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A Fan-type degree condition for the existence of disjoint chorded cycles in a graph

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Abstract

A chord of a cycle is an edge between two non-consecutive vertices of the cycle, and a chorded cycle is a cycle containing at least one chord. Let k be a positive integer. Finkel (2008) proved if G is a graph of order at least $4k$ with the minimum degree at least $3k$, then G contains k disjoint chorded cycles. In this paper, when the order of a graph is sufficiently large, we improve Finkel's result by considering a Fan-type degree condition.

1 Introduction

We consider only finite undirected simple graphs in this paper. A *chord* of a cycle is an edge between two non-consecutive vertices of the cycle, and a *chorded cycle* is a cycle containing at least one chord. The study of cycles has a long and important history. Building on these ideas, investigation of chorded cycles has seen a great deal of development over the last twenty years. Building on this development, we determine a Fan-type degree condition for the existence of disjoint chorded cycles in a graph.

For a graph G, $d_G(u)$ is the degree of a vertex u in G, $\delta(G)$ is the minimum degree of G, and we define $\sigma_2(G) = \min\{d_G(u) + d_G(v) \,|\, u, v \in V(G), uv \notin E(G)\}$ and $\sigma_2(G) = \infty$ when G is a complete graph. A set of subgraphs of G is said to be *disjoint* if no two of them have any common vertex in G. The distance between two vertices u and v in G is denoted by $\mathrm{dist}_G(u, v)$.

Let k be a positive integer. Corrádi and Hajnal [1] proved if G is a graph of order at least $3k$ with $\delta(G) \geq 2k$, then G contains k disjoint cycles. Enomoto [2] and Wang

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[9] independently extended the result of Corrádi and Hajnal showing, if $|G| \geq 3k$ and $\sigma_2(G) \geq 4k-1$, then G contains k disjoint cycles. Fan [3] considered a new degree condition to improve known sufficient conditions for a graph to be Hamiltonian, called a "Fan-condition": $\mu(G) = \min\{\max\{d_G(u), d_G(v)\} \mid u, v \in V(G), \mathrm{dist}_G(u,v) = 2\}$, and $\mu(G) = \infty$ when G is a complete graph. Zhang and Yan improved the result of Corrádi and Hajnal by considering a Fan-type condition:

$$\sigma_1^2(G) = \min\{\max\{d_G(u), d_G(v)\} \mid u, v \in V(G), uv \notin E(G)\},$$

and $\sigma_1^2(G) = \infty$ when G is a complete graph.

Theorem 1.1. (Zhang and Yan [10]) *Let k be a positive integer, and let G be a graph of order $n \geq 3k+2$. If $\sigma_1^2(G) \geq 2k$, then G contains k disjoint cycles.*

Finkel proved the following theorem on the existence of k disjoint chorded cycles.

Theorem 1.2. (Finkel [4]) *Let k be a positive integer, and let G be a graph of order $n \geq 4k$. If $\delta(G) \geq 3k$, then G contains k disjoint chorded cycles.*

In this paper, when the order of a graph is sufficiently large, we improve Finkel's result (Theorem 1.2) as Corrádi and Hajnal's result about $\delta(G)$ was improved by Zhang and Yan.

Theorem 1.3. *Let k be a positive integer, and let G be a graph of order $n \geq 12k-5$. If $\sigma_1^2(G) \geq 3k$, then G contains k disjoint chorded cycles.*

Remark 1.4. The degree condition of Theorem 1.3 is sharp. Consider a complete bipartite graph $G_1 = K_{3k-1,n-3k+1}$, with sufficiently large $n = |G_1|$. Then $\sigma_1^2(G_1) = 3k-1$. However, G_1 does not contain k disjoint chorded cycles, since any chorded cycle must contain at least three vertices from each partite set. Thus $\sigma_1^2(G) \geq 3k$ for a graph G is necessary.

Since $\sigma_1^2(G) \geq \delta(G)$ for a graph G, Theorem 1.3 strengthens Theorem 1.2 when the order of G is sufficiently large. Moreover, there exists a graph that satisfies the condition of Theorem 1.3, but does not satisfy the condition of Theorem 1.2, and still contains k disjoint chorded cycles. Consider a graph G_2 obtained by deleting an arbitrary edge from a complete bipartite graph $K_{3k,n-3k}$, with sufficiently large $n = |G_2|$. Then $\delta(G_2) = 3k-1$ and $\sigma_1^2(G_2) = 3k$. By Theorem 1.3, G_2 contains k disjoint chorded cycles, whereas Theorem 1.2 cannot be applied to G_2. In fact, G_2 contains k disjoint chorded cycles.

For other related results on disjoint chorded cycles in graphs, we refer the reader to [6, 8].

Let G be a graph, H a subgraph of G, and $X \subseteq V(G)$. For $u \in V(G)$, the set of neighbors of u in G is denoted by $N_G(u)$, and we denote $d_G(u) = |N_G(u)|$, $N_H(u) = N_G(u) \cap V(H)$, and $d_H(u) = |N_H(u)|$. Also we denote $d_H(X) = \sum_{u \in X} d_H(u)$. If $H = G$, then $d_G(X) = d_H(X)$. The subgraph of G induced by X is denoted by $\langle X \rangle_G$ or, more simply, by $\langle X \rangle$ if the graph G under discussion is clear. Let $G - X =$

$\langle V(G) - X\rangle_G$ and $G - H = \langle V(G) - V(H)\rangle_G$. If $X = \{x\}$, then we write $G - x$ for $G - X$. If there is no fear of confusion, then we use the same symbol for a graph and its vertex set. For two disjoint $X, Y \subseteq V(G)$, $E(X, Y)$ denotes the set of edges of G connecting a vertex in X and a vertex in Y. Let Q be a path or a cycle with a given orientation and $x \in V(Q)$. Then x^+ denotes the first successor of x on Q and x^- denotes the first predecessor of x on Q. If $x, y \in V(Q)$, then $Q[x, y]$ denotes the path of Q from x to y (including x and y) in the given orientation. We also write $Q(x, y] = Q[x^+, y]$, $Q[x, y) = Q[x, y^-]$, and $Q(x, y) = Q[x^+, y^-]$. If C is a cycle, say $C = x_1x_2 \cdots x_tx_1$, we assume that an orientation of C is given such that x_2 is the successor of x_1. For two disjoint graphs G_1 and G_2, $G_1 \cup G_2$ denotes the *union*. For an integer $\ell \geq 3$, a cycle of length ℓ is called an *ℓ-cycle*. For $x, y \in V(G)$, we denote by xPy a path connecting x and y on G. For terminology and notation not defined here, see [5].

2 Lemmas

For a positive integer r, let $C_1, \dots, C_r$ be r disjoint chorded cycles in a graph G. We say that $C_1, \dots, C_r$ is *minimal* if G does not contain r disjoint chorded cycles $C'_1, \dots, C'_r$ such that $|\bigcup_{i=1}^r V(C'_i)| < |\bigcup_{i=1}^r V(C_i)|$.

Lemma 2.1. *Let G be a graph of order at least 6, and let C be a 4-cycle in G. If $d_C(x) + d_C(y) \geq 7$ for $x, y \in V(G - C)$, then there exists a vertex $z \in V(C)$ such that $xz \in E(G)$ and $\langle V(C - z) \cup \{y\}\rangle$ contains a chorded 4-cycle.*

Proof. Let $C = z_1z_2z_3z_4z_1$. First, suppose that $d_C(x) = 4$. Then $d_C(y) \geq 3$. Without loss of generality, we may assume that $\{z_1, z_2, z_3\} \subseteq N_C(y)$. Then $xz_4 \in E(G)$, and $\langle V(C - z_4) \cup \{y\}\rangle$ contains a 4-cycle $yz_1z_2z_3y$ with chord yz_2. Next, suppose that $d_C(y) = 4$. Then $d_C(x) \geq 3$. Without loss of generality, we may assume that $xz_1 \in E(G)$. Then $\langle V(C - z_1) \cup \{y\}\rangle$ contains a 4-cycle $yz_2z_3z_4y$ with chord yz_3. Thus this lemma holds. □

Lemma 2.2. ([7, Lemma 3.2]) *Let $r \geq 1$ be an integer, and let $L = \{C_1, \dots, C_r\}$ be a minimal set of r disjoint chorded cycles in a graph G. If $|C_i| \geq 7$ for some $1 \leq i \leq r$, then C_i has at most two chords. Furthermore, if C_i has two chords, then these chords must be crossing.*

Lemma 2.3. ([7, Lemma 3.3]) *Let $r \geq 1$ be an integer, and let $L = \{C_1, \dots, C_r\}$ be a minimal set of r disjoint chorded cycles in a graph G. Then $d_{C_i}(x) \leq 4$ for any $1 \leq i \leq r$ and any $x \in V(G - \bigcup_{i=1}^r C_i)$. Furthermore, for some $C \in L$ and some $x \in V(G - \bigcup_{i=1}^r C_i)$, if $d_C(x) = 4$, then $|C| = 4$, and if $d_C(x) = 3$, then $|C| \leq 6$.*

Lemma 2.4. ([7, Lemma 3.6]) *Suppose that there exist at least five edges connecting two disjoint paths P_1 and P_2 with $|P_1 \cup P_2| \geq 7$. Then there exists a chorded cycle in $\langle P_1 \cup P_2\rangle$ not containing at least one vertex of $\langle P_1 \cup P_2\rangle$.*

Lemma 2.5. ([7, Lemma 3.7]) *Let P_1, P_2 be two disjoint paths, and let u_1, u_2 ($u_1 \neq u_2$) be in that order on P_1. Suppose that $d_{P_2}(u_i) \geq 2$ for each $i \in \{1, 2\}$. Then there exists a chorded cycle in $\langle P_1[u_1, u_2] \cup P_2\rangle$.*

Lemma 2.6. *Let $k \geq 2$ be an integer, and let G be a graph. Suppose that G does not contain k disjoint chorded cycles. Let $\{C_1, \ldots, C_{k-1}\}$ be a minimal set of $k-1$ disjoint chorded cycles in G, $L = \bigcup_{i=1}^{k-1} C_i$, $H = G - L$, and $X = \{x_1, x_2, x_3, x_4\} \subseteq V(H)$. Suppose that there exists a path connecting x_1 and x_2 in $H - \{x_3, x_4\}$ and there exists a path connecting x_3 and x_4 in $H - \{x_1, x_2\}$. Then $d_{C_i}(X) \leq 12$ for each $1 \leq i \leq k-1$.*

Proof. Suppose not, then $d_{C_{i_0}}(X) \geq 13$ for some $1 \leq i_0 \leq k-1$. By Lemma 2.3, $d_{C_{i_0}}(x_j) \leq 4$ for each $1 \leq j \leq 4$. Then there exists at least one vertex $x \in X$ with $d_{C_{i_0}}(x) = 4$. By Lemma 2.3, $|C_{i_0}| = 4$. Let $C_{i_0} = v_1v_2v_3v_4v_1$. We show the existence of two disjoint chorded cycles in $\langle H \cup C_{i_0} \rangle$, and then G contains k disjoint chorded cycles, a contradiction. It is sufficient to consider the following three cases.

