AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **95(2)** (2026), Pages 307–316

Forbidden configurations and boundary cases

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Abstract

Let F be a $k \times \ell$ (0,1)-matrix. Define a (0,1)-matrix A to have F as a configuration if there is a submatrix of A which is a row and column permutation of F. In the language of sets, a configuration is a trace. Define a matrix to be simple if it is a (0,1)-matrix with no repeated columns. Let $\text{Avoid}(m, F)$ be all simple m-rowed matrices A with no configuration F. Define $\text{forb}(m, F)$ as the maximum number of columns of any matrix in $\text{Avoid}(m, F)$. Determining $\text{forb}(m, F)$ requires determining bounds and constructions of matrices in $\text{Avoid}(m, F)$. This paper considers some column maximal k-rowed simple F's that have the bound $\Theta(m^{k-2})$ and yet adding a column increases the bound to $\Omega(m^{k-1})$. By a construction, $\text{forb}(m, F)$ is determined exactly.

1 Introduction

This paper considers some boundary forbidden configurations. We start with some notation.

[*] Research supported in part by NSERC.
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ISSN: 2202-3518

An $m \times n$ matrix A is said to be *simple* if it is a (0,1)-matrix with no repeated columns. There is a natural correspondence between columns of A and subsets of $[m]$. We consider an extremal set problem in matrix terminology as follows. Let $\|A\|$ be the number of columns of A. For a given matrix F, we say F is a *configuration* in A, denoted $F \prec A$, if there is a submatrix of A which is a row and column permutation of F. Define

$$\text{Avoid}(m, F) = \{A \mid A \text{ is } m\text{-rowed and simple}, F \not\prec A\},$$

$$\text{forb}(m, F) = \max_{A \in \text{Avoid}(m,F)} \|A\|.$$

A matrix $A \in \text{Avoid}(m, F)$ is called *extremal* if $\|A\| = \text{forb}(m, F)$ and let

$$\text{Ext}(m, F) = \{A \in \text{Avoid}(m, F) \colon \|A\| = \text{forb}(m, F)\}.$$

Some important matrices include I_k, the $k \times k$ identity matrix, T_k the $k \times k$ triangular matrix (with $1's$ in position i, j if $i \leq j$), and K_k the $k \times 2^k$ matrix of all possible (0,1)-columns on k rows. We denote by F^c the (0,1)-complement of F so that I_k^c is the complement of the identity. Define K_k^s to be the $k \times \binom{k}{s}$ matrix of columns of sum s. We define $\mathbf{1}_k$ as a $k \times 1$ column of 1's, $\mathbf{0}_k$ as a $k \times 1$ column of 0's and $\mathbf{1}_k\mathbf{0}_\ell$ as a $(k + \ell) \times 1$ column with k 1's on top of ℓ 0's.

For two simple matrices A, B where A is m_1-rowed and B is m_2-rowed, the *product* $A \times B$ is defined as the $(m_1 + m_2)$-rowed matrix of $\|A\|\|B\|$ columns consisting of each column of A on top of each column of B. For example $K_k = [0\,1] \times K_{k-1}$. For a subset S of rows, define A_S as the submatrix of A formed by those rows. We typically ignore row and column permutations of our matrices unless explicitly stated.

Many results are recorded in a survey [1]. Theorem 1.3 obtains exact bounds for four extremal configurations, cases of k-rowed simple F where $\text{forb}(m, F)$ is $O(m^{k-2})$ and adding any column α to F has $\text{forb}(m, [F \mid \alpha])$ being $\Omega(m^{k-1})$. The main conjecture in [1] suggests which matrices have this property. We discuss this after Theorem 1.1.

Define $f(m, k)$ for $k \geq 2$ by the recurrence

$$f(m, k) = f(m - 1, k) + f(m - 1, k - 1), \tag{1}$$

with the base cases $f(m, 2) = 2$ and $f(2, k) = 2$ for $k > 2$ (so that $f(m, 3) = 2m$ as in Theorem 2.2).

