sequences avoiding a single pattern σ .

2 Preliminaries

Let us begin our work with some basic properties of revised ascent sequences.

Lemma 2.1 (Zhou, [18]). *For $n, m \geq 1$, let $x = x_1x_2 \cdots x_n$ be a revised ascent sequence in $\hat{\mathcal{B}}_{n,m}$. Then we have $x_1 = m$, and m appears at least twice in x if $n \geq 2$.*

Let x be a revised ascent sequence in $\hat{\mathcal{B}}_n$. If $\max(x) = 1$, then $x = 1^n$, where a^n denotes a sequence of n consecutive occurrences of a . If $\max(x) > 1$, then $n \geq 2$ necessarily. We now describe a decomposition of x according to its leftmost 1.

Lemma 2.2. *For $n, m \geq 2$, let $x = x_1x_2 \cdots x_n$ be a revised ascent sequence in $\hat{\mathcal{B}}_{n,m}$. Then x can be decomposed uniquely as $x = L1R1^\ell$ for some $\ell \geq 0$, where L and R are non-empty sequences, and all entries in L and R are strictly greater than 1.*

Proof. According to Lemma 2.1, we have $x_1 = m \geq 2$. By the definition of revised ascent sequences, the entry immediately to the right of the leftmost 1 in x must be strictly greater than 1. Consequently, both L and R are non-empty sequences. Since only the leftmost 1 can serve as an ascent bottom in x , it follows that no larger number can appear after the second 1, if it exists. Therefore, x admits the decomposition described in the lemma. This completes the proof. ■

Lemma 2.3. *Let x be a sequence of positive integers. Then $\text{std}(x)$ is a Cayley permutation. Moreover, we have the following properties:*

- (1) *If $\text{Ascbot}(x) = \text{Nub}(x)$, then $\text{std}(x)$ is a revised ascent sequence.*

(2) If x avoids a pattern σ , then $\text{std}(x)$ also avoids σ .

Proof. By the definition of standardization, it is clear that $\text{std}(x)$ is a Cayley permutation. Furthermore, we have

$$\text{Ascbot}(\text{std}(x)) = \text{Ascbot}(x) = \text{Nub}(x) = \text{Nub}(\text{std}(x)).$$

Thus, statement (1) follows directly from the definition of revised ascent sequences, while statement (2) is evident. ■

Proposition 2.4. *Let m be a positive integer and let $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ be a Cayley permutation with $\max(\sigma) = m$ or $m - 1$. Then we have $m(m + 1)\sigma \simeq (m + 1)m\sigma$.*

Proof. Let x be a revised ascent sequence in $\hat{\mathcal{B}}_n$. Suppose that x contains the pattern $m(m + 1)\sigma$, i.e., there exists an occurrence $x_{i_1}x_{i_2}\cdots x_{i_{k+2}}$ of $m(m + 1)\sigma$ in x . By Lemma 2.1, we have $x_1 = \max(x)$. Since $\max(\sigma) \leq m$, it follows that the subsequence $x_1x_{i_1}x_{i_3}x_{i_4}\cdots x_{i_{k+2}}$ forms an occurrence of the pattern $(m + 1)m\sigma$ in x . Hence, x contains the pattern $(m + 1)m\sigma$.

Conversely, suppose that x contains the pattern $(m + 1)m\sigma$, i.e., there exists an occurrence $x_{j_1}x_{j_2}\cdots x_{j_{k+2}}$ of $(m + 1)m\sigma$ in x . Since $m + 1$ is the unique maximum value in the pattern $(m + 1)m\sigma$, we can always set $j_1 = 1$. Next, we choose x_{j_2} to be the leftmost entry that can play the role of m in the pattern $(m + 1)m\sigma$. This implies that $j_2 \in \text{Nub}(x)$. Since $\text{Ascbot}(x) = \text{Nub}(x)$, we have $x_{j_2} < x_{j_2+1}$. Moreover, because $\max(\sigma) \leq m$, it follows that $j_2 + 1 \neq j_3$. Consequently, the subsequence $x_{j_2}x_{j_2+1}x_{j_3}x_{j_4}\cdots x_{j_{k+2}}$ forms an occurrence of the pattern $m(m + 1)\sigma$ in x . In conclusion, a revised ascent sequence contains the pattern $m(m + 1)\sigma$ if and only if it contains the pattern $(m + 1)m\sigma$, as desired. ■

The following corollary follows directly from Proposition 2.4.

Corollary 2.5. *We have*

$$\begin{aligned} 1211 &\simeq 2111, 2311 \simeq 3211, 2312 \simeq 3212, \\ 2321 &\simeq 3221, 3412 \simeq 4312, 3421 \simeq 4321. \end{aligned}$$

We conclude this section by presenting some known results from [18] that will be utilized in this paper.

Proposition 2.6 (Zhou). *For $n, m \geq 1$, we have*

$$|\hat{\mathcal{B}}_{n,m}(112)| = \left\{ \begin{matrix} n - 1 \\ n - m \end{matrix} \right\},$$

where $\left\{ \begin{matrix} n - 1 \\ n - m \end{matrix} \right\}$ is the $(n - 1, n - m)$ -th Stirling number of the second kind.

Proposition 2.7 (Zhou). *For $n \geq 1$, we have*

$$|\hat{\mathcal{B}}_n(213)| = C_{n-1},$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n -th Catalan number.

Proposition 2.8 (Zhou). *We have*

$$\hat{\mathcal{B}}_{123}(t) = \sum_{n \geq 1} |\hat{\mathcal{B}}_n(123)|t^n = \frac{t^2 - 1 + \sqrt{t^4 + 2t^2 - 4t + 1}}{2(t - 1)}.$$

3 Easy patterns

We begin with some relatively simple patterns before tackling more challenging ones in the following sections.

3.1 Pattern 2213

Proposition 3.1. *For $n, m \geq 1$, we have*

$$|\hat{\mathcal{B}}_{n,m}(2213)| = \left\{ \begin{array}{l} n-1 \\ n-m \end{array} \right\}.$$

Consequently, $|\hat{\mathcal{B}}_n(2213)|$ is equal to the $(n-1)$ -th Bell number.

Proof. We proceed to prove the proposition by demonstrating that $2213 \simeq 112$. Let x be a revised ascent sequence in $\hat{\mathcal{B}}_n$. If x avoids the pattern 112, then x also avoids the pattern 2213. Conversely, if x contains the pattern 112, we will show that x must also contain the pattern 2213. Suppose $x_i x_j x_k$ is an occurrence of the pattern 112 in x . After fixing i and k , we select the rightmost j such that $x_i x_j x_k$ forms an occurrence of the pattern 112 in x . Then we have $x_i = x_j < x_k$. It follows that $j \notin \text{Nub}(x) = \text{Ascbot}(x)$. Thus, $x_j \geq x_{j+1}$. By our choice of j , we have $x_j > x_{j+1}$. This implies that $x_i x_j x_{j+1} x_k$ forms an occurrence of the pattern 2213 in x . Therefore, we have $2213 \simeq 112$. Then the conclusion follows directly from Proposition 2.6. ■

3.2 Pattern 2313

Proposition 3.2. *For $n \geq 1$, we have*

$$|\hat{\mathcal{B}}_n(2313)| = C_{n-1}.$$

Proof. From Proposition 2.7, to prove this proposition, it suffices to show that $2313 \simeq 213$. Let x be a revised ascent sequence in $\hat{\mathcal{B}}_n$. If x avoids the pattern 213, then x also avoids the pattern 2313. Conversely, assume that x contains the pattern 213. We select an occurrence $x_i x_j x_k$ of the pattern 213 in x with the following priorities: firstly, we maximize the value of x_k over all occurrences of 213 in x ; secondly, under this condition, x_i is maximized; and finally, we choose the leftmost such x_i .

We proceed to show that x must also contain the pattern 2313. Let x_ℓ be the leftmost entry in x such that $x_\ell = x_k$. By the definition of revised ascent sequences, if $\ell > 1$, then $x_\ell < x_{\ell+1}$. We consider three cases depending on the position of ℓ . If $\ell > j$, then $x_i x_j x_{\ell+1}$ forms an occurrence of the pattern 213 in x with $x_{\ell+1} > x_k$, which contradicts our choice of the pattern $x_i x_j x_k$. If $i < \ell < j$, then $x_i x_\ell x_j x_k$ forms an occurrence of the pattern 2313, as desired. Now we consider the case $\ell < i$. By our choice of i , we have $i \in \text{Nub}(x) = \text{Ascbot}(x)$, which implies that $x_i < x_{i+1}$. We claim that $x_{i+1} = x_k$. Assume for contradiction that $x_{i+1} \neq x_k$. If $x_{i+1} < x_k$, then $x_{i+1} x_j x_k$

forms an occurrence of the pattern 213 in x with $x_i < x_{i+1}$, contradicting our choice of $x_i x_j x_k$. If $x_{i+1} > x_k$, then $x_\ell x_i x_{i+1}$ forms an occurrence of the pattern 213 in x with $x_k < x_{i+1}$, again a contradiction. This proves the claim. Consequently, $x_i x_{i+1} x_j x_k$ forms an occurrence of the pattern 2313 in x , as desired. ■

3.3 Pattern 3123

Proposition 3.3. *We have*

$$\hat{\mathcal{B}}_{3123}(t) = \sum_{n \geq 1} |\hat{\mathcal{B}}_n(3123)|t^n = \frac{t^2 - 1 + \sqrt{t^4 + 2t^2 - 4t + 1}}{2(t - 1)}.$$