Case 1. X contains three vertices of degree 4 and a vertex of degree at least 1.

Without loss of generality, we may assume that $d_{C_{i_0}}(x_1) \geq 1$ and $d_{C_{i_0}}(x_j) = 4$ for each $2 \leq j \leq 4$. Let $x_1v_1 \in E(G)$. By the assumption of this lemma, there exists a path connecting x_1 and x_2 in $H - \{x_3, x_4\}$. Then $x_1Px_2v_2v_1x_1$ is a cycle with chord x_2v_1. Since $d_{C_{i_0}}(x_j) = 4$ for each $j \in \{3, 4\}$, $x_3v_3x_4v_4x_3$ is a cycle with chord v_3v_4. Thus there exist two disjoint chorded cycles in $\langle H \cup C_{i_0} \rangle$.

Case 2. X contains two vertices of degree 4, a vertex of degree 3, and a vertex of degree at least 2.

Without loss of generality, we may assume that $d_{C_{i_0}}(x_1) \geq 2$. First, suppose that $d_{C_{i_0}}(x_2) = 3$. Then, without loss of generality, we may assume that $v_j \in N_{C_{i_0}}(x_2)$ for each $1 \leq j \leq 3$. Assume that $v_1 \in N_{C_{i_0}}(x_1)$. Then $x_1Px_2v_2v_1x_1$ is a cycle with chord x_2v_1. Since $d_{C_{i_0}}(x_j) = 4$ for each $j \in \{3, 4\}$, $v_3, v_4 \in N_{C_{i_0}}(x_j)$. Then $x_3v_3x_4v_4x_3$ is a cycle with chord v_3v_4. Thus there exist two disjoint chorded cycles in $\langle H \cup C_{i_0} \rangle$. Hence $v_1 \notin N_{C_{i_0}}(x_1)$. By symmetry, $v_3 \notin N_{C_{i_0}}(x_1)$. Since $d_{C_{i_0}}(x_1) \geq 2$, $v_2 \in N_{C_{i_0}}(x_1)$. Then $x_1Px_2v_1v_2x_1$ is a cycle with chord x_2v_2. Since $v_3, v_4 \in N_{C_{i_0}}(x_j)$ for each $j \in \{3, 4\}$, $x_3v_3x_4v_4x_3$ is a cycle with chord v_3v_4. Thus there exist two disjoint chorded cycles in $\langle H \cup C_{i_0} \rangle$.

Next, suppose that $d_{C_{i_0}}(x_3) = 3$ or $d_{C_{i_0}}(x_4) = 3$. Without loss of generality, we may assume that $d_{C_{i_0}}(x_3) = 3$. Then $d_{C_{i_0}}(x_2) = 4$ and $d_{C_{i_0}}(x_4) = 4$. Without loss of generality, we may assume that $v_j \in N_{C_{i_0}}(x_3)$ for each $1 \leq j \leq 3$. Assume that $v_1 \in N_{C_{i_0}}(x_1)$. Then $x_1Px_2v_2v_1x_1$ is a cycle with chord x_2v_1. Since $d_{C_{i_0}}(x_4) = 4$, $x_3Px_4v_4v_3x_3$ is a cycle with chord x_4v_3. Thus there exist two disjoint chorded cycles in $\langle H \cup C_{i_0} \rangle$. Hence $v_1 \notin N_{C_{i_0}}(x_1)$. By symmetry, $v_3 \notin N_{C_{i_0}}(x_1)$. Since $d_{C_{i_0}}(x_1) \geq 2$, $v_2 \in N_{C_{i_0}}(x_1)$, and since $d_{C_{i_0}}(x_2) = 4$, $v_1, v_2 \in N_{C_{i_0}}(x_2)$. Then $x_1Px_2v_1v_2x_1$ is a cycle with chord x_2v_2. Since $d_{C_{i_0}}(x_4) = 4$, $v_3, v_4 \in N_{C_{i_0}}(x_4)$. Then $x_3Px_4v_4v_3x_3$ is a cycle with chord x_4v_3. Thus there exist two disjoint chorded cycles in $\langle H \cup C_{i_0} \rangle$.

Case 3. X contains a vertex of degree 4 and three vertices of degree 3.

Without loss of generality, we may assume that $d_{C_{i_0}}(x_1) = 4$. Then $d_{C_{i_0}}(x_j) = 3$ for each $2 \leq j \leq 4$. Since $d_{C_{i_0}}(x_3) = 3$, there exist two consecutive vertices $v_{j_0}, v_{j_0+1} \in N_{C_{i_0}}(x_3)$ for some $1 \leq j_0 \leq 4$, with $v_5 = v_1$. Let $v_1, v_2 \in N_{C_{i_0}}(x_3)$. Since $d_{C_{i_0}}(x_4) = 3$, $v_1 \in N_{C_{i_0}}(x_4)$ or $v_2 \in N_{C_{i_0}}(x_4)$. If $v_1 \in N_{C_{i_0}}(x_4)$, then $x_3Px_4v_1v_2x_3$ is a cycle with chord x_3v_1. If $v_2 \in N_{C_{i_0}}(x_4)$, then $x_3Px_4v_2v_1x_3$ is a cycle with chord x_3v_2. Since $d_{C_{i_0}}(x_1) = 4$, $v_3, v_4 \in N_{C_{i_0}}(x_1)$. Since $d_{C_{i_0}}(x_2) = 3$, $v_3 \in N_{C_{i_0}}(x_2)$ or $v_4 \in N_{C_{i_0}}(x_2)$. If $v_3 \in N_{C_{i_0}}(x_2)$, then $x_1Px_2v_3v_4x_1$ is a cycle with chord x_1v_3. If $v_4 \in N_{C_{i_0}}(x_2)$, then $x_1Px_2v_4v_3x_1$ is a cycle with chord x_1v_4. Thus there exist two disjoint chorded cycles in $\langle H \cup C_{i_0} \rangle$.

□

3 Proof of Theorem 1.3

Let k, G, n be as defined in Theorem 1.3. Since k is a positive integer and $n \geq 12k-5$, we have $n \geq 7$. Suppose, to the contrary, that G does not contain k disjoint chorded cycles. If G is a complete graph, then Theorem 1.3 holds. Thus we may assume that G is not a complete graph. Suppose that $k = 1$. Then, since $\sigma_1^2(G) \geq 3k$, we have $\sigma_1^2(G) \geq 3$. Thus there exists a vertex $u_0 \in V(G)$ with $d_G(u_0) \geq 3$. Let $\{u_1, u_2, u_3\} \subseteq N_G(u_0)$. Let $P = v_1v_2 \dots v_p$ be a longest path in G. Then $|P| \geq 3$. Suppose that $|P| = 3$. If $u_1u_2 \in E(G)$, then $u_1u_2u_0u_3$ is a longer path than P, a contradiction. Thus we may assume that $u_1u_2 \notin E(G)$. By the $\sigma_1^2(G)$ condition, $d_G(u_1) \geq 3$ or $d_G(u_2) \geq 3$. Since P is a longest path of order 3 in G, this is a contradiction. Thus $|P| \geq 4$. Assume that $v_1v_p \notin E(G)$. By the $\sigma_1^2(G)$ condition, without loss of generality, we may assume that $d_G(v_1) \geq 3$. Since P is a longest path in G, there exists a chorded cycle containing v_1 in $\langle V(P) \rangle$. Thus $v_1v_p \in E(G)$. Let $C = v_1v_2 \dots v_pv_1$. If $v_1v_{p-1} \in E(G)$, then C is a cycle with chord v_1v_{p-1} in G. Thus we may assume that $v_1v_{p-1} \notin E(G)$. By the $\sigma_1^2(G)$ condition, $d_G(v_1) \geq 3$ or $d_G(v_{p-1}) \geq 3$. Since $v_{p-1}v_{p-2} \dots v_1v_p$ is a longest path in G, we have $d_{G-P}(v_{p-1}) = 0$. Thus C is a cycle with chord in G. This is a contradiction.

Suppose that $k \geq 2$. Let G be an edge-maximal counter-example. Let $xy \notin E(G)$ for some $x, y \in V(G)$, and define $G' = G + xy$, the graph obtained from G by adding the edge xy. Since G' is not a counter-example by the edge-maximality of G, G' contains k disjoint chorded cycles $C_1, \dots, C_k$. Without loss of generality, we may assume that $xy \notin \bigcup_{i=1}^{k-1} E(C_i)$, that is, G contains $k-1$ disjoint chorded cycles. Let $L = \bigcup_{i=1}^{k-1} C_i$, $H = G - L$, and $P_1 = x_1x_2 \dots x_s$ be a longest path in H. We may assume that H does not contain a chorded cycle, otherwise G contains k disjoint chorded cycles, a contradiction. Since G' contains k disjoint chorded cycles, we have $|H| \geq 4$. Choose $C_1, \dots, C_{k-1}$ such that

- (A1) $|L|$ is as small as possible,
- (A2) subject to (A1), $|P_1|$ is as large as possible,
- (A3) subject to (A1) and (A2), $\sum_{x \in V(L)} d_G(x)$ is as large as possible, and
- (A4) subject to (A1)–(A3), $\sum_{x \in V(P_1)} d_G(x)$ is as small as possible.

Since G' is not a counter-example, we note that $|P_1| \geq 4$.

Claim 3.1. *If $d_G(u) \leq 3k-1$ and $d_G(v) \leq 3k-1$ for $u, v \in V(G)$, then $uv \in E(G)$.*

Proof. By the $\sigma_1^2(G)$ condition, this claim holds. □

Claim 3.2. *$|H| \geq 7$.*

Proof. Suppose that $|H| \leq 6$. Assume that $|C_i| \leq 12$ for each $1 \leq i \leq k-1$. Since $|G| \geq 12k-5$ by the assumption, it follows that $|H| \geq (12k-5) - 12(k-1) = 7$, a contradiction. Thus $|C_{i_0}| \geq 13$ for some $1 \leq i_0 \leq k-1$. Without loss of generality, we may assume that C_1 is a longest cycle in L with chord. Then $|C_1| \geq 13$, and $d_{C_1}(v) \leq 2$ for any $v \in V(H)$ by (A1) and Lemma 2.3. By Lemma 2.2, C_1 has at most two chords, and if C_1 has two chords, then these chords must be crossing. Thus, since $|H| \leq 6$ by our assumption, it follows that $|E(H, C_1)| \leq 12$. Let $|C_1| = t\,(\geq 13)$, and $X = \{v \in V(C_1) \mid d_G(v) \geq 3k\}$. If $|X| \leq t-4$, then there exists a 4-cycle with chords in $\langle V(C_1)\rangle$ by Claim 3.1. This contradicts (A1). Thus we may assume that $|X| \geq t-3$. Then we take a vertex set $X' \subseteq X$ with $|X'| = t-3$.