We can solve the recurrence to obtain by induction

$$f(m, k) = f(m - 1, k) + f(m - 1, k - 1)$$

$$= \binom{m-2}{k-2} + \sum_{i=0}^{k-2} \binom{m-1}{i} + \binom{m-2}{k-3} + \sum_{i=0}^{k-3} \binom{m-1}{i} = \binom{m-1}{k-2} + \sum_{i=0}^{k-2} \binom{m}{i}.$$

Note $f(m, 2) = 2$. Also

$$f(m, k) = 2\sum_{i=0}^{k-2} \binom{m-1}{i} = 2\binom{m}{k-2} + 2\binom{m}{k-4} + 2\binom{m}{k-6} + \cdots. \tag{2}$$

Theorem 1.1 *[1] [3] Let $k \geq 2$ be given.*

If $\mathcal{F}$ is a family of simple $k \times \ell$ matrices with the property that

- *there is an $F \in \mathcal{F}$ with a pair of rows that do not contain K_2^0,*
- *there is an $F \in \mathcal{F}$ with a pair of rows that do not contain K_2^2 and*
- *there is an $F \in \mathcal{F}$ with a pair of rows that do not contain the configuration $K_2^1 = I_2$,*

then $\text{forb}(m, \mathcal{F})$ *is $O(m^{k-2})$. More precisely* $\text{forb}(m, \mathcal{F}) \leq f(m, k)$ *defined by the recurrence (1) with bases cases $f(m, 2) = 2$ and $f(2, k) = 2$ for $k > 2$ so that $f(m, k) = \binom{m-1}{k-2} + \sum_{i=0}^{k-2} \binom{m}{i}$.*

If F is a simple $k \times \ell$ matrix with the property that

- *either every pair of rows has K_2^0*
- *or every pair of rows has K_2^2*
- *or every pair of rows has K_2^1,*

then $\text{forb}(m, F)$ *is $\Theta(m^{k-1})$.* □

The conjecture [1] asserts that the asymptotic bounds are achieved by products of I, I^c, T.

Conjecture 1.2 *Let F be given. Let t be the largest integer so that there is a t-fold product of matrices chosen from I, I^c, T that avoids F. Then* $\text{forb}(m, F)$ *is $\Theta(m^t)$.*

Applying Theorem 1.1 allows one to identify possible *boundary cases*, namely configurations F, with conjectured bound $\text{forb}(m, F)$, where adding any column to F forming F' has $\text{forb}(m, F')$ being $\Omega(m \cdot \text{forb}(m, F))$. Below in (3) (plus $F_{4,k}^c$) are all simple k-rowed F, for which $\text{forb}(m, F)$ is $O(m^{k-2})$ yet adding any column increases the bound to $\Omega(m^{k-1})$. This is proven in Lemma 2.1. Note each configuration F in (3) has only one pair of rows of F avoiding $\mathbf{1}_2$. To find F in the $(k-2)$-fold product construction $A = I_{m/(k-2)} \times I_{m/(k-2)} \times \cdots \times I_{m/(k-2)}$ one must either take three rows from one factor $I_{m/(k-2)}$ or one pair of rows from each of two factors $I_{m/(k-2)}$, neither works. So the $(k-2)$-fold product shows $\text{forb}(m, F)$ is $\Omega(m^{k-2})$ and hence $\Theta(m^{k-2})$.

$$F_{1,k} = \begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ \times \\ K_{k-3} \end{array}, \quad F_{2,k} = \begin{array}{c} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \times \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ \times \\ K_{k-4} \end{array},$$

$$F_{3,k} = \begin{array}{c} \begin{bmatrix} 0&0&0&0&1&0&0&1 \\ 0&0&0&1&0&1&1&0 \\ 0&1&1&0&0&1&1&1 \\ 1&0&1&1&1&0&1&1 \end{bmatrix} \\ \times \\ K_{k-4} \end{array}, \quad F_{4,k} = \begin{array}{c} \begin{bmatrix} 0&0&1&1&0 \\ 0&0&0&0&1 \\ 0&1&0&1&1 \end{bmatrix} \\ \times \\ \begin{bmatrix} 1&0&1 \\ 0&1&1 \end{bmatrix} \\ \times \\ K_{k-5} \end{array}, \tag{3}$$

$$F_{5,k} = F_{4,k}^c = \begin{array}{c} \begin{bmatrix} 1&0&0&1&1 \\ 0&1&1&1&1 \\ 0&0&1&0&1 \end{bmatrix} \\ \times \\ \begin{bmatrix} 0&1&0 \\ 0&0&1 \end{bmatrix} \\ \times \\ K_{k-5} \end{array}, \quad F_{6,k} = \begin{array}{c} \begin{bmatrix} 0&1&0 \\ 0&0&1 \end{bmatrix} \\ \times \\ \begin{bmatrix} 1&0&1 \\ 0&1&1 \end{bmatrix} \\ \times \\ \begin{bmatrix} 0&1&1 \\ 0&0&1 \end{bmatrix} \\ \times \\ K_{k-6} \end{array}$$

This paper establishes some exact bounds as follows.