Proof. From Proposition 2.8, to prove this proposition, it suffices to show that $3123 \simeq 123$. Let x be a revised ascent sequence in $\hat{\mathcal{B}}_n$. If x avoids the pattern 123, then x also avoids the pattern 3123. Conversely, assume that x contains the pattern 123. We proceed to show that x must also contain the pattern 3123. Select an occurrence $x_i x_j x_k$ of the pattern 123 in x such that the value of x_k is maximal among all occurrences of 123. Let x_ℓ be the leftmost occurrence of the value x_k in x . Then we have $\ell \in \text{Nub}(x) = \text{Ascbot}(x)$. We claim that $\ell < i$. If not, we have $x_\ell < x_{\ell+1}$. This implies that $x_i x_\ell x_{\ell+1}$ forms an occurrence of the pattern 123 in x with $x_{\ell+1} > x_k$, contradicting the maximality of x_k in our initial choice of $x_i x_j x_k$. This proves the claim, implying that $x_\ell x_i x_j x_k$ forms an occurrence of the pattern 3123 in x . Consequently, $3123 \simeq 123$. This completes the proof. ■

3.4 Pattern 3312

Proposition 3.4. *We have*

$$\hat{\mathcal{B}}_{3312}(t) = \sum_{n \geq 1} |\hat{\mathcal{B}}_n(3312)|t^n = \frac{t(1 - 2t)}{t^2 - 3t + 1}. \tag{3.1}$$

Proof. For convenience, let $f(t) = \hat{\mathcal{B}}_{3312}(t)$. Given two positive integers n, m , let x be a revised ascent sequence in $\hat{\mathcal{B}}_{n,m}(3312)$. If $m = 1$, then $x = 1^n$, which contributes $\frac{t}{1-t}$ to $f(t)$. If $m > 1$, by Lemma 2.2, x can be decomposed uniquely as $x = L1R1^\ell$ for some $\ell \geq 0$, where L and R are non-empty sequences, and all entries in L and R are strictly greater than 1. We consider two cases according to the value x_2 .

Case 1. $x_2 = 1$.

From Lemma 2.1, L only contains one element m in this case. Let $R' = \text{std}(mR)$. Note that x is a revised ascent sequence. We deduce $\text{Ascbot}(mR) = \text{Nub}(mR)$. By Lemma 2.3, R' is a revised ascent sequence. Moreover, it is straightforward to verify that x avoids the pattern 3312 if and only if R' avoids 3312. Note that $|R'| \geq 2$. Hence, this case contributes $\frac{t(f(t)-t)}{1-t}$ to $f(t)$.

Case 2. $x_2 > 1$.

We claim that R consists solely of m , that is, $R = m^i$ for some $i \geq 1$. Since x avoids the pattern 3312, to prove the claim, it suffices to show that L contains at least two

occurrences of m . From Lemma 2.1, we have $x_1 = m$. If L consists solely of the element m , then clearly it contains at least two occurrences of m . Otherwise, if L contains elements other than m , let b be the largest element in L that is less than m . Given that x is a revised ascent sequence, the entry immediately to the right of the leftmost b in x must be m . This proves the claim. By Lemma 2.3, $\text{std}(L)$ is a revised ascent sequence. Moreover, x avoids the pattern 3312 if and only if $\text{std}(L)$ avoids 3312. Note that $|L| \geq 2$ in this case. Hence, this case contributes $\frac{t^2(f(t)-t)}{(1-t)^2}$ to $f(t)$.

From the argument above, we derive the following functional equation for $f(t)$:

$$f(t) = \frac{t}{1-t} + \frac{t(f(t)-t)}{1-t} + \frac{t^2(f(t)-t)}{(1-t)^2}.$$

Solving the above equation gives the desired expression (3.1). ■

From (3.1), we obtain the sequence

$$(|\hat{\mathcal{B}}_n(3312)|)_{n \geq 1} = (1, 1, 2, 5, 13, 34, 89, 233, 610, 1597, \dots),$$

which corresponds to A001519 in the OEIS [15]. This sequence is a bisection of the Fibonacci sequence, that is, $|\hat{\mathcal{B}}_n(3312)| = F_{2n-1}$, where F_n denotes the n -th Fibonacci number.

3.5 Pattern 4213

Lemma 3.5. *Given two integers $n, m \geq 2$, let x be a revised ascent sequence in $\hat{\mathcal{B}}_{n,m}(4213)$ with the decomposition $x = L1R1^\ell$ as described in Lemma 2.2. Then for any entry $a < m$ in L and any entry $b < m$ in R , we have $a \geq b$.*

Proof. Suppose, for contradiction, that there exist $a < m$ in L and $b < m$ in R such that $a < b$. Note that $a, b > 1$ by Lemma 2.2. By Lemma 2.1, we have $x_1 = m$. Then the sequence $ma1b$ forms an occurrence of the pattern 4213 in x , which contradicts the assumption that x is 4213-avoiding. This completes the proof. ■

Proposition 3.6. *We have*

$$\hat{\mathcal{B}}_{4213}(t) = \sum_{n \geq 1} |\hat{\mathcal{B}}_n(4213)|t^n = \frac{t}{2} \left(3 - \sqrt{\frac{1-5t}{1-t}} \right). \tag{3.2}$$

Proof. For convenience, let $g(t) = \hat{\mathcal{B}}_{4213}(t)$. Given two positive integers n, m , let x be a revised ascent sequence in $\hat{\mathcal{B}}_{n,m}(4213)$. If $m = 1$, then $x = 1^n$, which contributes $\frac{t}{1-t}$ to $g(t)$. If $m > 1$, by Lemma 2.2, x can be decomposed uniquely as $x = L1R1^\ell$ for some $\ell \geq 0$, where L and R are non-empty sequences, and all entries in L and R are strictly greater than 1. Let $i = |L|$ and let $s = \min(L)$. Let $L' = \text{std}(L)$. Note that

$$\text{Ascbot}(L) = \text{Ascbot}(x) \cap [i] = \text{Nub}(x) \cap [i] = \text{Nub}(L).$$

By Lemma 2.3, L' is a revised ascent sequence. Now we consider three cases.

Case 1. $s = m$.

In this case, we have $L = m^i$ and $L' = 1^i$. Since x is a revised ascent sequence avoiding the pattern 4213, it follows that mR also avoids 4213, and moreover, $\text{Ascbot}(mR) = \text{Nub}(mR)$. Let $R' = \text{std}(mR)$. By Lemma 2.3, R' is a 4213-avoiding revised ascent sequence. Note that $|R'| \geq 2$. Therefore, this case contributes $\frac{t(g(t)-t)}{(1-t)^2}$ to $g(t)$.

Case 2. $s < m$ and s does not occur in R .

From Lemma 3.5, we also have $\text{Ascbot}(mR) = \text{Nub}(mR)$. Then by an argument similar to Case 1, $R' = \text{std}(mR)$ is a 4213-avoiding revised ascent sequence. Conversely, Lemma 3.5 guarantees that one can uniquely recover such an $x \in \hat{\mathcal{B}}_n(4213)$ from L' and R' . Note that $L' \neq 1^i$ and $|R'| \geq 2$. Therefore, this case contributes $\frac{(g(t)-t)(g(t)-\frac{t}{1-t})}{1-t}$ to $g(t)$.

Case 3. $s < m$ and s occurs in R .

By inserting the sequence $m^{\ell+1}$, where ℓ is the number of trailing 1's in $x = L1R1^\ell$, immediately after the leftmost occurrence of s in mR and standardizing, we obtain a sequence denoted by R'' . We deduce that R'' can be any revised ascent sequence in $\hat{\mathcal{B}}_{n+1-i}(4213)$ except 1^{n+1-i} . Similarly, L' can be any revised ascent sequence in $\hat{\mathcal{B}}_i(4213)$ except 1^i . Conversely, Lemma 3.5 guarantees that one can uniquely recover such an $x \in \hat{\mathcal{B}}_n(4213)$ from L' and R'' . Therefore, this case contributes $\frac{(g(t)-\frac{t}{1-t})^2}{t}$ to $g(t)$.

From the argument above, we derive the following functional equation for $g(t)$:

$$g(t) = \frac{t}{1-t} + \frac{t(g(t)-t)}{(1-t)^2} + \frac{(g(t)-t)(g(t)-\frac{t}{1-t})}{1-t} + \frac{(g(t)-\frac{t}{1-t})^2}{t}.$$

Solving the above equation gives the desired expression (3.2). ■

From (3.2), we obtain the sequence

$$(|\hat{\mathcal{B}}_n(4213)|)_{n \geq 1} = (1, 1, 2, 5, 15, 51, 188, 731, 2950, \dots),$$

which corresponds to A181768 in the OEIS [15]. If the initial term 1 is omitted, the sequence coincides with the binomial transform of the Catalan numbers.

3.6 Patterns 4312 and 3412

The following lemma is analogous to Lemma 3.5, and its proof follows a similar line of reasoning, so we omit it here.