Suppose that $k = 2$. Then L contains only one cycle C_1, and since $t \geq 13$,

$$\begin{aligned}|E(X', H)| &= d_G(X') - d_{C_1}(X') \geq 3k(t-3) - (2(t-3)+4)\\ &= 3kt - 9k - 2t + 2 = 6t - 18 - 2t + 2\\ &= 4t - 16 \geq 52 - 16 = 36.\end{aligned}$$

Since $12 \geq |E(C_1, H)| \geq |E(X', H)|$, this is a contradiction.

Next, suppose that $k \geq 3$. Since $|H| \leq 6$ and $d_{C_1}(v) \leq 2$ for any $v \in V(H)$,

$$\begin{aligned}|E(X', L - C_1)| &= d_G(X') - d_{C_1}(X') - d_H(X')\\ &\geq 3k(t-3) - (2(t-3)+4) - 2 \cdot 6\\ &= 3t(k-1) + t - 9k - 10,\end{aligned}$$

and since $t \geq 13$,

$$\begin{aligned}3t(k-1) + t - 9k - 10 &= (3t-9)(k-1) + t - 19 \geq (3t-9)(k-1) - 6\\ &> (3t-9)(k-1) - (3t-9)\\ &= (3t-9)(k-2).\end{aligned}$$

Since $L - C_1$ contains $k-2$ disjoint chorded cycles, we have $|E(X', C')| > 3t-9$ for some C' in $L - C_1$. Let $h = \max\{d_{C'}(v) \mid v \in X'\}$. Let v^* be a vertex of X' such that $d_{C'}(v^*) = h$. If $h \leq 3$, then $|E(X', C')| \leq 3 \times (t-3) = 3t-9$, a contradiction. Thus we may assume that $h \geq 4$. By the maximality of C_1, $|C'| \leq |C_1| = t$. It follows that $h = d_{C'}(v^*) \leq |C'| \leq t$. Since $t \geq 13$,

$$\begin{aligned}|E(X' - \{v^*\}, C')| &= |E(X', C')| - d_{C'}(v^*) \geq (3t-8) - t\\ &= 2t - 8 \geq 18.\end{aligned} \tag{1}$$

Since $h = d_{C'}(v^*) \geq 4$, let v_1, v_2, v_3, v_4 be four distinct neighbors of v^* in that order on C'. Note that v_1, v_2, v_3, v_4 partition C' into four intervals $C'[v_i, v_{i+1})$ for each $1 \leq i \leq 4$, with $v_5 = v_1$. By (1), there exist at least 18 edges from $C_1 - v^*$ to C'. Thus $C'[v_{i_0}, v_{i_0+1})$ for some $1 \leq i_0 \leq 4$ contains at least 5 of these edges. Without loss of generality, we may assume that $i_0 = 4$, that is, $C'[v_4, v_1)$. Then by Lemma 2.4, $\langle (C_1 - v^*) \cup C'[v_4, v_1)\rangle$ contains a chorded cycle C_1' not containing at least one vertex of $\langle (C_1 - v^*) \cup C'[v_4, v_1) \rangle$. Note that $C_1'' = v^* C'[v_1, v_3] v^*$ is a cycle with chord $v^* v_2$, and it uses no vertices from $C'[v_4, v_1)$. Thus there exist two disjoint chorded cycles C_1', C_1'' in $\langle C_1 \cup C' \rangle$ such that $|C_1'| + |C_1''| < |C_1| + |C'|$, contradicting (A1). Hence Claim 3.2 holds. □

Claim 3.3. $|P_1| = |H|$.

Proof. Suppose that $|P_1| < |H|$. Let $P_2 = y_1 y_2 \dots y_t$ $(t \geq 1)$ be a longest path in $H - P_1$. We first claim that

$$\text{there exists a vertex } y_0 \in V(H - P_1) \text{ with } d_H(y_0) \leq 3. \tag{2}$$

By the maximality of P_2, $d_{H-(P_1 \cup P_2)}(y_i) = 0$ for each $i \in \{1, t\}$. Suppose that $|P_2| \geq 3$. Since H does not contain a chorded cycle, $d_{P_2}(y_i) \leq 2$ for each $i \in \{1, t\}$. By Lemma 2.5, $d_{P_1}(y_{i_0}) \leq 1$ for some $i_0 \in \{1, t\}$. Thus $d_H(y_{i_0}) \leq 2 + 1 = 3$. Suppose that $|P_2| = 2$. Then $t = 2$ and $d_{P_2}(y_i) = 1$ for each $i \in \{1, 2\}$. By Lemma 2.5, $d_{P_1}(y_{i_0}) \leq 1$ for some $i_0 \in \{1, 2\}$. Thus $d_H(y_{i_0}) \leq 1 + 1 = 2$. Suppose that $|P_2| = 1$. Then $t = 1$, $d_{P_2}(y_1) = 0$, and $d_{P_1}(y_1) \leq 2$. Thus $d_H(y_1) \leq 2$. Hence (2) holds.

Next, we claim that

$$d_G(x_i) \geq 3k \text{ for each } i \in \{1, s\}, \text{ and } d_G(y_0) \leq 3k - 1. \tag{3}$$

Suppose that $d_G(y_0) + d_G(x_{i_0}) \geq 6k$ for some $i_0 \in \{1, s\}$. Without loss of generality, we may assume that $i_0 = 1$. Since $d_H(y_0) \leq 3$ by (2), and $d_H(x_1) \leq 2$,

$$\begin{aligned} d_L(y_0) + d_L(x_1) &= d_G(y_0) + d_G(x_1) - (d_H(y_0) + d_H(x_1)) \\ &\geq 6k - (3 + 2) = 6(k - 1) + 1. \end{aligned}$$

Thus $d_{C_{j_0}}(y_0) + d_{C_{j_0}}(x_1) \geq 7$ for some $1 \leq j_0 \leq k - 1$. Then $d_{C_{j_0}}(y_0) \geq 4$ or $d_{C_{j_0}}(x_1) \geq 4$. By (A1) and Lemma 2.3, we have $|C_{j_0}| = 4$. By Lemma 2.1, there exists a vertex $z \in V(C_{j_0})$ such that $x_1 z \in E(G)$ and $\langle V(C_{j_0} - z) \cup \{y_0\} \rangle$ contains a chorded 4-cycle, say C_{j_0}'. Replacing C_{j_0} in L by C_{j_0}', we consider the new H'. Then $z x_1 x_2 \dots x_s$ is a longer path in H' than P_1. This contradicts (A2). Thus $d_G(y_0) + d_G(x_i) \leq 6k - 1$ for each $i \in \{1, s\}$. If $d_G(y_0) \leq 3k - 1$, then since $\sigma_1^2(G) \geq 3k$, $d_G(x_i) \geq 3k$ for each $i \in \{1, s\}$. Thus (3) holds. Hence we may assume that $d_G(y_0) \geq 3k$. Since $d_G(y_0) + d_G(x_i) \leq 6k - 1$ for each $i \in \{1, s\}$, we have $d_G(x_i) \leq 3k - 1$. By Claim 3.1, $x_1 x_s \in E(G)$. Note that $|P_1| \geq 4$. Since $x_{s-1} x_{s-2} \dots x_1 x_s$ is a longest path in H, we have $d_G(x_{s-1}) \leq 3k - 1$ by the above arguments. By Claim 3.1, $x_1 x_{s-1} \in E(G)$. Then $x_1 x_2 \dots x_s x_1$ is a cycle with chord $x_1 x_{s-1}$, a contradiction. Thus (3) holds.

Next, we claim that

$$d_G(y) \leq 3k - 1 \text{ for any } y \in V(H - P_1), \text{ and } |H - P_1| \leq 3. \tag{4}$$

Suppose that $d_H(y') \geq 4$ for some $y' \in V(H - P_1)$. Since H does not contain a chorded cycle, we have $d_{P_1}(y') \leq 2$, and $d_{H-P_1}(y') \geq 2$. Thus there exists a path of order at least 3 in $H - P_1$. Since $P_2 (= y_1 y_2 \dots y_t)$ is a longest path in $H - P_1$, we have

$|P_2| \geq 3$. Since H does not contain a chorded cycle, $d_{P_2}(y_i) \leq 2$ for each $i \in \{1, t\}$. By Lemma 2.5, $d_{P_1}(y_{i_0}) \leq 1$ for some $i_0 \in \{1, t\}$. Note that $d_{H-(P_1 \cup P_2)}(y_{i_0}) = 0$. Thus $d_H(y_{i_0}) \leq 2 + 1 = 3$. Without loss of generality, we may assume that $i_0 = 1$. If $d_G(y_1) \geq 3k$, then since $d_G(x_1) \geq 3k$ by (3), $d_G(y_1) + d_G(x_1) \geq 6k$. Since $d_H(y_1) \leq 3$ and $d_H(x_1) \leq 2$, by the same arguments as the proof of (3), we have $d_{C_{j_0}}(y_1) + d_{C_{j_0}}(x_1) \geq 7$ for some $1 \leq j_0 \leq k-1$. By Lemma 2.1, there exists a longer path than P_1, a contradiction. Thus we may assume that $d_G(y_1) \leq 3k - 1$. Since P_1 is a longest path in H and $|P_2| \geq 3$, we have $x_i y_1 \notin E(G)$ for each $i \in \{2, s-1\}$, and since $d_G(y_1) \leq 3k - 1$, $d_G(x_i) \geq 3k$. We show that

there exist two distinct vertices $u_1, u_2 \in V(H) - \{x_1, x_s\}$ such that $d_G(u_\ell) \geq 3k$, $d_H(u_\ell) \leq 3$ for each $\ell \in \{1, 2\}$, and there exists a path connecting x_1 and u_1 in $H - \{u_2, x_s\}$ and a path connecting u_2 and x_s in $H - \{x_1, u_1\}$. (5)

Suppose that $d_{H-P_1}(x_2) = 0$ and $d_{H-P_1}(x_{s-1}) = 0$. Then we let $u_1 = x_2$ and $u_2 = x_{s-1}$. Since $d_G(y_1) \leq 3k-1$, $d_G(u_\ell) \geq 3k$ for each $\ell \in \{1, 2\}$. Since H does not contain a chorded cycle, $d_H(u_\ell) \leq 3$ for each $\ell \in \{1, 2\}$.