Theorem 1.3 $\text{forb}(m, F_{1,4}) = f(m, 4)$ *and* $\text{forb}(m, F_{3,k}) = f(m, k)$ *for* $k = 4, 5, 6$.

Note that $f(m,k)$ is $\Theta(m^{k-2})$. Now $A = I_{m/(k-2)} \times I_{m/(k-2)} \times \cdots \times I_{m/(k-2)}$ is an $m \times (m/(k-2))^{k-2}$ simple matrix and $\|A\|/m^{k-2} = \frac{1}{(k-2)^{k-2}}$. Note that $\lim_{m\to\infty} f(m,k)/m^{k-2} = \frac{2}{(k-2)!}$ leaving a wide gap for exact bounds for cases in (3).

Coding theory helps to give a better construction for $\text{Avoid}(m, F_{i,k})$ than the one given by the product construction. One has to observe that $I_2 \times \mathbf{1}_{k-2} \prec F_{i,k}$ for all $1 \le i \le 6$. Let $A(m, 4, k-1)$ be a constant weight code of minimum distance 4, length m and weight $k-1$. Thus, if A is an m-rowed matrix consisting of all columns with at most $k-2$ 1's and columns of codewords of $A(m, 4, k-1)$, then $A \in \text{Avoid}(m, I_2 \times \mathbf{1}_{k-2}) \subset \text{Avoid}(m, F_{i,k})$. Indeed, A does not have two columns with $k-1$ 1's on the same k-set of rows. By a theorem of Graham and Sloane [6] this gives the lower bound $\text{forb}(m, F_{i,k}) \ge \frac{1}{m}\binom{m}{k-1} + \sum_{i=0}^{k-2}\binom{m}{i}$. Thus we have, applying Theorem 1.1,

$$\frac{1}{k-1}\binom{m-1}{k-2} + \sum_{i=0}^{k-2}\binom{m}{i} \le \text{forb}(m, F_{i,k}) \le \binom{m-1}{k-2} + \sum_{i=0}^{k-2}\binom{m}{i}. \tag{4}$$

The lower bound of (4) divided by m^{k-2} decreases the wide gap mentioned above to

$$\frac{1}{(k-1)!} + \frac{1}{(k-2)!} \le \lim_{m\to\infty} \frac{1}{m^{k-2}} \text{forb}(m, F_{i,k}) \le \frac{2}{(k-2)!}. \tag{5}$$

In fact, for $k = 4, 5$, Dukes [4] gives better lower bounds using nested block designs. Let $d(m, k) = \frac{1}{m^{k-2}}\mathrm{forb}(m, I_2 \times \mathbf{1}_{k-2})$. His constructions give $d(m, 4) \geq 0.6909$ and $d(m, 5) \geq 0.25138$, while (5) results in only $d(m, 4) \geq \frac{2}{3}$ and $d(m, 5) \geq \frac{5}{24} \approx 0.2083$. For $k > 5$, the difficult part of Dukes' method is finding good $t - (n, \ell, 1)$ packings where n is not too much larger than ℓ for $t > 3$. These results highlight the difficulty of finding the best, or indeed good constructions.

There are other functions that satisfy the recurrence (1) with different base cases including of course the binomial coefficients (with base cases $\binom{m}{k} = 0$ for $m < k$ and $\binom{m}{0} = 1$). One standard proof for $\mathrm{forb}(m, K_k)$ uses the recurrence $\mathrm{forb}(m, K_k) = \mathrm{forb}(m-1, K_k) + \mathrm{forb}(m-1, K_{k-1})$ but in this case the base cases are $\mathrm{forb}(m, K_1) = 1$ and $\mathrm{forb}(1, K_k) = 2$ for $k \geq 2$.

The recurrence (1) appears in geometry counting arguments [5] (see their Fact 2). For example $\mathrm{forb}(m, K_k)$ gives a bound on the number of regions in $\mathbf{R}^{k-1}$ when divided by m hyperplanes. The k in $f(m, k)$ is typically the dimension while m is the number of some geometric objects.