Lemma 3.7. *Given two integers $n, m \geq 2$, let x be a revised ascent sequence in $\hat{\mathcal{B}}_{n,m}(4312)$ with the decomposition $x = L1R1^\ell$ as described in Lemma 2.2. Then for any entry $a < m$ in L and any entry $b < m$ in R , we have $a \leq b$.*

Proposition 3.8. *We have*

$$\hat{\mathcal{B}}_{4312}(t) = \sum_{n \geq 1} |\hat{\mathcal{B}}_n(4312)|t^n = \frac{2t^2 - 2t + 1 - \sqrt{8t^4 - 20t^3 + 20t^2 - 8t + 1}}{2(1-t)}. \tag{3.3}$$

Proof. For convenience, let $h(t) = \hat{\mathcal{B}}_{4312}(t)$. Given two positive integers n, m , let x be a revised ascent sequence in $\hat{\mathcal{B}}_{n,m}(4312)$. If $m = 1$, then $x = 1^n$, which contributes $\frac{t}{1-t}$ to $h(t)$. If $m > 1$, by Lemma 2.2, x can be decomposed uniquely as $x = L1R1^\ell$ for some $\ell \geq 0$, where L and R are non-empty sequences, and all entries in L and R are strictly greater than 1. Let $i = |L|$. If L consists solely of m , set $s = m$; otherwise, let s be the largest element of L other than m . Let $L' = \text{std}(L)$. It is straightforward to verify that L' is a revised ascent sequence in $\hat{\mathcal{B}}_i(4312)$. We consider the following three cases.

Case 1. $s = m$.

This case follows a similar argument to Case 1 of Proposition 3.6, and contributes $\frac{t(h(t)-t)}{(1-t)^2}$ to $h(t)$.

Case 2. $s < m$ and s does not occur in R .

This case follows a similar argument to Case 2 of Proposition 3.6, and contributes $\frac{(h(t)-t)(h(t)-\frac{t}{1-t})}{1-t}$ to $h(t)$.

Case 3. $s < m$ and s occurs in R .

By Lemma 3.7, all entries in R are greater than or equal to s . Since the leftmost s of x lies in L , it follows that no occurrence of s in R can serve as an ascent bottom in R . Consequently, all these occurrences of s must appear at the end of R . By removing these trailing s from mR and standardizing, we obtain a sequence denoted by R' . We deduce that R' can be any 4312-avoiding revised ascent sequence. Similarly, L' can be any revised ascent sequence in $\hat{\mathcal{B}}_i(4312)$ except 1^i . Therefore, this case contributes $\frac{th(t)(h(t)-\frac{t}{1-t})}{(1-t)^2}$ to $h(t)$.

From the argument above, we derive the following functional equation for $h(t)$:

$$h(t) = \frac{t}{1-t} + \frac{t(h(t)-t)}{(1-t)^2} + \frac{(h(t)-t)(h(t)-\frac{t}{1-t})}{1-t} + \frac{th(t)(h(t)-\frac{t}{1-t})}{(1-t)^2}.$$

Solving the above equation gives the desired expression (3.3). ■

From (3.3), we obtain the sequence

$$(|\hat{\mathcal{B}}_n(4312)|)_{n \geq 1} = (1, 1, 2, 5, 15, 50, 176, 639, 2369, \dots).$$

The following proposition follows directly from Corollary 2.5 and Proposition 3.8.

Proposition 3.9. *We have*

$$\hat{\mathcal{B}}_{3412}(t) = \sum_{n \geq 1} |\hat{\mathcal{B}}_n(3412)|t^n = \frac{2t^2 - 2t + 1 - \sqrt{8t^4 - 20t^3 + 20t^2 - 8t + 1}}{2(1-t)}.$$

4 Patterns 3212 and 2312

In this section, we concentrate on the enumeration of revised ascent sequences that avoid the pattern 3212 or 2312. Owing to Corollary 2.5, it suffices to consider the pattern 3212.

Lemma 4.1. *Let $a_{n,m} = |\hat{\mathcal{B}}_{n,m}(3212)|$. For $n, m \geq 2$, the numbers $a_{n,m}$ satisfy the recurrence relation*

$$a_{n,m} = \sum_{k=1}^{m-1} \binom{m-2}{k-1} \sum_{i=1}^{n-2} \sum_{j=1}^{n-1-i} a_{i,k} a_{j+1,m-k}, \tag{4.1}$$

with the initial condition $a_{n,1} = 1$ for $n \geq 1$.

Proof. The only sequence in $\hat{\mathcal{B}}_{n,1}(3212)$ is 1^n . Thus, we have $a_{n,1} = 1$ for $n \geq 1$. For $n, m \geq 2$, let x be a revised ascent sequence in $\hat{\mathcal{B}}_{n,m}(3212)$. By Lemma 2.2, we have $x = L1R1^\ell$ for some $\ell \geq 0$, where L and R are non-empty sequences, and all entries in L and R are strictly greater than 1. Let $i = |L|$ and $j = |R|$. From Lemma 2.1, we have $x_1 = m$. Since x avoids the pattern 3212, it follows that for any $a < m$ in L and any $b < m$ in R , we have $a \neq b$. We claim that m occurs in R . Assume to the contrary that $\max(R) = s < m$. Then s does not occur in L . Since x is a revised ascent sequence, the entry immediately to the right of the leftmost s must be greater than s . It follows that R contains an entry greater than s , contradicting the assumption that $\max(R) = s$. This proves the claim.

Suppose that L contains k different integers. Note that x is a Cayley permutation with $\max(x) = m$, and L and R share exactly one common element, namely, m . It follows that R must contain $m - k$ different integers. Let $L' = \text{std}(L)$ and $R' = \text{std}(mR)$. One can easily check that $L' \in \hat{\mathcal{B}}_{i,k}(3212)$ and $R' \in \hat{\mathcal{B}}_{j+1,m-k}(3212)$. Conversely, given $L' \in \hat{\mathcal{B}}_{i,k}(3212)$ and $R' \in \hat{\mathcal{B}}_{j+1,m-k}(3212)$, we can reconstruct the original sequence $x \in \hat{\mathcal{B}}_{n,m}(3212)$ through the following way:

- Select $k-1$ integers from $\{2, 3, \dots, m-1\}$ for L and assign the remaining $m-k-1$ integers to R (with $\binom{m-2}{k-1}$ possibilities).
- Given that m appears in both L and R , we can uniquely recover L and R from L' and R' , respectively.
- Finally, combine them to obtain the sequence $L1R1^\ell$, where $\ell = n - 1 - i - j$.

Summing over all possible values of i, j, k , we obtain the recurrence relation (4.1). ■

Let

$$A(t, u) = \sum_{n=1}^{\infty} t^n \sum_{m=1}^{\infty} a_{n,m} \frac{u^{m-1}}{(m-1)!},$$

$$A_n(u) = \sum_{m=1}^{\infty} a_{n,m} \frac{u^{m-1}}{(m-1)!}.$$

It is obvious that

$$A(t, u) = \sum_{n=1}^{\infty} A_n(u)t^n.$$

Lemma 4.2. *The generating function $A(t, u)$, for brevity denoted by A , satisfies the following partial differential equation*

$$\frac{\partial A}{\partial u} = \frac{A(A-t)}{1-t},$$

with the initial condition $A(t, 0) = \frac{t}{1-t}$.

Proof. Multiplying both sides of (4.1) by $\frac{t^n u^{m-2}}{(m-2)!}$ and summing over all $n, m \geq 2$, The left-hand side is just $\frac{\partial A}{\partial u}$. For the right-hand side, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} t^n \sum_{m=2}^{\infty} \frac{u^{m-2}}{(m-2)!} \sum_{k=1}^{m-1} \binom{m-2}{k-1} \sum_{i=1}^{n-2} \sum_{j=1}^{n-1-i} a_{i,k} a_{j+1,m-k} \\ &= \sum_{n=2}^{\infty} t^n \sum_{i=1}^{n-2} \sum_{j=1}^{n-1-i} \sum_{m=2}^{\infty} \sum_{k=1}^{m-1} \binom{m-2}{k-1} a_{i,k} a_{j+1,m-k} \frac{u^{m-2}}{(m-2)!} \\ &= \sum_{n=2}^{\infty} \sum_{i=1}^{n-2} \sum_{j=1}^{n-1-i} A_i(u) A_{j+1}(u) t^n \\ &= \sum_{n=2}^{\infty} \sum_{i=1}^{n-2} \sum_{j=2}^{n-i} A_i(u) A_j(u) t^n \\ &= \frac{A(A-t)}{1-t}, \end{aligned}$$

where in the last step we used the fact that $A_1(u) = 1$. Thus, we have established the desired partial differential equation. The initial condition $A(t, 0) = \frac{t}{1-t}$ is readily verified from the definition of $A(t, u)$. This completes the proof. ■

Proposition 4.3. *We have*

$$A(t, u) = \frac{t}{1 - te^{\frac{tu}{1-t}}}.$$

Furthermore, for $n, m \geq 1$, we have

$$|\hat{\mathcal{B}}_{n,m}(3212)| = \sum_{k=0}^{n-m} k^{m-1} \binom{n-2-k}{n-m-k}.$$

Proof. From Lemma 4.2, the generating function $A(t, u)$ satisfies the following initial value problem for a partial differential equation:

$$\frac{\partial A}{\partial u} = \frac{A(A-t)}{1-t}, \quad A(t, 0) = \frac{t}{1-t}.$$

Through separation of variables and partial fraction decomposition, we have

$$\begin{aligned} \int \frac{dA}{A(A-t)} &= \int \frac{du}{1-t}, \\ \frac{1}{t} \ln \left| \frac{A-t}{A} \right| &= \frac{u}{1-t} + C. \end{aligned}$$

The initial condition gives $C = \frac{1}{t} \ln t$, leading to the desired solution for A .