Suppose that $d_{H-P_1}(x_2) \geq 1$ and $d_{H-P_1}(x_{s-1}) = 0$ or $d_{H-P_1}(x_2) = 0$ and $d_{H-P_1}(x_{s-1}) \geq 1$. By symmetry, we may assume that $d_{H-P_1}(x_2) \geq 1$ and $d_{H-P_1}(x_{s-1}) = 0$. Then there exists a vertex $u \in V(H - P_1)$ with $x_2 u \in E(G)$, and we let $u_1 = u$ and $u_2 = x_{s-1}$. Since P_1 is a longest path in H and $|P_2| \geq 3$, we have $u_1 \notin V(P_2)$. Since $u_1 y_1 \notin E(G)$, $d_G(u_1) \geq 3k$. Since $u_1 x_2 x_3 \dots x_s$ is a longest path in H, we have $d_H(u_1) \leq 2$. Also, we have $d_G(u_2) \geq 3k$ and $d_H(u_2) \leq 3$.

Suppose that $d_{H-P_1}(x_2) \geq 1$ and $d_{H-P_1}(x_{s-1}) \geq 1$. Then there exist vertices $u', u'' \in V(H - P_1)$ such that $x_2 u' \in E(G)$ and $x_{s-1} u'' \in E(G)$. If $u' \neq u''$, then we let $u_1 = u'$ and $u_2 = u''$. Thus we may assume that $u' = u''$. If $|P_1| = 4$ ($s = 4$), then $x_1 x_2 u' x_3 x_4$ is a longer path than P_1, a contradiction. Thus we may assume that $|P_1| \geq 5$. Suppose that $d_{H-P_1}(x_3) \geq 1$. Let $x_3 w \in E(G)$ for $w \in V(H - P_1)$. If $w = u'$, then $x_1 x_2 w x_3 x_4 \dots x_s$ is a longer path than P_1. Thus we may assume that $w \neq u'$. Then $x_1 x_2 u' x_{s-1} x_{s-2} \dots x_3 w$ is a longer path than P_1, a contradiction. Thus $d_{H-P_1}(x_3) = 0$. Since $x_3 y_1 \notin E(G)$, $d_G(x_3) \geq 3k$. We prove that $d_{P_1}(x_3) = 2$. If $x_3 x_1 \in E(G)$, then $x_1 x_2 u' x_{s-1} x_{s-2} \dots x_3 x_1$ is a cycle with chord $x_2 x_3$, a contradiction. If $x_3 x_{r_0} \in E(G)$ for some $5 \leq r_0 \leq s-1$, then $x_2 x_3 \dots x_{s-1} u' x_2$ is a cycle with chord $x_3 x_{r_0}$, a contradiction. If $x_3 x_s \in E(G)$, then $x_1 x_2 u' x_{s-1} x_{s-2} \dots x_3 x_s$ is a longer path than P_1, a contradiction. Thus $d_{P_1}(x_3) = 2$. Since $d_{H-P_1}(x_3) = 0$, $d_H(x_3) = 2$. Then we let $u_1 = u'$ and $u_2 = x_3$.

By the above choices of u_1 and u_2, we note that there exists a path connecting x_1 and u_1 in $H - \{u_2, x_s\}$ and a path connecting u_2 and x_s in $H - \{x_1, u_1\}$. Therefore, (5) holds.

Let $X = \{x_1, u_1, u_2, x_s\}$. Then, by (5),

$$\begin{aligned} d_L(X) &= d_G(X) - d_H(X) \\ &\geq 4 \cdot 3k - (2 + 3 + 3 + 2) = 12k - 10 \\ &= 12(k-1) + 2. \end{aligned}$$

Hence $d_{C_{i_0}}(X) \geq 13$ for some $1 \leq i_0 \leq k-1$. Since X satisfies the conditions of Lemma 2.6 by (5), this inequality contradicts Lemma 2.6. Hence $d_H(y) \leq 3$ for any $y \in V(H - P_1)$. Similar to (3), we have $d_G(y) \leq 3k-1$ for any $y \in V(H - P_1)$, and by Claim 3.1, $H - P_1$ is a complete graph. Since H does not contain a chorded cycle, we have $|H - P_1| \leq 3$. Thus (4) holds.

Since $d_H(x_i) \leq 2$ for each $i \in \{1, s\}$, we have

$$\begin{aligned} d_L(x_1) + d_L(x_s) &= d_G(x_1) + d_G(x_s) - (d_H(x_1) + d_H(x_s)) \\ &\geq 2 \cdot 3k - (2+2) = 6(k-1) + 2. \end{aligned}$$

Thus $d_{C_{i_0}}(x_1) + d_{C_{i_0}}(x_s) \geq 7$ for some $1 \leq i_0 \leq k-1$. Without loss of generality, we may assume that $i_0 = k-1$. Then $d_{C_{k-1}}(x_1) \geq 4$ or $d_{C_{k-1}}(x_s) \geq 4$. By (A1) and Lemma 2.3, we have $|C_{k-1}| = 4$. Let $C_{k-1} = z_1z_2z_3z_4z_1$. By symmetry, we may assume that $d_{C_{k-1}}(x_1) = 4$ and $\{z_2, z_3, z_4\} \subseteq N_{C_{k-1}}(x_s)$. Then $x_sz_2z_3z_4x_s$ is a 4-cycle with chord x_sz_3, and $z_1x_1x_2 \dots x_{s-1}$ is a path of the same length as P_1. Note that $d_G(x_s) \geq 3k$ by (3). Thus, by (A3), we have

$$d_G(z_1) \geq 3k. \tag{6}$$

Subclaim 3.3.1. *The following statements hold.*

(i) *For some* $2 \leq i_0 \leq s-1$, $N_{C_{k-1} \cup P_1}(z_1) \subseteq \{z_2, z_3, z_4, x_1, x_{i_0}, x_s\}$, *and* $N_{H-P_1}(z_1) = \emptyset$.

(ii) *For each* $i \in \{2, s-1\}$, $N_{H-P_1}(x_i) = \emptyset$ *and* $d_G(x_i) \geq 3k$.

(iii) *Let* $H_1 = \langle H \cup C_{k-1} \rangle$. *Then* $d_{H_1}(x_2) + d_{H_1}(x_{s-1}) \leq 10$.

Proof. If $\{x_{i_0}, x_{j_0}\} \subseteq N_{P_1}(z_1)$ for some $2 \leq i_0 < j_0 \leq s-1$, then $\langle P_1 \cup C_{k-1} \rangle$ contains two disjoint chorded cycles, that is, one is a cycle $x_1x_2 \dots x_{j_0}z_1x_1$ with chord $x_{i_0}z_1$ and the other is a cycle $x_sz_2z_3z_4x_s$ with chord x_sz_3, a contradiction. Thus $N_{C_{k-1} \cup P_1}(z_1) \subseteq \{z_2, z_3, z_4, x_1, x_{i_0}, x_s\}$ for some $2 \leq i_0 \leq s-1$. Suppose that $N_{H-P_1}(z_1) \neq \emptyset$, and let $z_1y \in E(G)$ for $y \in V(H - P_1)$. Then $x_sz_2z_3z_4x_s$ is a 4-cycle with chord x_sz_3, and $yz_1x_1x_2 \dots x_{s-1}$ is a longer path than P_1. This contradicts (A2). Thus $N_{H-P_1}(z_1) = \emptyset$, and (i) holds.

Suppose that $N_{H-P_1}(x_2) \neq \emptyset$. Let $x_2y \in E(G)$ for $y \in V(H - P_1)$. Then $P_1' = yx_2x_3 \dots x_s$ is a path in H with $|P_1'| = |P_1|$. Recall that $d_G(x_1) \geq 3k$ by (3), and $d_G(y) \leq 3k-1$ by (4). Since $\sum_{x \in V(P_1')} d_G(x) < \sum_{x \in V(P_1)} d_G(x)$, this contradicts (A4). Thus $N_{H-P_1}(x_2) = \emptyset$. By the $\sigma_1^2(G)$ condition and (4), we have $d_G(x_2) \geq 3k$. By symmetry, $N_{H-P_1}(x_{s-1}) = \emptyset$ and $d_G(x_{s-1}) \geq 3k$. Thus (ii) holds.

Suppose that $d_{H_1}(x_2) + d_{H_1}(x_{s-1}) \geq 11$. Since H does not contain a chorded cycle, we have $d_{P_1}(x_i) \leq 3$ for each $i \in \{2, s-1\}$, and since $N_{H-P_1}(x_i) = \emptyset$ by (ii), we have $d_H(x_i) \leq 3$. Thus

$$\begin{aligned} d_{C_{k-1}}(x_2) + d_{C_{k-1}}(x_{s-1}) &= d_{H_1}(x_2) + d_{H_1}(x_{s-1}) - (d_H(x_2) + d_H(x_{s-1})) \\ &\geq 11 - (3+3) = 5. \end{aligned}$$

Since $|C_{k-1}| = 4$, $N_{C_{k-1}}(x_2) \cap N_{C_{k-1}}(x_{s-1}) \neq \emptyset$. Thus $z_{i_0} \in N_{C_{k-1}}(x_2) \cap N_{C_{k-1}}(x_{s-1})$ for some $1 \leq i_0 \leq 4$. If $i_0 = 1$, then $x_1x_2 \dots x_{s-1}z_1x_1$ is a cycle with chord x_2z_1, and $x_sz_2z_3z_4x_s$ is a cycle with chord x_sz_3, a contradiction. If $i_0 = 2$, then $x_2x_3 \dots x_sz_2x_2$ is a cycle with chord $x_{s-1}z_2$, and $x_1z_3z_4z_1x_1$ is a cycle with chord x_1z_4, a contradiction. If $i_0 \in \{3, 4\}$, then we get a contradiction similar to the case where $i_0 = 2$. Thus (iii) holds.

□

Subclaim 3.3.2. *For each* $1 \leq i \leq k-2$, $2(d_{C_i}(z_1)+d_{C_i}(x_2)+d_{C_i}(x_{s-1}))+d_{C_i}(x_1)+d_{C_i}(x_s) \leq 24$.

Proof. Suppose not. Then, for some $1 \leq i_0 \leq k-2$,

$$2(d_{C_{i_0}}(z_1) + d_{C_{i_0}}(x_2) + d_{C_{i_0}}(x_{s-1})) + d_{C_{i_0}}(x_1) + d_{C_{i_0}}(x_s) \geq 25. \tag{7}$$

Thus $d_{C_{i_0}}(z_1) + d_{C_{i_0}}(x_2) + d_{C_{i_0}}(x_{s-1}) \geq 10$ or $d_{C_{i_0}}(x_1) + d_{C_{i_0}}(x_s) \geq 7$.