2 Constructions

When we use induction to establish the bound Theorem 1.1, we forbid the three 2-rowed matrices

$$\mathcal{F} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \right\},$$

for which $\mathrm{forb}(m, \mathcal{F}) = 2$. This is the case $k = 2$ in our recurrence. Now consider $k \geq 3$.

Lemma 2.1 *Assume $k \geq 3$. The six configurations in (3) are the only column maximal k-rowed simple matrices F with* $\mathrm{forb}(m, F)$ *being* $\Theta(m^{k-1})$.

Proof: This was noted in [1] without proof. To obtain the list of (3) we consider a simple k-rowed matrix F satisfying the hypotheses of Theorem 1.1 so that F has a pair of rows i_1, i_2 which has no $K_2^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, a pair of rows i_3, i_4 which has no $K_2^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and a pair of rows i_5, i_6 which has no $K_2^1 = I_2$. Assume F is column maximal with these properties.

First we show that the pairs (i_1, i_2) (i_3, i_4) (i_5, i_5) cannot overlap. If $i_1 = i_3 = i_5$ and $i_2 = i_4 = i_6$ then the k-rowed simple matrix would be $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \times K_{k-2}$ which is contained in $F_{1,k}$ using 2nd and 3rd columns from first product. If $i_1 = i_3$ and $i_2 = i_4$ and $i_5 = i_1$, and $i_6 \notin \{i_1, i_2\}$ then we obtain

$$\begin{matrix} i_1 \\ i_2 \\ i_6 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \times K_{k-3} \quad \text{or} \quad \begin{matrix} i_1 \\ i_2 \\ i_6 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \times K_{k-3}$$

but these are both contained in $F_{1,k}$. If $i_1 = i_3$ and $i_2 = i_4$ and $i_5, i_6 \notin \{i_1, i_2\}$ the column maximal k-rowed matrix is $F_{2,k}$ with the first four rows being i_1, i_2, i_3, i_4 in that order. If $i_1 = i_5$ and $i_2 = i_6$ and $i_1 = i_3$ and $i_4 \notin \{i_1, i_2\}$ the column maximal k-rowed simple matrix is

$$\begin{matrix} i_1 \\ i_2 \\ i_4 \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times K_{k-3}$$

which is $F_{1,k}$. If $i_1 = i_5$ and $i_2 = i_6$ and $i_3, i_4 \notin \{i_1, i_2\}$, the column maximal k-rowed simple matrix is

$$\begin{matrix} i_1 \\ i_2 \end{matrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{matrix} i_3 \\ i_4 \end{matrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times K_{k-4}$$

which is columns 2,3,6,7,8,5 of $F_{3,k}$.

Now consider $|\{i_1, i_2, i_3, i_4, i_5, i_6\}| = n$ when the pairs of rows are distinct. Then regardless of n, when $i_1 = i_3$ then we have one column maximal simple construction on three rows (ignoring positions of i_5, i_6)

$$\begin{matrix} i_1 \\ i_2 \\ i_4 \end{matrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = F_{1,3}$$

As a consequence, when $i_1 = i_3$, $i_4 \notin \{i_1, i_2\}$ then the column maximal k-rowed simple matrix is contained in $F_{1,k} = F_{1,3} \times K_{k-3}$ regardless of positioning of i_5, i_6 (which might reduce the number of columns).

When $n = 3$, the only possibility is $i_1 = i_3$, $i_2 = i_5$ and $i_4 = i_6$ and we obtain $F_{1,k}$ as noted above.

When $n = 4$ there are more cases. If $i_1 = i_3$, we appeal to the above observation that the column maximal matrix is contained in $F_{1,k}$. So we may assume $\{i_1, i_2\} \cap \{i_3, i_4\} = \emptyset$ and $i_1 = i_5$ and $i_3 = i_6$ for which the column maximal k-rowed simple matrix is $F_{3,k}$.

When $n = 5$ with the three pairs of rows are distinct, there are three cases. When $i_1 = i_3$ appeal to the above observation to obtain $F_{1,k}$. When $i_1 = i_5$, this yields the column maximal k-rowed simple matrix $F_{4,k}$ (row order i_1, i_2, i_6, i_3, i_4). When $i_3 = i_5$, this yields the column maximal k-rowed simple matrix $F_{5,k} = F^c_{4,k}$ (row order i_3, i_4, i_6, i_1, i_2).