For the second part of the proposition, we expand the solution as follows:

$$\begin{aligned} A(t, u) &= t \sum_{k=0}^{\infty} t^k e^{\frac{ktu}{1-t}} \\ &= t \sum_{k=0}^{\infty} t^k \sum_{m=0}^{\infty} \frac{(kt)^m}{(1-t)^m} \frac{u^m}{m!} \\ &= \sum_{m=1}^{\infty} \frac{u^{m-1}}{(m-1)!} \sum_{k=0}^{\infty} \frac{k^{m-1} t^{k+m}}{(1-t)^{m-1}}, \end{aligned} \tag{4.2}$$

where we reindexed m with $m - 1$ in the last step. Using the binomial expansion of $(1 - t)^{-(m-1)}$, we have

$$(1 - t)^{-(m-1)} = \sum_{i=0}^{\infty} \binom{m-2+i}{i} t^i.$$

Combining terms and extracting the coefficient of $\frac{t^n u^{m-1}}{(m-1)!}$ in (4.2) gives

$$a_{n,m} = \sum_{\substack{k \geq 0, i \geq 0 \\ k+m+i=n}} k^{m-1} \binom{m-2+i}{i} = \sum_{k=0}^{n-m} k^{m-1} \binom{n-2-k}{n-m-k}.$$

This completes the proof. ■

From Proposition 4.3, we have

$$|\hat{\mathcal{B}}_n(3212)| = \sum_{m=1}^n \sum_{k=0}^{n-m} k^{m-1} \binom{n-2-k}{n-m-k}.$$

It follows that the sequence

$$(|\hat{\mathcal{B}}_n(3212)|)_{n \geq 2} = (1, 2, 5, 14, 43, 144, 523, 2048, \dots),$$

which corresponds to A047970 in the OEIS [15]. This sequence also counts the modified ascent sequences of length n avoiding the pattern 1232 [5].

The following proposition follows directly from Corollary 2.5 and Proposition 4.3.

Proposition 4.4. *For $n, m \geq 1$, we have*

$$|\hat{\mathcal{B}}_{n,m}(2312)| = \sum_{k=0}^{n-m} k^{m-1} \binom{n-2-k}{n-m-k}.$$

5 Pattern 3122

In this section, we enumerate $\hat{\mathcal{B}}_n(3122)$ by constructing a bijection Γ between $\hat{\mathcal{B}}_n(3122)$ and $\hat{\mathcal{B}}_n(3212)$. The idea behind this bijection Γ is as follows: we repeatedly swap two adjacent components in a revised ascent sequence $x \in \hat{\mathcal{B}}_n(3122)$. Each such swap converts at least one occurrence of the pattern 3212 in x into an occurrence of

the pattern 3122 in x . By choosing a specific order for these swaps, we ensure that the process is reversible and that the resulting sequence eventually avoids 3212.

For $x \in \hat{\mathcal{B}}_n$, define the following two sets:

$$p(x) = \{a \mid \text{there exists an occurrence } x_i x_j x_k x_\ell \text{ of } 3212 \text{ in } x \text{ such that } x_j = a\},$$

$$q(x) = \{a \mid \text{there exists an occurrence } x_i x_j x_k x_\ell \text{ of } 3122 \text{ in } x \text{ such that } x_k = a\}.$$

Example 5.1. Let $x = 85854836866542787121 \in \hat{\mathcal{B}}_{20}$. We have $p(x) = \{2, 4, 5\}$ and $q(x) = \{6, 7\}$.

Our bijection Γ depends on the following decomposition of x . Let $x \in \hat{\mathcal{B}}_{n,m}$ and let $a \in p(x) \setminus q(x)$. It is clear that $a < m = x_1$. Denote by x_ℓ and x_r the leftmost and rightmost occurrences of a in x , respectively. By the definition of revised ascent sequences, we have $a = x_\ell < x_{\ell+1}$. Let j be the largest index such that all entries in $x_{\ell+1}x_{\ell+2}\cdots x_j$ are greater than or equal to a . This implies that $x_{j+1} < a$. Since $a \in p(x)$, we deduce that $\ell < j < r$. Consequently, x admits a unique a -based decomposition of the form $x = LPQR$, where

$$L = x_1x_2\cdots x_{\ell-1}, \quad P = x_\ell x_{\ell+1}\cdots x_j, \quad Q = x_{j+1}x_{j+2}\cdots x_{r-1}, \quad \text{and } R = x_r x_{r+1}\cdots x_n.$$

We denote this a -based decomposition of x by

$$\text{Dec}_a(x) = (L, P, Q, R).$$

Continuing with the same sequence $x = 85854836866542787121$ from Example 5.1, recall that $p(x) = \{2, 4, 5\}$ and $q(x) = \{6, 7\}$. Choose $a = 5 \in p(x) \setminus q(x)$. The 5-based decomposition of x is given by

$$\text{Dec}_5(x) = (8, 585, 4836866, 542787121).$$

Property 5.2. Given $x \in \hat{\mathcal{B}}_n$ and $a \in p(x) \setminus q(x)$, let $\text{Dec}_a(x) = (L, P, Q, R)$. Then we have

- (i) All entries in L are strictly greater than a .
- (ii) P starts with the leftmost a of x , and all entries in P are greater than or equal to a .
- (iii) Q starts with an entry strictly less than a and contains no entries equal to a .
- (iv) R starts with the rightmost a of x .
- (v) Furthermore, if $\max(p(x)) = a$, then P and Q have no common elements other than possibly m .

Proof. Since $a \notin q(x)$, it follows that all entries in L are strictly greater than a and no entry in Q equals a . Properties (i)–(iv) then follow directly from the construction of the decomposition. For (v), assume to the contrary that P and Q have a common

element $b < m$. By (ii) and (iii), we have $b > a$. It is then easily seen that $b \in p(x)$, which contradicts the assumption that $\max(p(x)) = a$. Hence, (v) follows. ■

The following transformation will play an essential role in the construction of Γ .

The Transformation γ_a

Let $x \in \hat{\mathcal{B}}_n$ and let $a \in p(x) \setminus q(x)$ with $\text{Dec}_a(x) = (L, P, Q, R)$. Define $\gamma_a(x)$ as the sequence obtained by rearranging the components as follows:

$$\gamma_a(x) = LQPR.$$

Again consider the same sequence $x = 85854836866542787121$ from Example 5.1. Recall that $\text{Dec}_5(x) = (8, 585, 4836866, 542787121)$. Hence, we have

$$\gamma_5(x) = 84836866585542787121.$$

Lemma 5.3. *Let $x \in \hat{\mathcal{B}}_{n,m}$ be a revised ascent sequence with $\max(p(x)) = a$. If $a \notin q(x)$, then we have $y = \gamma_a(x) \in \hat{\mathcal{B}}_{n,m}$.*

Proof. Let $\text{Dec}_a(x) = (L, P, Q, R)$. Then, by the definition of γ_a , we have $y = \gamma_a(x) = LQPR$. This implies that $\text{Im}(y) = \text{Im}(x) = [m]$, meaning y is a Cayley permutation. Suppose that under the mapping γ_a , the entry x_i in x corresponds to y_{j_i} in y . From Property 5.2, P and Q have no common elements other than possibly m . Therefore, we have x_i is a leftmost occurrence of x if and only if y_{j_i} is a leftmost occurrence of y . Moreover, we claim that x_i is an ascent bottom of x if and only if y_{j_i} is an ascent bottom of y . To prove the claim, it suffices to show that under the transformation γ_a , whether the last element of each of L , P , and Q is an ascent bottom remains unchanged. This follows directly from Property 5.2. Noting that $\text{Ascbot}(x) = \text{Nub}(x)$, we have $\text{Ascbot}(y) = \text{Nub}(y)$. Therefore, y is a revised ascent sequence in $\hat{\mathcal{B}}_{n,m}$, completing the proof. ■

Lemma 5.4. *Let $x \in \hat{\mathcal{B}}_n$ be a revised ascent sequence satisfying $\max(p(x)) = a$, and either $q(x) = \emptyset$ or $\min(q(x)) > a$. Then for $y = \gamma_a(x)$, the following properties hold:*

- (i) $p(y) = p(x) \setminus \{a\}$.
- (ii) $q(y) = q(x) \cup \{a\}$.

Proof. By hypothesis, we have $a \in p(x) \setminus q(x)$. Let $\text{Dec}_a(x) = (L, P, Q, R)$. Then, by the definition of γ_a , we have $y = \gamma_a(x) = LQPR$. Applying Property 5.2, it is clear that $a \in q(y)$. Moreover, since there is no entry strictly less than a between the leftmost and rightmost occurrences of a in y , we have $a \notin p(y)$.

For (i), we first claim that $\max(p(y)) < a$. Assume to the contrary that there exists some $b > a$ such that $b \in p(y)$. By the definition of $p(y)$, there exists an occurrence $y_i y_j y_k y_\ell$ of the pattern 3212 in y , where $y_j = y_\ell = b$. Since $\max(p(x)) = a < b$, we have $b \notin p(x)$. Note that $x = LPQR$ and $y = LQPR$. Consequently, any occurrence of the pattern 3212 in y with the second entry equal to b must involve elements from both P and Q . From Lemma 5.3, we know y is a revised ascent

sequence. Lemma 2.1 then establishes that $y_1 = \max(y)$, which implies that $y_1y_jy_ky_\ell$ is an occurrence of the pattern 3212 in y . Note that both P and Q must contribute to the subsequence $y_1y_jy_ky_\ell$. Hence, there are two possible cases: either $y_\ell \in R$ or $y_\ell \in P$. By abuse of notation, we will always write $y_\ell \in R$ to denote the element y_ℓ appears in the segment R , both here and in what follows. If $y_\ell \in R$, we must have $y_j \in Q$ and $y_k \in P$. By Property 5.2, replacing y_k with the first entry of R in $y_1y_jy_ky_\ell$ still yields an occurrence of the pattern 3212 in y . However, in this case, P contributes nothing to this new occurrence, which is a contradiction. If $y_\ell \in P$, then by Property 5.2, P and Q have no common elements other than possibly m , which implies that $y_j \notin Q$. Hence, we must have $y_j \in L$ and $y_k \in Q$. By Property 5.2, replacing y_k with the first entry of P in $y_1y_jy_ky_\ell$ still results in an occurrence of the pattern 3212 in y . In this case, Q contributes nothing to this new occurrence, which is again a contradiction. This proves the claim.