First, suppose that $d_{C_{i_0}}(x_1) + d_{C_{i_0}}(x_s) \geq 7$. Then $d_{C_{i_0}}(x_1) \geq 4$ or $d_{C_{i_0}}(x_s) \geq 4$. By (A1) and Lemma 2.3, we have $|C_{i_0}| = 4$. Let $C_{i_0} = w_1w_2w_3w_4w_1$. Without loss of generality, we may assume that $d_{C_{i_0}}(x_1) = 4$ and $\{w_2, w_3, w_4\} \subseteq N_{C_{i_0}}(x_s)$. We claim that $N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_{s-1}) = \emptyset$. Suppose not. We prove that there exist two disjoint chorded cycles in $\langle P_1 \cup C_{i_0} \rangle$, and then G contains k disjoint chorded cycles, a contradiction. Assume that $w_1 \in N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_{s-1})$. Then $x_1x_2 \dots x_{s-1}w_1x_1$ is a cycle with chord x_2w_1, and $x_sw_2w_3w_4x_s$ is a cycle with chord x_sw_3. Assume that $w_2 \in N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_{s-1})$. Then $x_2x_3 \dots x_sw_2x_2$ is a cycle with chord $x_{s-1}w_2$, and $x_1w_3w_4w_1x_1$ is a cycle with chord x_1w_4. If $w_{j_0} \in N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_{s-1})$ for some $j_0 \in \{3, 4\}$, then we can prove the existence of two disjoint chorded cycles in $\langle P_1 \cup C_{i_0} \rangle$, similarly. Thus the claim holds. Then $d_{C_{i_0}}(x_2) + d_{C_{i_0}}(x_{s-1}) \leq 4$. Since $|C_{i_0}| = 4$, we have $d_{C_{i_0}}(z_1) \leq 4$ and $d_{C_{i_0}}(x_j) \leq 4$ for each $j \in \{1, s\}$. Hence,

$$2(d_{C_{i_0}}(z_1) + d_{C_{i_0}}(x_2) + d_{C_{i_0}}(x_{s-1})) + d_{C_{i_0}}(x_1) + d_{C_{i_0}}(x_s) \leq 2(4+4) + 4 + 4 = 24.$$

This contradicts (7).

Next, suppose that $d_{C_{i_0}}(z_1) + d_{C_{i_0}}(x_2) + d_{C_{i_0}}(x_{s-1}) \geq 10$. Then $d_{C_{i_0}}(v) \geq 4$ for some $v \in \{z_1, x_2, x_{s-1}\}$. We show that $|C_{i_0}| = 4$. If $v \in \{x_2, x_{s-1}\}$, then by (A1) and Lemma 2.3, we have $|C_{i_0}| = 4$. Thus we may assume that $v = z_1$. Recall that $d_{C_{k-1}}(x_1) = 4$. Then $C'_{k-1} = x_1z_2z_3z_4x_1$ is a 4-cycle with chord x_1z_3. Replacing C_{k-1} in L by C'_{k-1}, we consider the new L' not containing z_1. Since $d_{C_{i_0}}(z_1) \geq 4$, we have $|C_{i_0}| = 4$ by (A1) and Lemma 2.3. Let $C_{i_0} = w_1w_2w_3w_4w_1$.

We prove that there exist three disjoint chorded cycles in $\langle V(P_1) \cup V(C_{i_0}) \cup V(C_{k-1}) \rangle$. Then G contains k disjoint chorded cycles, a contradiction. Since $|C_{i_0}| = 4$, $d_{C_{i_0}}(z_1) + d_{C_{i_0}}(x_2) + d_{C_{i_0}}(x_{s-1}) \leq 12$.

CASE (A). $d_{C_{i_0}}(z_1) + d_{C_{i_0}}(x_2) + d_{C_{i_0}}(x_{s-1}) = 12$.

In this case, $d_{C_{i_0}}(v) = 4$ for each $v \in \{z_1, x_2, x_{s-1}\}$, and by (7), we have

$$d_{C_{i_0}}(x_1) + d_{C_{i_0}}(x_s) \geq 1. \tag{8}$$

First, we claim that $d_{C_{i_0}}(x_1) = 0$. Assume that $x_1w_1 \in E(G)$. Then $x_1x_2 \dots x_{s-1}w_1x_1$ is a cycle with chord x_2w_1, $z_1w_2w_3w_4z_1$ is a cycle with chord z_1w_3, and $x_sz_2z_3z_4x_s$ is a cycle with chord x_sz_3. Thus $x_1w_1 \notin E(G)$. Similarly, $x_1w_i \notin E(G)$ for each $2 \le i \le 4$. Thus the claim holds. Next, we claim that $d_{C_{i_0}}(x_s) = 0$. Assume that $x_sw_1 \in E(G)$. Then $x_2x_3 \dots x_sw_1x_2$ is a cycle with chord $x_{s-1}w_1$, $x_1z_2z_3z_4x_1$ is a cycle with chord x_1z_3, and $z_1w_2w_3w_4z_1$ is a cycle with chord z_1w_3. Thus $x_sw_1 \notin E(G)$. Similarly, $x_sw_i \notin E(G)$ for each $2 \le i \le 4$. Thus the claim holds. Since $d_{C_{i_0}}(x_i) = 0$ for each $i \in \{1, s\}$, this contradicts (8).

CASE (B). $d_{C_{i_0}}(z_1) + d_{C_{i_0}}(x_2) + d_{C_{i_0}}(x_{s-1}) = 11$.

In this case, $\{d_{C_{i_0}}(z_1), d_{C_{i_0}}(x_2), d_{C_{i_0}}(x_{s-1})\} = \{4, 4, 3\}$, and by (7), we have

$$d_{C_{i_0}}(x_1) + d_{C_{i_0}}(x_s) \ge 3. \tag{9}$$

Case 1. $d_{C_{i_0}}(z_1) = 3$.

In this case, $d_{C_{i_0}}(x_2) = d_{C_{i_0}}(x_{s-1}) = 4$. Without loss of generality, we may assume $\{w_1, w_2, w_3\} \subseteq N_{C_{i_0}}(z_1)$. We claim that $d_{C_{i_0}}(x_1) = 0$. Assume that $x_1w_1 \in E(G)$. Then $x_1x_2w_1z_1x_1$ is a cycle with chord x_1w_1, $x_{s-1}w_2w_3w_4x_{s-1}$ is a cycle with chord $x_{s-1}w_3$, and $x_sz_2z_3z_4x_s$ is a cycle with chord x_sz_3. Thus $x_1w_1 \notin E(G)$. Similarly, $x_1w_i \notin E(G)$ for each $i \in \{2, 3\}$. If $x_1w_4 \in E(G)$, then $x_1x_2 \dots x_{s-1}w_4x_1$ is a cycle with chord x_2w_4, $z_1w_1w_2w_3z_1$ is a cycle with chord z_1w_2, and $x_sz_2z_3z_4x_s$ is a cycle with chord x_sz_3. Thus the claim holds. If $x_sw_4 \in E(G)$, then $x_2x_3 \dots x_sw_4x_2$ is a cycle with chord $x_{s-1}w_4$, $x_1z_2z_3z_4x_1$ is a cycle with chord x_1z_3, and $z_1w_1w_2w_3z_1$ is a cycle with chord z_1w_2. Thus $x_sw_4 \notin E(G)$. Since $d_{C_{i_0}}(x_1) = 0$, we have $d_{C_{i_0}}(x_s) \ge 3$ by (9). Then $N_{C_{i_0}}(x_s) = \{w_1, w_2, w_3\}$. Thus $x_2x_3 \dots x_{s-1}w_3w_4x_2$ is a cycle with chord $x_{s-1}w_4$, $x_1z_2z_3z_4x_1$ is a cycle with chord x_1z_3, and $x_sw_1z_1w_2x_s$ is a cycle with chord w_1w_2.

Case 2. $d_{C_{i_0}}(x_2) = 3$.

In this case, $d_{C_{i_0}}(z_1) = d_{C_{i_0}}(x_{s-1}) = 4$. Without loss of generality, we may assume $\{w_1, w_2, w_3\} \subseteq N_{C_{i_0}}(x_2)$. We claim that $d_{C_{i_0}}(x_1) = 0$. Assume that $x_1w_1 \in E(G)$. Then $x_1x_2 \dots x_{s-1}w_1x_1$ is a cycle with chord x_2w_1, $z_1w_2w_3w_4z_1$ is a cycle with chord z_1w_3, and $x_sz_2z_3z_4x_s$ is a cycle with chord x_sz_3. Thus $x_1w_1 \notin E(G)$. Similarly, $x_1w_j \notin E(G)$ for each $j \in \{2, 3\}$. If $x_1w_4 \in E(G)$, then $x_1z_1w_3w_4x_1$ is a cycle with chord z_1w_4, $x_2x_3 \dots x_{s-1}w_2w_1x_2$ is a cycle with chord $x_{s-1}w_1$, and $x_sz_2z_3z_4x_s$ is a cycle with chord x_sz_3. Thus the claim holds. If $x_sw_1 \in E(G)$, then $x_2x_3 \dots x_sw_1x_2$ is a cycle with chord $x_{s-1}w_1$, $x_1z_2z_3z_4x_1$ is a cycle with chord x_1z_3, and $z_1w_2w_3w_4z_1$ is a cycle with chord z_1w_3. Thus $x_sw_1 \notin E(G)$. Similarly, $x_sw_i \notin E(G)$ for each $i \in \{2, 3\}$. Thus $d_{C_{i_0}}(x_s) \le 1$. Since $d_{C_{i_0}}(x_1) = 0$, $d_{C_{i_0}}(x_s) \ge 3$ by (9). This is a contradiction.

Case 3. $d_{C_{i_0}}(x_{s-1}) = 3$.

We can prove this case similar to Case 2.

CASE (C). $d_{C_{i_0}}(z_1) + d_{C_{i_0}}(x_2) + d_{C_{i_0}}(x_{s-1}) = 10$.

In this case, by (7), we have

$$d_{C_{i_0}}(x_1) + d_{C_{i_0}}(x_s) \geq 5. \tag{10}$$

Case 1. $\{d_{C_{i_0}}(z_1), d_{C_{i_0}}(x_2), d_{C_{i_0}}(x_{s-1})\} = \{4, 4, 2\}$.

Subcase 1.1. $d_{C_{i_0}}(z_1) = 2$.

In this case, $d_{C_{i_0}}(x_2) = d_{C_{i_0}}(x_{s-1}) = 4$.

(i) The vertex z_1 is adjacent to consecutive vertices on C_{i_0}.

In this case, without loss of generality, we may assume that $w_1, w_2 \in N_{C_{i_0}}(z_1)$. Since $x_s z_2 z_3 z_4 x_s$ is a cycle with chord $x_s z_3$, we show the existence of two disjoint chorded cycles in $\langle (V(P_1) - \{x_s\}) \cup V(C_{i_0}) \cup \{z_1\} \rangle$. Assume that $x_1 w_1 \in E(G)$. Then $x_1 z_1 w_2 w_1 x_1$ is a cycle with chord $z_1 w_1$, and $x_2 w_3 x_{s-1} w_4 x_2$ is a cycle with chord $w_3 w_4$. Thus $x_1 w_1 \notin E(G)$. Similarly, $x_1 w_2 \notin E(G)$. Assume that $x_1 w_3 \in E(G)$. Then $x_1 w_3 w_4 x_2 x_1$ is a cycle with chord $x_2 w_3$, and $x_{s-1} w_1 z_1 w_2 x_{s-1}$ is a cycle with chord $w_1 w_2$. Thus $x_1 w_3 \notin E(G)$. Similarly, $x_1 w_4 \notin E(G)$. Thus $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

(ii) The vertex z_1 is not adjacent to consecutive vertices on C_{i_0}.

In this case, without loss of generality, we may assume that $w_1, w_3 \in N_{C_{i_0}}(z_1)$. Assume that $x_1 w_1 \in E(G)$. Then $x_1 x_2 w_1 z_1 x_1$ is a cycle with chord $x_1 w_1$, and $x_{s-1} w_2 w_3 w_4 x_{s-1}$ is a cycle with chord $x_{s-1} w_3$. Also, $x_s z_2 z_3 z_4 x_s$ is a cycle with chord $x_s z_3$. Thus $x_1 w_1 \notin E(G)$. Similarly, $x_1 w_3 \notin E(G)$. Then $d_{C_{i_0}}(x_1) \leq 2$. By (10), $d_{C_{i_0}}(x_s) \geq 3$. Suppose that $x_1 w_2 \in E(G)$. Then $x_1 x_2 w_1 w_2 x_1$ is a cycle with chord $x_2 w_2$. Since $d_{C_{i_0}}(x_s) \geq 3$, $x_s w_3 \in E(G)$ or $x_s w_4 \in E(G)$. If $x_s w_3 \in E(G)$, then $x_{s-1} x_s w_3 w_4 x_{s-1}$ is a cycle with chord $x_{s-1} w_3$, and if $x_s w_4 \in E(G)$, then $x_{s-1} x_s w_4 w_3 x_{s-1}$ is a cycle with chord $x_{s-1} w_4$. Also, we consider a chorded cycle C_{k-1}. Thus $x_1 w_2 \notin E(G)$. By symmetry, $x_1 w_4 \notin E(G)$. Thus $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

Subcase 1.2. $d_{C_{i_0}}(x_2) = 2$.

In this case, $d_{C_{i_0}}(z_1) = d_{C_{i_0}}(x_{s-1}) = 4$. Since $x_s z_2 z_3 z_4 x_s$ is a cycle with chord $x_s z_3$, we show the existence of two disjoint chorded cycles in $\langle (V(P_1) - \{x_s\}) \cup V(C_{i_0}) \cup \{z_1\} \rangle$.

(i) The vertex x_2 is adjacent to consecutive vertices on C_{i_0}.

In this case, without loss of generality, we may assume that $w_1, w_2 \in N_{C_{i_0}}(x_2)$. Assume that $x_1 w_1 \in E(G)$. Then $x_1 x_2 \ldots x_{s-1} w_1 x_1$ is a cycle with chord $x_2 w_1$, and $z_1 w_2 w_3 w_4 z_1$ is a cycle with chord $z_1 w_3$. Thus $x_1 w_1 \notin E(G)$. Similarly, $x_1 w_2 \notin E(G)$. Assume that $x_1 w_3 \in E(G)$. Then $x_1 w_3 w_4 z_1 x_1$ is a cycle with chord $z_1 w_3$, and $x_2 w_1 x_{s-1} w_2 x_2$ is a cycle with chord $w_1 w_2$. Thus $x_1 w_3 \notin E(G)$. Similarly, $x_1 w_4 \notin E(G)$. Thus $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

(ii) The vertex x_2 is not adjacent to consecutive vertices on C_{i_0}.

In this case, without loss of generality, we may assume that $w_1, w_3 \in N_{C_{i_0}}(x_2)$. Assume that $x_1 w_1 \in E(G)$. Then $x_1 x_2 \ldots x_{s-1} w_1 x_1$ is a cycle with chord $x_2 w_1$, and

$z_1w_2w_3w_4z_1$ is a cycle with chord z_1w_3. Thus $x_1w_1 \notin E(G)$. Similarly, $x_1w_3 \notin E(G)$. Assume that $x_1w_2 \in E(G)$. Then $x_1w_2w_3z_1x_1$ is a cycle with chord z_1w_2, and $x_2x_3 \ldots x_{s-1}w_4w_1x_2$ is a cycle with chord $x_{s-1}w_1$. Thus $x_1w_2 \notin E(G)$. Similarly, $x_1w_4 \notin E(G)$. Thus $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

Subcase 1.3. $d_{C_{i_0}}(x_{s-1}) = 2$.

In this case, we get a contradiction by the same arguments as Subcase 1.2.

Case 2. $\{d_{C_{i_0}}(z_1), d_{C_{i_0}}(x_2), d_{C_{i_0}}(x_{s-1})\} = \{4, 3, 3\}$.

Subcase 2.1. $d_{C_{i_0}}(z_1) = 4$.

In this case, $d_{C_{i_0}}(x_2) = d_{C_{i_0}}(x_{s-1}) = 3$. Without loss of generality, we may assume that $N_{C_{i_0}}(x_2) = \{w_1, w_2, w_3\}$. Since $x_sz_2z_3z_4x_s$ is a cycle with chord x_sz_3, we show the existence of two disjoint chorded cycles in $\langle (V(P_1) - \{x_s\}) \cup V(C_{i_0}) \cup \{z_1\} \rangle$.

(i) $N_{C_{i_0}}(x_{s-1}) = \{w_1, w_2, w_3\}$.

Assume that $x_1w_1 \in E(G)$. Then $x_1x_2 \ldots x_{s-1}w_1x_1$ is a cycle with chord x_2w_1, and $z_1w_2w_3w_4z_1$ is a cycle with chord z_1w_3. Thus $x_1w_1 \notin E(G)$. Similarly, $x_1w_3 \notin E(G)$. Assume that $x_1w_2 \in E(G)$. Then $x_1x_2 \ldots x_{s-1}w_2x_1$ is a cycle with chord x_2w_2, and $z_1w_3w_4w_1z_1$ is a cycle with chord z_1w_4. Thus $x_1w_2 \notin E(G)$. Assume that $x_1w_4 \in E(G)$. Then $x_1z_1w_3w_4x_1$ is a cycle with chord z_1w_4, and $x_2w_1x_{s-1}w_2x_2$ is a cycle with chord w_1w_2. Thus $x_1w_4 \notin E(G)$. Therefore, $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

(ii) $N_{C_{i_0}}(x_{s-1}) = \{w_1, w_2, w_4\}$ or $N_{C_{i_0}}(x_{s-1}) = \{w_2, w_3, w_4\}$.

By symmetry, it is sufficient to prove the case where $N_{C_{i_0}}(x_{s-1}) = \{w_1, w_2, w_4\}$. Assume that $x_1w_1 \in E(G)$. Then $x_1x_2 \ldots x_{s-1}w_1x_1$ is a cycle with chord x_2w_1, and $z_1w_2w_3w_4z_1$ is a cycle with chord z_1w_3. Thus $x_1w_1 \notin E(G)$. Assume that $x_1w_2 \in E(G)$. Then $x_1x_2 \ldots x_{s-1}w_2x_1$ is a cycle with chord x_2w_2, and $z_1w_3w_4w_1z_1$ is a cycle with chord z_1w_4. Thus $x_1w_2 \notin E(G)$. Assume that $x_1w_3 \in E(G)$. Then $x_1x_2w_2w_3x_1$ is a cycle with chord x_2w_3, and $z_1w_4x_{s-1}w_1z_1$ is a cycle with chord w_1w_4. Thus $x_1w_3 \notin E(G)$. Assume that $x_1w_4 \in E(G)$. Then $x_1z_1w_3w_4x_1$ is a cycle with chord z_1w_4, and $x_2x_3 \ldots x_{s-1}w_2w_1x_2$ is a cycle with chord $x_{s-1}w_1$. Thus $x_1w_4 \notin E(G)$. Therefore, $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

(iii) $N_{C_{i_0}}(x_{s-1}) = \{w_1, w_3, w_4\}$.

Assume that $x_1w_1 \in E(G)$. Then $x_1x_2w_2w_1x_1$ is a cycle with chord x_2w_1, and $z_1w_3x_{s-1}w_4z_1$ is a cycle with chord w_3w_4. Thus $x_1w_1 \notin E(G)$. Similarly, $x_1w_3 \notin E(G)$. Assume that $x_1w_2 \in E(G)$. Then $x_1x_2w_3w_2x_1$ is a cycle with chord x_2w_2, and $z_1w_1x_{s-1}w_4z_1$ is a cycle with chord w_1w_4. Thus $x_1w_2 \notin E(G)$. Assume that $x_1w_4 \in E(G)$. Then $x_1z_1w_3w_4x_1$ is a cycle with chord z_1w_4, and $x_2x_3 \ldots x_{s-1}w_1w_2x_2$ is a cycle with chord x_2w_1. Thus $x_1w_4 \notin E(G)$. Therefore, $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

Subcase 2.2. $d_{C_{i_0}}(x_2) = 4$.

In this case, $d_{C_{i_0}}(x_{s-1}) = d_{C_{i_0}}(z_1) = 3$. Without loss of generality, we may assume that $N_{C_{i_0}}(x_{s-1}) = \{w_1, w_2, w_3\}$. Since $x_s z_2 z_3 z_4 x_s$ is a cycle with chord $x_s z_3$, we show the existence of two disjoint chorded cycles in $\langle (V(P_1) - \{x_s\}) \cup V(C_{i_0}) \cup \{z_1\} \rangle$.

(i) $N_{C_{i_0}}(z_1) = \{w_1, w_2, w_3\}$.