By Property 5.2, all entries in P are greater than or equal to a . Consequently, for any $b < a$, any occurrence of the pattern 3212 in x (respectively, y) with the 3 corresponding to x_1 (respectively, y_1) and the 2 corresponding to b cannot involve elements from P . It follows immediately that for any $b < a$, we have $b \in p(x)$ if and only if $b \in p(y)$. Note that $\max(p(x)) = a$ and $\max(p(y)) < a$. Therefore, $p(y) = p(x) \setminus \{a\}$.

For (ii), it suffices to show that for any $c \neq a$, we have $c \in q(x)$ if and only if $c \in q(y)$. We first prove the forward direction: if $c \in q(x)$, then $c \in q(y)$. Suppose to the contrary that there exists some $c \neq a$ such that $c \in q(x)$ but $c \notin q(y)$. By the definition of $q(x)$, there exists an occurrence $x_r x_s x_u x_v$ of the pattern 3122 in x , where $x_u = x_v = c$. It is clear that $x_1 x_s x_u x_v$ is also an occurrence of the pattern 3122 in x . Since $c \in q(x)$ and $c \notin q(y)$, any occurrence of the pattern 3122 in x with the third entry equal to c must involve elements from both P and Q . We consider two cases.

Case (1): $c < a$.

From Property 5.2, all entries in P are greater than or equal to a . Hence, P cannot contribute anything to the occurrence $x_1 x_s x_u x_v$, which is a contradiction.

Case (2): $c > a$.

Since $x = LPQR$ and $x_1 x_s x_u x_v$ involves elements from both P and Q , there are two cases: either $x_v \in R$ or $x_v \in Q$. If $x_v \in R$, we must have $x_s \in P$ and $x_u \in Q$. If $x_v \in Q$, since P and Q have no common elements other than possibly m , we must have $x_s \in P$ and $x_u \in Q$. In either case, by Property 5.2, replacing x_s with the first entry of Q in $x_1 x_s x_u x_v$ yields an occurrence of the pattern 3122 in x . However, in this new occurrence, P contributes nothing, which is a contradiction.

Therefore, for any $c \neq a$, if $c \in q(x)$, then $c \in q(y)$. Conversely, the implication that for any $c \neq a$, if $c \in q(y)$ then $c \in q(x)$ can be proved similarly. Therefore, we obtain $q(y) = q(x) \cup \{a\}$. ■

Now we are in a position to define our map $\Gamma : \hat{\mathcal{B}}_n(3122) \rightarrow \hat{\mathcal{B}}_n(3212)$. Given $x \in \hat{\mathcal{B}}_n(3122)$, define $\Gamma(x)$ through the following iterative process:

1. Initialization: Set $i := 0$, $x^{(0)} := x$, and $S := p(x) \setminus q(x)$.

2. Iteration: While $S \neq \emptyset$:

- Increment $i := i + 1$.
- Select the maximum element: $a_i := \max(S)$.
- Apply the transformation: $x^{(i)} := \gamma_{a_i}(x^{(i-1)})$.
- Refresh $S := p(x^{(i)}) \setminus q(x^{(i)})$.

3. Termination: The procedure stops when $S = \emptyset$. The final output is $\Gamma(x) := x^{(i)}$.

Example 5.5. Let $x = 87868658548365427121 \in \hat{\mathcal{B}}_{20}(3122)$. Then we have $\Gamma(x) = 83686658554841278721$ with the detailed process below, where the a_{i+1} -based decomposition of $x^{(i)}$ is performed by placing vertical bars at specific positions of $x^{(i)}$.

$$\begin{aligned}
 x &= 8|78|6865854836542|7121 \\
 &\quad \downarrow \gamma_7 \\
 x^{(1)} &= 8|686|585483|6542787121 \\
 &\quad \downarrow \gamma_6 \\
 x^{(2)} &= 8|585|4836866|542787121 \\
 &\quad \downarrow \gamma_5 \\
 x^{(3)} &= 8|48|368665855|42787121 \\
 &\quad \downarrow \gamma_4 \\
 x^{(4)} &= 8368665855484|2787|1|21 \\
 &\quad \downarrow \gamma_2 \\
 x^{(5)} &= 83686658554841278721 = \Gamma(x).
 \end{aligned}$$

Let $x \in \hat{\mathcal{B}}_n(3122)$ be a revised ascent sequence with $p(x) = \{a_1, a_2, \dots, a_k\}$ where $a_1 > a_2 > \dots > a_k$. By the definition of Γ and Lemma 5.4, we obtain that

$$\Gamma(x) = \gamma_{a_k} \circ \dots \circ \gamma_{a_2} \circ \gamma_{a_1}(x).$$

Let $x^{(i)} = \gamma_{a_i}(x^{(i-1)})$ for $1 \leq i \leq k$ with the convention that $x^{(0)} = x$. From Lemma 5.4, we have

$$\begin{aligned}
 p(x^{(i)}) &= \{a_{i+1}, a_{i+2}, \dots, a_k\}, \\
 q(x^{(i)}) &= \{a_1, a_2, \dots, a_i\},
 \end{aligned}$$

for $0 \leq i \leq k$. In particular, we have $p(x^{(k)}) = \emptyset$.

The next lemma follows immediately by iteratively applying Lemmas 5.3 and 5.4.

Lemma 5.6. *Let $x \in \hat{\mathcal{B}}_{n,m}(3122)$ be a revised ascent sequence and let $y = \Gamma(x)$. Then we have $y \in \hat{\mathcal{B}}_{n,m}(3212)$.*

To prove that Γ is a bijection between $\hat{\mathcal{B}}_n(3122)$ and $\hat{\mathcal{B}}_n(3212)$, we construct its inverse map Θ . Let $x \in \hat{\mathcal{B}}_{n,m}$ and let $a \in q(x) \setminus p(x)$. It is clear that $a < m = x_1$. Denote by x_ℓ and x_r the leftmost and rightmost occurrences of a in x , respectively. Since $a \in q(x) \setminus p(x)$, it follows that all entries between x_ℓ and x_r in x are greater than or equal to a , and there exists at least one entry to the left of x_ℓ that is strictly less than a . Let x_j be the leftmost entry in x such that $x_j < a$. Consequently, x admits a unique a -based decomposition of the form $x = LQPR$, where

$$L = x_1x_2 \cdots x_{j-1}, \quad Q = x_jx_{j+1} \cdots x_{\ell-1}, \quad P = x_\ell x_{\ell+1} \cdots x_{r-1} \quad \text{and} \quad R = x_r x_{r+1} \cdots x_n.$$

We denote this a -based decomposition of x by

$$\text{Dec}'_a(x) = (L, Q, P, R).$$

Example 5.7. Let $x = 84836866585542787121 \in \hat{\mathcal{B}}_{20}$. We have $q(x) = \{5, 6, 7\}$ and $p(x) = \{2, 4\}$. Choose $a = 5 \in q(x) \setminus p(x)$. The 5-based decomposition of x is given by

$$\text{Dec}'_5(x) = (8, 4836866, 585, 542787121).$$

The above argument directly yields the following property.

Property 5.8. *Given $x \in \hat{\mathcal{B}}_n$ and $a \in q(x) \setminus p(x)$, let $\text{Dec}'_a(x) = (L, Q, P, R)$. Then we have*

- (i) *All entries in L are strictly greater than a .*
- (ii) *Q starts with an entry strictly less than a .*
- (iii) *P starts with the leftmost a of x , and all entries in P are greater than or equal to a .*
- (iv) *R starts with the rightmost a of x .*
- (v) *Furthermore, if $p(x) = \emptyset$ or $\max(p(x)) < a$, then P and Q have no common elements other than possibly m .*

The following transformation will play an essential role in the construction of Θ .

The Transformation θ_a

Let $x \in \hat{\mathcal{B}}_n$ and let $a \in q(x) \setminus p(x)$ with $\text{Dec}'_a(x) = (L, Q, P, R)$. Define $\theta_a(x)$ as the sequence obtained by rearranging the components as follows:

$$\theta_a(x) = LPQR.$$

Consider the same sequence $x = 84836866585542787121$ from Example 5.7. Recall that $\text{Dec}'_5(x) = (8, 4836866, 585, 542787121)$. Hence, we have

$$\theta_5(x) = 85854836866542787121.$$

Combining Property 5.2 and Property 5.8, we immediately obtain the following conclusion.

Lemma 5.9. *The maps γ_a and θ_a are inverses of each other.*

Lemma 5.10. *Let $x \in \hat{\mathcal{B}}_{n,m}$ be a revised ascent sequence and let $a \in q(x)$. If $p(x) = \emptyset$ or $\max(p(x)) < a$, then we have $y = \theta_a(x) \in \hat{\mathcal{B}}_{n,m}$.*

Proof. Let $\text{Dec}'_a(x) = (L, Q, P, R)$. By Property 5.8, P and Q share no common elements other than possibly m . The conclusion then follows by an argument similar to that of Lemma 5.3. ■

The proof of the following lemma is essentially the same as that of Lemma 5.4, and hence omitted.