Assume that $x_1 w_1 \in E(G)$. Then $x_1 z_1 w_2 w_1 x_1$ is a cycle with chord $z_1 w_1$, and $x_2 x_3 \ldots x_{s-1} w_3 w_4 x_2$ is a cycle with chord $x_2 w_3$. Thus $x_1 w_1 \notin E(G)$. Similarly, $x_1 w_3 \notin E(G)$. Assume that $x_1 w_2 \in E(G)$. Then $x_1 z_1 w_1 w_2 x_1$ is a cycle with chord $z_1 w_2$, and $x_2 x_3 \ldots x_{s-1} w_3 w_4 x_2$ is a cycle with chord $x_2 w_3$. Thus $x_1 w_2 \notin E(G)$. Assume that $x_1 w_4 \in E(G)$. Then $x_1 x_2 w_3 w_4 x_1$ is a cycle with chord $x_2 w_4$, and $x_{s-1} w_1 z_1 w_2 x_{s-1}$ is a cycle with chord $w_1 w_2$. Thus $x_1 w_4 \notin E(G)$. Therefore, $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

(ii) $N_{C_{i_0}}(z_1) = \{w_1, w_2, w_4\}$ or $N_{C_{i_0}}(z_1) = \{w_2, w_3, w_4\}$.

By symmetry, it is sufficient to prove the case where $N_{C_{i_0}}(z_1) = \{w_1, w_2, w_4\}$. Assume that $x_1 w_1 \in E(G)$. Then $x_1 z_1 w_4 w_1 x_1$ is a cycle with chord $z_1 w_1$, and $x_2 x_3 \ldots x_{s-1} w_3 w_2 x_2$ is a cycle with chord $x_{s-1} w_2$. Thus $x_1 w_1 \notin E(G)$. Assume that $x_1 w_2 \in E(G)$. Then $x_1 w_2 w_1 z_1 x_1$ is a cycle with chord $z_1 w_2$, and $x_2 x_3 \ldots x_{s-1} w_3 w_4 x_2$ is a cycle with chord $x_2 w_3$. Thus $x_1 w_2 \notin E(G)$. Assume that $x_1 w_3 \in E(G)$. Then $x_1 x_2 w_4 w_3 x_1$ is a cycle with chord $x_2 w_3$, and $x_{s-1} w_1 z_1 w_2 x_{s-1}$ is a cycle with chord $w_1 w_2$. Thus $x_1 w_3 \notin E(G)$. Assume that $x_1 w_4 \in E(G)$. Then $x_1 x_2 w_3 w_4 x_1$ is a cycle with chord $x_2 w_4$, and $x_{s-1} w_1 z_1 w_2 x_{s-1}$ is a cycle with chord $w_1 w_2$. Thus $x_1 w_4 \notin E(G)$. Therefore, $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

(iii) $N_{C_{i_0}}(z_1) = \{w_1, w_3, w_4\}$.

Assume that $x_1 w_1 \in E(G)$. Then $x_1 z_1 w_4 w_1 x_1$ is a cycle with chord $z_1 w_1$, and $x_2 x_3 \ldots x_{s-1} w_2 w_3 x_2$ is a cycle with chord $x_{s-1} w_3$. Thus $x_1 w_1 \notin E(G)$. Similarly, $x_1 w_3 \notin E(G)$. Assume that $x_1 w_2 \in E(G)$. Then $x_1 x_2 \ldots x_{s-1} w_2 x_1$ is a cycle with chord $x_2 w_2$, and $z_1 w_3 w_4 w_1 z_1$ is a cycle with chord $z_1 w_4$. Thus $x_1 w_2 \notin E(G)$. Assume that $x_1 w_4 \in E(G)$. Then $x_1 z_1 w_1 w_4 x_1$ is a cycle with chord $z_1 w_4$, and $x_2 x_3 \ldots x_{s-1} w_3 w_2 x_2$ is a cycle with chord $x_{s-1} w_2$. Thus $x_1 w_4 \notin E(G)$. Therefore, $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

Subcase 2.3. $d_{C_{i_0}}(x_{s-1}) = 4$.

In this case, $d_{C_{i_0}}(x_2) = d_{C_{i_0}}(z_1) = 3$. Without loss of generality, we may assume that $N_{C_{i_0}}(x_2) = \{w_1, w_2, w_3\}$.

(i) $N_{C_{i_0}}(z_1) = \{w_1, w_2, w_3\}$.

Assume that $x_1 w_1 \in E(G)$. Then $x_1 z_1 w_2 w_1 x_1$ is a cycle with chord $z_1 w_1$, and $x_2 x_3 \ldots x_{s-1} w_4 w_3 x_2$ is a cycle with chord $x_{s-1} w_3$. Also, $x_s z_2 z_3 z_4 x_s$ is a cycle with chord $x_s z_3$. Thus $x_1 w_1 \notin E(G)$. Similarly, $x_1 w_3 \notin E(G)$. Assume that $x_1 w_2 \in E(G)$. Then $x_1 z_1 w_1 w_2 x_1$ is a cycle with chord $z_1 w_2$, and $x_2 x_3 \ldots x_{s-1} w_4 w_3 x_2$ is a cycle with chord $x_{s-1} w_3$. Also, $x_s z_2 z_3 z_4 x_s$ is a cycle with chord $x_s z_3$. Thus $x_1 w_2 \notin E(G)$. Therefore, $d_{C_{i_0}}(x_1) \leq 1$. By (10), $d_{C_{i_0}}(x_s) \geq 4$. Then $x_1 z_2 z_3 z_4 x_1$ is a cycle with chord x_1z_3, $x_2w_1z_1w_2x_2$ is a cycle with chord w_1w_2, and $x_{s-1}x_sw_3w_4x_{s-1}$ is a cycle with chord $x_{s-1}w_3$, a contradiction.

(ii) $N_{C_{i_0}}(z_1) = \{w_1, w_2, w_4\}$ or $N_{C_{i_0}}(z_1) = \{w_2, w_3, w_4\}$.

By symmetry, it is sufficient to prove the case where $N_{C_{i_0}}(z_1) = \{w_1, w_2, w_4\}$. Since $x_sz_2z_3z_4x_s$ is a cycle with chord x_sz_3, we show the existence of two disjoint chorded cycles in $\langle (V(P_1) - \{x_s\}) \cup V(C_{i_0}) \cup \{z_1\}\rangle$. Assume that $x_1w_1 \in E(G)$. Then $x_1z_1w_2w_1x_1$ is a cycle with chord z_1w_1, and $x_2x_3 \dots x_{s-1}w_4w_3x_2$ is a cycle with chord $x_{s-1}w_3$. Thus $x_1w_1 \notin E(G)$. Assume that $x_1w_2 \in E(G)$. Then $x_1x_2w_3w_2x_1$ is a cycle with chord x_2w_2, and $z_1w_1x_{s-1}w_4z_1$ is a cycle with chord w_1w_4. Thus $x_1w_2 \notin E(G)$. Assume that $x_1w_3 \in E(G)$. Then $x_1x_2w_2w_3x_1$ is a cycle with chord x_2w_3, and $z_1w_1x_{s-1}w_4z_1$ is a cycle with chord w_1w_4. Thus $x_1w_3 \notin E(G)$. Assume that $x_1w_4 \in E(G)$. Then $x_1z_1w_1w_4x_1$ is a cycle with chord z_1w_4, and $x_2w_2x_{s-1}w_3x_2$ is a cycle with chord w_2w_3. Thus $x_1w_4 \notin E(G)$. Therefore, $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

(iii) $N_{C_{i_0}}(z_1) = \{w_1, w_3, w_4\}$.

Since $x_sz_2z_3z_4x_s$ is a cycle with chord x_sz_3, we show the existence of two disjoint chorded cycles in $\langle (V(P_1) - \{x_s\}) \cup V(C_{i_0}) \cup \{z_1\}\rangle$. Assume that $x_1w_1 \in E(G)$. Then $x_1x_2w_1z_1x_1$ is a cycle with chord x_1w_1, and $x_{s-1}w_2w_3w_4x_{s-1}$ is a cycle with chord $x_{s-1}w_3$. Thus $x_1w_1 \notin E(G)$. Similarly, $x_1w_3 \notin E(G)$. Assume that $x_1w_2 \in E(G)$. Then $x_1x_2w_1w_2x_1$ is a cycle with chord x_2w_2, and $z_1w_3x_{s-1}w_4z_1$ is a cycle with chord w_3w_4. Thus $x_1w_2 \notin E(G)$. Assume that $x_1w_4 \in E(G)$. Then $x_1z_1w_1w_4x_1$ is a cycle with chord z_1w_4, and $x_2w_2x_{s-1}w_3x_2$ is a cycle with chord w_2w_3. Thus $x_1w_4 \notin E(G)$. Therefore, $d_{C_{i_0}}(x_1) = 0$. By (10), $d_{C_{i_0}}(x_s) \geq 5$. This contradicts $|C_{i_0}| = 4$.

□

Let

$$\begin{aligned} H_1 &= \langle H \cup C_{k-1}\rangle, \\ d_1 &= \sum_{i=1}^{k-2}\{2(d_{C_i}(z_1) + d_{C_i}(x_2) + d_{C_i}(x_{s-1})) + d_{C_i}(x_1) + d_{C_i}(x_s)\}, \text{ and} \\ d_2 &= 2(d_{H_1}(z_1) + d_{H_1}(x_2) + d_{H_1}(x_{s-1})) + d_{H_1}(x_1) + d_{H_1}(x_s). \end{aligned}$$

By (3), (6), and Subclaim 3.3.1 (ii), we have $d_G(z_1) \geq 3k$ and $d_G(x_j) \geq 3k$ for each $j \in \{1, 2, s-1, s\}$. By Subclaim 3.3.2, $d_1 \leq 24(k-2)$. Since $|C_{k-1}| = 4$, $d_{H_1}(x_1) = d_H(x_1) + d_{C_{k-1}}(x_1) \leq 2 + 4 = 6$. Similarly, $d_{H_1}(x_s) \leq 6$. By Subclaims 3.3.1 (i) and (iii), $d_2 \leq 2(6 + 10) + 6 + 6 = 44$. Thus we have

$$\begin{aligned} 24k &\leq 2(d_G(z_1) + d_G(x_2) + d_G(x_{s-1})) + d_G(x_1) + d_G(x_s) \\ &= d_1 + d_2 \leq 24(k-2) + 44 = 24k - 4. \end{aligned}$$

This is a contradiction.

■

Claim 3.4. *Let* $X = \{x \in V(P_1) \mid d_G(x) \geq 3k\}$. *Then* $|X| \leq 3$.

Proof. Suppose that $|X| \geq 4$. We claim that we may assume that $d_G(x_1) \geq 3k$. Suppose that $x_1x_s \notin E(G)$. By the $\sigma_1^2(G)$ condition, without loss of generality, we may assume that $d_G(x_1) \geq 3k$. Suppose that $x_1x_s \in E(G)$. By Claims 3.2 and 3.3, we have $|P_1| \geq 7$. Let $C' = x_1x_2 \dots x_sx_1$. Then $|C'| \geq 7$. Since H does not contain a chorded cycle, C' does not have a chord. Thus $x_1x_{s-1} \notin E(G)$. By the $\sigma_1^2(G)$ condition, without loss of generality, we may assume that $d_G(x_1) \geq 3k$. Thus the claim holds. Since $x_1 \in X$, there exist at least three vertices x_p, x_q, x_r $(2 \leq p < q < r \leq s)$ in X. We choose x_r such that r is as large as possible. If $r \leq s-4$, then $\langle\{x_{s-3}, x_{s-2}, x_{s-1}, x_s\}\rangle$ is a complete subgraph of order 4 in H by Claim 3.1, and contains a chorded cycle, a contradiction. Thus we may assume that $r \geq s-3$.

Case 1. $r \in \{s-3, s-1, s\}$.

We prove that $d_{P_1}(x_r) \leq 3$ for each $r \in \{s-3, s-1, s\}$. Suppose that $r \in \{s-1, s\}$. Since H does not contain a chorded cycle, we have $d_{P_1}(x_r) \leq 3$. Suppose that $r = s-3$. Since $d_G(x_{r'}) \leq 3k-1$ for each $s-2 \leq r' \leq s$, we have $x_{s-2}x_s \in E(G)$ by Claim 3.1. If $x_{s-3}x_{s-1} \in E(G)$, then $x_{s-3}x_{s-2}x_sx_{s-1}x_{s-3}$ is a cycle with chord $x_{s-2}x_{s-1}$. If $x_{s-3}x_s \in E(G)$, then $x_{s-3}x_{s-2}x_{s-1}x_sx_{s-3}$ is a cycle with chord $x_{s-2}x_s$. Since H does not contain a chorded cycle, we have $d_{P_1}(x_{s-3}) \leq 3$. Thus $d_{P_1}(x_r) \leq 3$ for each $r \in \{s-3, s-1, s\}$, and by Claim 3.3, $d_H(x_r) = d_{P_1}(x_r) \leq 3$. Then

$$\begin{aligned} d_L(x_1) + d_L(x_r) &= d_G(x_1) + d_G(x_r) - (d_H(x_1) + d_H(x_r)) \\ &\geq 3k + 3k - (2+3) = 6(k-1) + 1. \end{aligned}$$

Thus $d_{C_{i_0}}(x_1) + d_{C_{i_0}}(x_r) \geq 7$ for some $1 \leq i_0 \leq k-1$. Without loss of generality, we may assume that $i_0 = k-1$. Then $d_{C_{k-1}}(x_1) \geq 4$ or $d_{C_{k-1}}(x_r) \geq 4$. By (A1) and Lemma 2.3, we have $|C_{k-1}| = 4$. Let $C_{k-1} = z_1z_2z_3z_4z_1$. Suppose that $d_{C_{k-1}}(x_1) \geq 4$ (in fact, $d_{C_{k-1}}(x_1) = 4$) and $\{z_2, z_3, z_4\} \subseteq N_{C_{k-1}}(x_r)$. If $d_{C_{k-1}}(x_r) \geq 4$, then we can prove this case by the same arguments as $d_{C_{k-1}}(x_1) \geq 4$.

We prove that $d_G(z_1) \geq 3k$. Suppose that $d_G(z_1) < 3k$. Assume that $r \in \{s-3, s-1\}$. Then $C'_{k-1} = x_1z_2z_3z_4x_1$ is a 4-cycle with chord x_1z_3, and we consider the new L' containing C'_{k-1} instead of C_{k-1}. Let $H' = G - L'$. By the maximality of r, we have $d_G(x_s) < 3k$. By Claim 3.1, $x_sz_1 \in E(G)$. Thus $x_2x_3 \dots x_sz_1$ is a longest path (the same length as P_1) in H'. Since $d_G(x_1) \geq 3k$ and $d_G(z_1) < 3k$, this contradicts (A3). Assume that $r = s$. Then $C''_{k-1} = x_sz_2z_3z_4x_s$ is a 4-cycle with chord x_sz_3, and we consider the new L'' containing C''_{k-1} instead of C_{k-1}. Let $H'' = G - L''$. Then $z_1x_1x_2 \dots x_{s-1}$ is a longest path (the same length as P_1) in H''. Since $d_G(x_s) \geq 3k$ and $d_G(z_1) < 3k$, this contradicts (A3). Thus $d_G(z_1) \geq 3k$. Let $H_1 = \langle H \cup C_{k-1}\rangle$. We prove that $d_{H_1}(z_1) \leq 6$. Suppose that $d_{P_1}(z_1) \geq 4$. Let $x_{i_1}, x_{i_2}, x_{i_3} \in N_{P_1 - x_1}(z_1)$ for $i_1 < i_2 < i_3$. Then $z_1x_{i_1}x_{i_1+1} \dots x_{i_3}z_1$ is a cycle with chord $z_1x_{i_2}$, and $x_1z_2z_3z_4x_1$ is a cycle with chord x_1z_3, a contradiction. Thus $d_{P_1}(z_1) \leq 3$. Since $d_{C_{k-1}}(z_1) \leq 3$, $d_{H_1}(z_1) \leq 6$.

We have two observations as follows. Since $d_{P_1}(x_i) \leq 3$ for each $i \in \{2, s-1\}$, we have

$$d_{P_1}(x_2) + d_{P_1}(x_{s-1}) \leq 6. \tag{11}$$

We prove that for each $1 \le j \le s-1$,

$$d_{P_1}(x_j) + d_{P_1}(x_{j+1}) \le 7. \tag{12}$$

Suppose that $d_{P_1}(x_{j_0}) + d_{P_1}(x_{j_0+1}) \ge 8$ for some $1 \le j_0 \le s-1$. Then $d_{P_1}(x_{j_0}) = 4$ and $d_{P_1}(x_{j_0+1}) = 4$. Let $x_{j_0}x_{t_2}, x_{j_0+1}x_{t_1} \in E(G)$ for $j_0+1 < t_2$ and $t_1 < j_0$. Then $x_{t_1}x_{t_1+1}\dots x_{j_0}x_{t_2}x_{t_2-1}\dots x_{j_0+1}x_{t_1}$ is a cycle with chord $x_{j_0}x_{j_0+1}$, a contradiction. Thus (12) holds. We show that

$$x_p \text{ and } x_q \text{ can be chosen so that } d_{P_1}(x_p) + d_{P_1}(x_q) \le 7. \tag{13}$$

Let $Y = \{y \in V(P_1) \mid d_G(y) \le 3k-1\}$. By Claim 3.1, $|Y| \le 3$. Suppose that $r = s$. If $|Y| = 0$, then we let $x_p = x_2$ and $x_q = x_{s-1}$ by (11). If $|Y| \in \{1, 2\}$, then since $|P_1| \ge 7$, we can choose the desired vertices x_p, x_q by (11) and (12). Suppose that $|Y| = 3$. Then Y consists of three consecutive vertices on P_1, otherwise, there exists a chorded cycle in H by Claim 3.1. Since $|P_1| \ge 7$, we can choose the desired vertices x_p, x_q by (11) and (12). Thus (13) holds.

Suppose that $r = s-1$. Then $x_s \in Y$. Let $Y' = Y - \{x_s\}$. If $|Y'| \in \{0, 1\}$, then since $|P_1| \ge 7$, we can choose the desired vertices x_p, x_q by (12). If $|Y'| = 2$, then since $r = s-1$, there exists a chorded cycle in H by Claim 3.1. Suppose that $r = s-3$. Then $x_j \in Y$ for each $s-2 \le j \le s$. Since $|P_1| \ge 7$, we can choose the desired vertices x_p, x_q by (12).

Since H_1 does not contain two disjoint chorded cycles, $N_{C_{k-1}}(x_p) \cap N_{C_{k-1}}(x_q) = \emptyset$ similar to the proof of Subclaim 3.3.1 (iii). Thus $d_{C_{k-1}}(x_p) + d_{C_{k-1}}(x_q) \le 4$. By (13), $d_{H_1}(x_p) + d_{H_1}(x_q) \le 7 + 4 = 11$. By regarding x_p, x_q, x_r as x_2, x_{s-1}, x_s, we have $2(d_{C_i}(z_1) + d_{C_i}(x_p) + d_{C_i}(x_q)) + d_{C_i}(x_1) + d_{C_i}(x_r) \le 24$ for each $1 \le i \le k-2$ similar to Subclaim 3.3.2. Also, $d_{H_1}(x_1) = d_{P_1}(x_1) + d_{C_{k-1}}(x_1) \le 2 + 4 = 6$ and $d_{H_1}(x_r) = d_{P_1}(x_r) + d_{C_{k-1}}(x_r) \le 3 + 4 = 7$. Thus we have $2(d_{H_1}(z_1) + d_{H_1}(x_p) + d_{H_1}(x_q)) + d_{H_1}(x_1) + d_{H_1}(x_r) \le 2(6 + 11) + 6 + 7 = 47$. Thus $2(d_G(z_1) + d_G(x_p) + d_G(x_q)) + d_G(x_1) + d_G(x_r) \le 24(k-2) + 47 = 24k - 1$. Since $d_G(z_1) \ge 3k$ and $d_G(x_j) \ge 3k$ for each $j \in \{1, p, q, r\}$, this is a contradiction.

Case 2. $r = s-2$.

If $d_G(x_{i_0}) \le 3k-1$ for some $1 \le i_0 \le s-3$, then $x_{i_0}x_{s-1}, x_{i_0}x_s \in E(G)$ by Claim 3.1, and $x_{i_0}x_{i_0+1}\dots x_s x_{i_0}$ is a cycle with chord $x_{i_0}x_{s-1}$, a contradiction. Thus we may assume that $d_G(x_i) \ge 3k$ for each $1 \le i \le s-3$. If $d_{P_1}(x_{s-2}) \le 3$, then we get a contradiction by the same arguments as Case 1. Thus we may assume that $d_{P_1}(x_{s-2}) = 4$. Then $x_{s-2}x_s \in E(G)$. If $d_{P_1}(x_{s-3}) = 4$, then $x_{s-3}x_{s-1} \in E(G)$ or $x_{s-3}x_s \in E(G)$. In both cases, $\langle \{x_{s-3}, x_{s-2}, x_{s-1}, x_s\} \rangle$ contains a chorded cycle. Thus we may assume that $d_{P_1}(x_{s-3}) \le 3$. Then we get a contradiction by the same arguments as Case 1.

□

Let $Y = \{y \in V(P_1) \mid d_G(y) \le 3k-1\}$. By Claims 3.2, 3.3, and 3.4, we have $|Y| \ge 4$. By Claim 3.1, $\langle Y \rangle$ is a complete subgraph in H. Thus H contains a chorded cycle, a contradiction.

■

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