Lemma 5.11. *Let $x \in \hat{\mathcal{B}}_n$ be a revised ascent sequence satisfying $\min(q(x)) = a$, and either $p(x) = \emptyset$ or $\max(p(x)) < a$. Then for $y = \theta_a(x)$, the following properties hold:*

- (i) $q(y) = q(x) \setminus \{a\}$.
- (ii) $p(y) = p(x) \cup \{a\}$.

Now we are ready to define $\Theta : \hat{\mathcal{B}}_n(3212) \rightarrow \hat{\mathcal{B}}_n(3122)$. Given $x \in \hat{\mathcal{B}}_n(3212)$, define $\Theta(x)$ through the following iterative process:

1. Initialization: Set $i := 0$, $x^{(0)} := x$, and $S := q(x) \setminus p(x)$.
2. Iteration: While $S \neq \emptyset$:
 - Increment $i := i + 1$.
 - Select the minimum element: $a_i := \min(S)$.
 - Apply the transformation: $x^{(i)} := \theta_{a_i}(x^{(i-1)})$.
 - Refresh $S := q(x^{(i)}) \setminus p(x^{(i)})$.
3. Termination: The procedure stops when $S = \emptyset$. The final output is $\Theta(x) := x^{(i)}$.

For example, let $x = 83686658554841278721 \in \hat{\mathcal{B}}_{20}(3212)$. We have $\Theta(x) = 87868658548365427121$. See the procedure in Example 5.5 backward for an illustration.

Let $x \in \hat{\mathcal{B}}_n(3212)$ be a revised ascent sequence with $q(x) = \{a_1, a_2, \dots, a_k\}$ where $a_1 < a_2 < \dots < a_k$. By the definition of Θ and Lemma 5.11, we obtain that

$$\Theta(x) = \theta_{a_k} \circ \dots \circ \theta_{a_2} \circ \theta_{a_1}(x).$$

Let $x^{(i)} = \theta_{a_i}(x^{(i-1)})$ for $1 \leq i \leq k$ with the convention that $x^{(0)} = x$. From Lemma 5.11, we have

$$\begin{aligned} p(x^{(i)}) &= \{a_1, a_2, \dots, a_i\}, \\ q(x^{(i)}) &= \{a_{i+1}, a_{i+2}, \dots, a_k\}, \end{aligned}$$

for $0 \leq i \leq k$. In particular, we have $q(x^{(k)}) = \emptyset$.

The next lemma follows immediately by iteratively applying Lemmas 5.10 and 5.11.

Lemma 5.12. *Let $x \in \hat{\mathcal{B}}_{n,m}(3212)$ be a revised ascent sequence and let $y = \Theta(x)$. Then we have $y \in \hat{\mathcal{B}}_{n,m}(3122)$.*

Proposition 5.13. *The map Γ is a bijection between $\hat{\mathcal{B}}_{n,m}(3122)$ and $\hat{\mathcal{B}}_{n,m}(3212)$.*

Proof. By the definitions of Γ and Θ , and in light of Lemmas 5.4, 5.9, and 5.11, it follows that Γ and Θ are inverses of each other. The proposition then follows directly from Lemmas 5.6 and 5.12. ■

By Proposition 4.3 and Proposition 5.13, we immediately obtain the following proposition.

Proposition 5.14. *For $n, m \geq 1$, we have*

$$|\hat{\mathcal{B}}_{n,m}(3122)| = \sum_{k=0}^{n-m} k^{m-1} \binom{n-2-k}{n-m-k}.$$

6 Pattern 2132

In this section, we enumerate $\hat{\mathcal{B}}_n(2132)$ by constructing a bijection Φ between $\hat{\mathcal{B}}_n(2132)$ and $\hat{\mathcal{B}}_n(112)$. The construction of this mapping is very similar to the bijection Θ between $\hat{\mathcal{B}}_n(3212)$ and $\hat{\mathcal{B}}_n(3122)$. Many of the proofs of the corresponding properties are also analogous to those for Θ , and therefore some details will be omitted.

For $x \in \hat{\mathcal{B}}_n$, define the following two sets:

$$\begin{aligned} u(x) &= \{a \mid \text{there exists an occurrence } x_i x_j x_k \text{ of } 112 \text{ in } x \text{ such that } x_i = a\}, \\ v(x) &= \{a \mid \text{there exists an occurrence } x_i x_j x_k x_\ell \text{ of } 2132 \text{ in } x \text{ such that } x_i = a\}. \end{aligned}$$

Example 6.1. Let $x = 9589386879274965133 \in \hat{\mathcal{B}}_{19}$. We have $u(x) = \{7, 8\}$ and $v(x) = \{3, 5, 6\}$.

Our bijection depends on the following decomposition of x . Let $x \in \hat{\mathcal{B}}_{n,m}$ and let $a \in u(x) \setminus v(x)$. It is clear that $a < m = x_1$. Denote by x_ℓ and x_r the leftmost and rightmost occurrences of a in x , respectively. By the definition of revised ascent sequences, we have $a = x_\ell < x_{\ell+1}$ and $a = x_r > x_{r+1}$. Let j be the largest index such that all entries in $x_{\ell+1}x_{\ell+2}\cdots x_j$ are strictly greater than a . It is clear that $\ell < j < r$. Let x_k be the rightmost entry in x such that $x_k > a$.

We claim that all entries in $x_{j+1}x_{j+2}\cdots x_r$ are less than or equal to a . If not, suppose that there exists some $j + 1 \leq t \leq r$ such that $x_t > a$. By our choice of j , we have $x_{j+1} \leq a$. If $x_{j+1} < a$, then $x_\ell x_{j+1} x_t x_r$ forms an occurrence of the pattern 2132 in x , contradicting the fact that $a \notin v(x)$. If $x_{j+1} = a$, let s be the smallest index such that $s > j + 1$ and $x_s \neq a$. Clearly, we have $s \leq t$. Since x is a revised ascent sequence and $x_{s-1} = a$, we deduce $x_s < a$. This implies that $x_\ell x_s x_t x_r$ forms an occurrence of the pattern 2132 in x , again contradicting the fact that $a \notin v(x)$. This

proves the claim. Since $a \in u(x)$, we have $k > r$. Consequently, x admits a unique a -based decomposition of the form $x = ABCD$, where

$$A = x_1x_2 \cdots x_j, B = x_{j+1}x_{j+2} \cdots x_r, C = x_{r+1}x_{r+2} \cdots x_k, \text{ and } D = x_{k+1}x_{k+2} \cdots x_n.$$

We denote this a -based decomposition of x by

$$\text{Dec}_a''(x) = (A, B, C, D).$$

Continuing with the same sequence $x = 9589386879274965133$ from Example 6.1, recall that $u(x) = \{7, 8\}$ and $v(x) = \{3, 5, 6\}$. Choose $a = 7 \in u(x) \setminus v(x)$. The 7-based decomposition of x is given by

$$\text{Dec}_7''(x) = (9589386879, 27, 49, 65133).$$

Property 6.2. *Given $x \in \hat{\mathcal{B}}_n$ and $a \in u(x) \setminus v(x)$, let $\text{Dec}_a''(x) = (A, B, C, D)$. Then we have*

- (i) *A contains the leftmost a of x and all entries to the right of this a in A are strictly greater than a.*
- (ii) *B ends with the rightmost a of x, and all entries in B are less than or equal to a.*
- (iii) *C starts with an entry strictly less than a and ends with an entry strictly greater than a.*
- (iv) *All entries in D (may be empty) are strictly less than a.*
- (v) *Furthermore, if $\min(u(x)) = a$, then B and C have no common elements.*

Proof. Properties (i)–(iv) follows directly from the argument above. For (v), assume to the contrary that B and C have a common element b . By (ii) and (iii), we have $b < a$. It is then easily seen that $b \in u(x)$, which contradicts the assumption that $\min(u(x)) = a$. Hence, (v) follows. ■

The following transformation will play an essential role in the construction of Φ .

The Transformation ϕ_a

Let $x \in \hat{\mathcal{B}}_n$ and let $a \in u(x) \setminus v(x)$ with $\text{Dec}_a''(x) = (A, B, C, D)$. Define $\phi_a(x)$ as the sequence obtained by rearranging the components as follows:

$$\phi_a(x) = ACBD.$$

Again consider the same sequence $x = 9589386879274965133$ from Example 6.1. Recall that $\text{Dec}_7''(x) = (9589386879, 27, 49, 65133)$. Hence, we have

$$\phi_7(x) = 9589386879492765133.$$

The following conclusion can be demonstrated through reasoning similar to Lemma 5.3. We omit the proof here.

Lemma 6.3. *Let $x \in \hat{\mathcal{B}}_{n,m}$ be a revised ascent sequence with $\min(u(x)) = a$. If $a \notin v(x)$, then we have $y = \phi_a(x) \in \hat{\mathcal{B}}_{n,m}$.*

Lemma 6.4. *Let $x \in \hat{\mathcal{B}}_n$ be a revised ascent sequence satisfying $\min(u(x)) = a$, and either $v(x) = \emptyset$ or $\max(v(x)) < a$. Then for $y = \phi_a(x)$, the following properties hold:*

- (i) *Either $u(y) = \emptyset$ or $\min(u(y)) > a$.*
- (ii) *$\max(v(y)) = a$.*

Proof. By hypothesis, we have $a \in u(x) \setminus v(x)$. Let $\text{Dec}_a''(x) = (A, B, C, D)$. Then, by the definition of ϕ_a , we have $y = \phi_a(x) = ACBD$. Applying Property 6.2, we have $a \in v(y)$. Moreover, since there is no entry strictly greater than a after the second occurrence of a in y , it follows that $a \notin u(y)$.

For (i), assume to the contrary that there exists some $b < a$ such that $b \in u(y)$. By the definition of $u(y)$, there exists an occurrence $y_i y_j y_k$ of the pattern 112 in y , where $y_i = y_j = b$. Since $\min(u(x)) = a > b$, we have $b \notin u(x)$. Note that $x = ABCD$ and $y = ACBD$. Consequently, any occurrence of the pattern 112 in y with the first entry equal to b must involve elements from both B and C . Since $\min(u(x)) = a$, it follows from Property 6.2 that, B and C contain no common elements. This implies that $y_k \in B$ and $y_i, y_j \notin B$. By Property 6.2, replacing y_k with the last entry of C in $y_i y_j y_k$ still yields an occurrence of the pattern 112 in y . However, in this case, B contributes nothing to this new occurrence, which is a contradiction. Therefore, property (i) follows.

For (ii), assume to the contrary that there exists some $c > a$ such that $c \in v(y)$. By the definition of $v(y)$, there exists an occurrence $y_r y_s y_u y_v$ of the pattern 2132 in y , where $y_r = y_v = c$. Given that $v(x) = \emptyset$ or $\max(v(x)) < a < c$, it follows that $c \notin v(x)$. Consequently, any occurrence of the pattern 2132 in y with the first entry equal to c must involve elements from both B and C . Since all entries in B are less than or equal to a , we deduce that B contains the only element y_s of $y_r y_s y_u y_v$. This implies that $y_r \in C$ and $y_u, y_v \in D$. Note that $y_v = c > a$. However, by Property 6.2, all entries in D are strictly less than a , which is a contradiction. Therefore, property (ii) follows. ■

Remark 6.5. We keep the notation and conditions of Lemma 6.4. By Property 6.2, all entries in B are less than or equal to a . Consequently, any occurrence of the pattern 112 in x (or y) whose first entry is d with $d > a$ contains no entry from B . Hence, for any $d > a$, we have $d \in u(x)$ if and only if $d \in u(y)$. Combining this with Lemma 6.4, we obtain $u(y) = u(x) \setminus \{a\}$. However, the claim $v(y) = v(x) \cup \{a\}$ is not correct. For example, take $x = 434231421 \in \hat{\mathcal{B}}_9$. We have $u(x) = \{3\}$ and $v(x) = \{2\}$. Moreover, we have $y = \phi_3(x) = 434142321$ with $u(y) = \emptyset$ and $v(y) = \{3\}$.

We can now define our map $\Phi : \hat{\mathcal{B}}_n(2132) \rightarrow \hat{\mathcal{B}}_n(112)$. Given $x \in \hat{\mathcal{B}}_n(2132)$, define $\Phi(x)$ through the following iterative process:

1. Initialization: Set $i := 0$, $x^{(0)} := x$, and $S := u(x) \setminus v(x)$.
2. Iteration: While $S \neq \emptyset$:
 - Increment $i := i + 1$.
 - Select the minimum element: $a_i := \min(S)$.
 - Apply the transformation: $x^{(i)} := \phi_{a_i}(x^{(i-1)})$.
 - Refresh $S := u(x^{(i)}) \setminus v(x^{(i)})$.
3. Termination: The procedure stops when $S = \emptyset$. The final output is $\Phi(x) := x^{(i)}$.

Example 6.6. Let $x = 9589538687961332749 \in \hat{\mathcal{B}}_{19}(2132)$. Then we have $\Phi(x) = 9589794938682765133$ with the detailed process below, where the a_{i+1} -based decomposition of $x^{(i)}$ is performed by placing vertical bars at specific positions of $x^{(i)}$.

$$\begin{aligned}
 x &= 958953868796|133|2749| \\
 &\quad \downarrow \phi_3 \\
 x^{(1)} &= 9589|5|38687962749|133 \\
 &\quad \downarrow \phi_5 \\
 x^{(2)} &= 9589386879|6|2749|5133 \\
 &\quad \downarrow \phi_6 \\
 x^{(3)} &= 9589386879|27|49|65133 \\
 &\quad \downarrow \phi_7 \\
 x^{(4)} &= 9589|3868|7949|2765133 \\
 &\quad \downarrow \phi_8 \\
 x^{(5)} &= 9589794938682765133 = \Phi(x).
 \end{aligned}$$

Let $x \in \hat{\mathcal{B}}_n(2132)$ be a revised ascent sequence with $u(x) = \{a_1, a_2, \dots, a_k\}$ where $a_1 < a_2 < \dots < a_k$. By the definition of Φ , Lemma 6.4 and Remark 6.5, we obtain that

$$\Phi(x) = \phi_{a_k} \circ \dots \circ \phi_{a_2} \circ \phi_{a_1}(x).$$

The next lemma is an immediate consequence of the definition of Φ combined with iterative applications of Lemma 6.4.

Lemma 6.7. *Let $x \in \hat{\mathcal{B}}_n(2132)$ be a revised ascent sequence with $u(x) \neq \emptyset$. Assume that $\Phi(x) = \phi_{a_k} \circ \dots \circ \phi_{a_2} \circ \phi_{a_1}(x)$, and let $x^{(i)} = \phi_{a_i}(x^{(i-1)})$ for $1 \leq i \leq k$ with the convention that $x^{(0)} = x$. Then the following statements hold:*

- (i) $\min(u(x^{(i)})) > a_i$ for $1 \leq i \leq k - 1$, and $u(x^{(k)}) = \emptyset$.
- (ii) $\max(v(x^{(i)})) = a_i$ for $1 \leq i \leq k$.

According to Lemma 6.7, for $1 \leq i \leq k + 1$, we have

$$u(x^{(i-1)}) \cap v(x^{(i-1)}) = \emptyset.$$

Consequently,

$$a_i = \min(u(x^{(i-1)}) \setminus v(x^{(i-1)})) = \min(u(x^{(i-1)}))$$

for $1 \leq i \leq k$. This demonstrates that in the construction of Φ , the value of a_i depends solely on $u(x^{(i-1)})$ and is independent of $v(x^{(i-1)})$.

The following lemma follows directly from Lemma 6.3 and Lemma 6.7.

Lemma 6.8. *Let $x \in \hat{\mathcal{B}}_{n,m}(2132)$ be a revised ascent sequence and let $y = \Phi(x)$. Then we have $y \in \hat{\mathcal{B}}_{n,m}(112)$.*

To prove that Φ is a bijection between $\hat{\mathcal{B}}_n(2132)$ and $\hat{\mathcal{B}}_n(112)$, we construct its inverse map Ψ . Let $x \in \hat{\mathcal{B}}_{n,m}$, and let $a \in v(x) \setminus u(x)$. Denote by x_ℓ and x_r the leftmost and rightmost occurrences of a in x , respectively. By the definition of revised ascent sequences, we have $a = x_\ell < x_{\ell+1}$. Let j be the largest index such that all entries in $x_{\ell+1}x_{\ell+2}\cdots x_j$ are strictly greater than a . Let k be the largest index such that $x_k > a$. Since $a \in v(x) \setminus u(x)$, we have $\ell < j < k < r$. Moreover, we have $x_{j+1} < a$, and no entry in the subsequence $x_{j+1}x_{j+2}\cdots x_k$ equals a . Consequently, x admits a unique a -based decomposition of the form $x = ACBD$, where

$$A = x_1x_2\cdots x_j, C = x_{j+1}x_{j+2}\cdots x_k, B = x_{k+1}x_{k+2}\cdots x_r \text{ and } D = x_{r+1}x_{r+2}\cdots x_n.$$

We denote this a -based decomposition of x by

$$\text{Dec}_a'''(x) = (A, C, B, D).$$

Example 6.9. Let $x = 9589386879274965133 \in \hat{\mathcal{B}}_{19}$. We have $u(x) = \{7, 8\}$ and $v(x) = \{3, 5, 6\}$. Choose $a = 6 \in v(x) \setminus u(x)$. The 6-based decomposition of x is given by

$$\text{Dec}_6'''(x) = (9589386879, 2749, 6, 5133).$$

By the argument above, we immediately obtain the following property.

Property 6.10. *Given $x \in \hat{\mathcal{B}}_n$ and $a \in v(x) \setminus u(x)$, let $\text{Dec}_a'''(x) = (A, C, B, D)$. Then we have*

- (i) *A contains the leftmost a of x and all entries to the right of this a in A are strictly greater than a.*
- (ii) *C starts with an entry strictly less than a and ends with an entry strictly greater than a. Moreover, C contains no entries equal to a.*
- (iii) *B ends with the rightmost a of x, and all entries in B are less than or equal to a.*

- (iv) All entries in D (may be empty) are strictly less than a .
- (v) Furthermore, if $u(x) = \emptyset$ or $\min(u(x)) > a$, then B and C have no common elements.

The following transformation will play an essential role in the construction of Ψ .

The Transformation ψ_a

Let $x \in \hat{\mathcal{B}}_n$ and let $a \in v(x) \setminus u(x)$ with $\text{Dec}_a'''(x) = (A, C, B, D)$. Define $\psi_a(x)$ as the sequence obtained by rearranging the components as follows:

$$\psi_a(x) = ABCD.$$

Consider the same sequence $x = 9589386879274965133$ from Example 6.9. Recall that $\text{Dec}_6'''(x) = (9589386879, 2749, 6, 5133)$. Hence, we have

$$\psi_6(x) = 9589386879627495133.$$

Combining Properties 6.2 and 6.10, we immediately obtain the following conclusion.

Lemma 6.11. *The maps ϕ_a and ψ_a are inverses of each other.*

Lemma 6.12. *Let $x \in \hat{\mathcal{B}}_{n,m}$ be a revised ascent sequence and let $a \in v(x)$. If $u(x) = \emptyset$ or $\min(u(x)) > a$, then we have $y = \psi_a(x) \in \hat{\mathcal{B}}_{n,m}$.*

Proof. Let $\text{Dec}_a'''(x) = (A, C, B, D)$. Since $u(x) = \emptyset$ or $\min(u(x)) > a$, it follows from Property 6.10 that, B and C contain no common elements. Then the conclusion can be demonstrated through reasoning similar to Lemma 5.3. ■

The proof of the following lemma is essentially the same as that of Lemma 6.4, and hence omitted.

Lemma 6.13. *Let $x \in \hat{\mathcal{B}}_n$ be a revised ascent sequence satisfying $\max(v(x)) = a$, and either $u(x) = \emptyset$ or $\min(u(x)) > a$. Then for $y = \psi_a(x)$, the following properties hold:*

- (i) *Either $v(y) = \emptyset$ or $\max(v(y)) < a$.*
- (ii) *$\min(u(y)) = a$.*

Now we are ready to define $\Psi : \hat{\mathcal{B}}_n(112) \rightarrow \hat{\mathcal{B}}_n(2132)$. Given $x \in \hat{\mathcal{B}}_n(112)$, define $\Psi(x)$ through the following iterative process:

1. Initialization: Set $i := 0$, $x^{(0)} := x$, and $S := v(x) \setminus u(x)$.
2. Iteration: While $S \neq \emptyset$:
 - Increment $i := i + 1$.
 - Select the maximum element: $a_i := \max(S)$.

- Apply the transformation: $x^{(i)} := \psi_{a_i}(x^{(i-1)})$.
- Refresh $S := v(x^{(i)}) \setminus u(x^{(i)})$.

3. Termination: The procedure stops when $S = \emptyset$. The final output is $\Psi(x) := x^{(i)}$.

For example, let $x = 9589794938682765133 \in \hat{\mathcal{B}}_{19}(112)$. We have $\Psi(x) = 9589538687961332749$. See the procedure in Example 6.6 backward for an illustration.

The next lemma is an immediate consequence of the definition of Ψ combined with iterative applications of Lemma 6.13.

Lemma 6.14. *Let $x \in \hat{\mathcal{B}}_n(112)$ be a revised ascent sequence with $v(x) \neq \emptyset$. Assume that $\Psi(x) = \psi_{a_k} \circ \dots \circ \psi_{a_2} \circ \psi_{a_1}(x)$, and let $x^{(i)} = \psi_{a_i}(x^{(i-1)})$ for $1 \leq i \leq k$ with the convention that $x^{(0)} = x$. Then the following statements hold:*

- (i) $\max(v(x^{(i)})) < a_i$ for $1 \leq i \leq k - 1$, and $v(x^{(k)}) = \emptyset$.
- (ii) $\min(u(x^{(i)})) = a_i$ for $1 \leq i \leq k$.

According to Lemma 6.14, for $1 \leq i \leq k + 1$, we have

$$u(x^{(i-1)}) \cap v(x^{(i-1)}) = \emptyset.$$

Consequently,

$$a_i = \max(v(x^{(i-1)}) \setminus u(x^{(i-1)})) = \max(v(x^{(i-1)}))$$

for $1 \leq i \leq k$. This demonstrates that in the construction of Ψ , the value of a_i depends solely on $v(x^{(i-1)})$ and is independent of $u(x^{(i-1)})$.

The following lemma follows immediately from Lemmas 6.12 and 6.14.

Lemma 6.15. *Let $x \in \hat{\mathcal{B}}_{n,m}(112)$ be a revised ascent sequence and let $y = \Psi(x)$. Then we have $y \in \hat{\mathcal{B}}_{n,m}(2132)$.*

Proposition 6.16. *The map Φ is a bijection between $\hat{\mathcal{B}}_{n,m}(2132)$ and $\hat{\mathcal{B}}_{n,m}(112)$.*

Proof. By the definitions of Φ and Ψ , and in light of Lemmas 6.7, 6.11, and 6.14, it follows that Φ and Ψ are inverses of each other. The proposition then follows directly from Lemmas 6.8 and 6.15. ■

By Propositions 2.6 and 6.16, we immediately obtain the following proposition.

Proposition 6.17. *For $n, m \geq 1$, we have*

$$|\hat{\mathcal{B}}_{n,m}(2132)| = \left\{ \begin{matrix} n - 1 \\ n - m \end{matrix} \right\}.$$

Consequently, $|\hat{\mathcal{B}}_n(2132)|$ is equal to the $(n - 1)$ -th Bell number.

7 Final remarks and future directions

In this paper, we systematically investigate revised ascent sequences avoiding a single pattern of length 4 and derive a variety of enumeration results. Building upon the current findings, we further propose the following potential research directions and suggestions.

Several enumerations for pattern avoiding revised ascent sequences coincide with well-known number sequences, including the Catalan numbers, Bell numbers, Fibonacci numbers, and the binomial transform of the Catalan numbers (see Table 1). However, our proofs are mainly based on recurrence relations and generating functions, and do not offer direct combinatorial explanations for why these particular sequences arise. This naturally leads to the following question.

Problem 7.1. Can we establish bijections between these pattern avoiding revised ascent sequences and other combinatorial objects counted by the same numbers?

Conjecture 7.2. We have

$$\begin{aligned} 1112 \sim 1121, 2122 \sim 2212, \\ 1223 \sim 2123, 2331 \sim 4231. \end{aligned}$$

Numerical evidence shows that the Wilf-equivalence classes for revised ascent sequences in Conjecture 7.2 encompass all remaining possible Wilf-equivalence classes for patterns of length 4. In other words, a proof of Conjecture 7.2 would complete the classification of Wilf-equivalence classes for revised ascent sequences with respect to all 4-length patterns.

A closer examination reveals that the bijections between $\hat{\mathcal{B}}_n(3122)$ and $\hat{\mathcal{B}}_n(3212)$ in Section 5, as well as between $\hat{\mathcal{B}}_n(2132)$ and $\hat{\mathcal{B}}_n(112)$ in Section 6, exhibit remarkable structural similarities in both their construction methods and proof techniques. This observation naturally leads us to investigate two fundamental questions:

- (1) Does there exist a unified proof framework that can simultaneously explain both bijections?
- (2) Can we establish a more general conclusion about Wilf-equivalence on revised ascent sequences?

We have attempted to adapt this proof technique to demonstrate the Wilf-equivalence $1223 \sim 2123$, but have not yet achieved a breakthrough.

Conjecture 7.3. For $n \geq 1$, we have

$$|\hat{\mathcal{B}}_{n+1}(3121)| = |\hat{\mathcal{A}}_n(221)|,$$

where $\hat{\mathcal{A}}_n(221)$ denotes the set of 221-avoiding modified ascent sequences of length n .

Conjecture 7.4. For $n \geq 1$, we have

$$|\hat{\mathcal{B}}_n(1233)| = |\mathcal{S}_n(31\bar{5}\bar{4}2)|,$$

where $\mathcal{S}_n(31\bar{5}\bar{4}2)$ denotes the set of permutations on $[n]$ which avoid $31\bar{5}\bar{4}2$ (i.e. every occurrence of 312 is contained in an occurrence of $31\bar{5}\bar{4}2$).

All the above conjectures have been verified up to $n = 11$.

Revised ascent sequences can be characterized using the language of ordered set partitions. For a Cayley permutation $x = x_1x_2\cdots x_n$ with $\max(x) = m$, define an ordered partition $\alpha(x) = (B_1, B_2, \dots, B_m)$ by $B_j = \{i : x_i = j\}$. It is well known that α is a bijection between Cayley permutations of length n and ordered set partitions of $[n]$. Let $\mathcal{OP}_{n,m}$ be the set of ordered set partitions of $[n]$ with m blocks. For an ordered set partition $\pi = (B_1, B_2, \dots, B_m) \in \mathcal{OP}_{n,m}$, define

$$\begin{aligned} \text{Nub}(\pi) &= \{\min(B_i) \mid 1 \leq i \leq m\}, \\ \text{Ascbot}(\pi) &= \{1\} \cup \{i \mid 1 \leq i \leq n-1 \text{ and } b(i) < b(i+1)\}, \end{aligned}$$

where $b(i) = j$ if $i \in B_j$. Let

$$\mathcal{C}_{n,m} = \{\pi \in \mathcal{OP}_{n,m} \mid \text{Ascbot}(\pi) = \text{Nub}(\pi)\},$$

and let $\mathcal{C}_n = \bigcup_m \mathcal{C}_{n,m}$. It is obvious that α induces a bijection between $\hat{\mathcal{B}}_{n,m}$ and $\mathcal{C}_{n,m}$ which preserves the statistics Ascbot and Nub . A systematic investigation of pattern avoiding ordered set partitions within \mathcal{C}_n would be an interesting direction for future work.

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