When $n = 6$ then the pairs are disjoint and yields $F_{6,k}$. All cases are now handled. □

Let

$$F_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, F_4 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, F_5 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that $F_3 \prec F_4 \prec F_{1,3}$ and $F_3^c \prec F_5 \prec F_{1,3}$. In [1], $F_{1,3}$ is called F_1.

Theorem 2.2 *[1]* $\text{forb}(m, F_{1,3}) = 2m = \text{forb}(m, F_3) = \text{forb}(m, F_3^c)$.

Proof: The upper bound $\text{forb}(m, F_{1,3}) \leq 2m$ is Theorem 1.1. We can form an m-rowed simple $m \times 2m$ matrix $A = [\mathbf{0} \,|\, I_m \,|\, T_m^c \backslash \mathbf{0}_{m-1}\mathbf{1}_1]$ satisfying $F_3 \nprec A$. □

There is more to be said in consideration of $F_{1,3}$'s role in Theorem 2.2. We explore the 'what is missing' idea ([1]). One way to avoid $F_{1,3}$ is to have for each triple of rows i, j, k (with $i < j < k$)

$$\begin{array}{c} \\ i \\ j \\ k \end{array} \overset{\text{no}}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \quad \text{and} \quad \begin{array}{c} \\ i \\ j \\ k \end{array} \overset{\text{no}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

which yields an $m \times 2m$ simple matrix $[T_m\, T_m^c]$ avoiding $F_{1,3}$. This particular construction has $F_{1,3} \nprec [T\, T^c]$ and it usefully generalizes.

Lemma 2.3 *Let $k \geq 3$. Let $A(k)$ be the m-rowed matrix of all columns such that for each k-tuple $i_1 < i_2 < \cdots < i_k$, the columns satisfy*

$$\begin{array}{c} i_1 \\ i_2 \\ i_3 \\ i_4 \\ \vdots \\ i_k \end{array} \overset{\textit{no}}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ \ \end{bmatrix}} \quad \textit{and} \quad \begin{array}{c} i_1 \\ i_2 \\ i_3 \\ i_4 \\ \vdots \\ i_k \end{array} \overset{\textit{no}}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \\ \ \end{bmatrix}} . \tag{6}$$

Then 'what is missing' in each k-tuple of rows of $A(k)$ is a pair of two complementary columns where for k even, the complementary columns have $k/2$ 1's and for k odd, the columns have $\lceil k/2 \rceil$ 1's and $\lfloor k/2 \rfloor$ 1's. Also $\|A(k)\| = f(m, k)$.

Proof: The observation for 'what is missing' follows from (6). Computing $\|A(k)\|$ is a little more work. In a column, define a *transition* at row i in the adjacent rows $i, i+1$ if

$$\begin{array}{c} i \\ i+1 \end{array} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{array}{c} i \\ i+1 \end{array} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The 'what is missing' conditions (6) force there to be at most $k-2$ transitions in any column. If we choose $\binom{m-1}{k-2}$ rows $j_1 < j_2 < \cdots < j_{k-2}$ to be the rows for transitions then there are exactly two columns α, β on m rows with this pattern. Define α as $\alpha|_{j_1} = 0$, $\alpha|_{j_1+1} = 1$ (a transition at row j_1) and $\alpha|_{j_2} = 1$, $\alpha|_{j_2+1} = 0$ (a transition at row j_2)and $\alpha|_{j_3} = 0$, $\alpha|_{j_3+1} = 1$ (a transition at row j_3) etc. They must alternate else there will be additional transitions. The remaining entries of α are forced if there are no other transitions. Column β has $\beta = \alpha^c$. Thus the number of columns with $k-2$ transitions is $2\binom{m-1}{k-2}$. Similarly the number of columns with (exactly) t transitions is $2\binom{m-1}{t}$ and so the number of columns satisfying (6) is

$$\sum_{i=0}^{k-2} 2\binom{m-1}{i} = f(m,k),$$

establishing $\|A(k)\| = f(m,k)$ using (2). □

As an example, consider $k=3$ which has $\binom{k}{2}=3$ complementary pairs of sum 1 and 2. We note that F_3, F_4 both have the property of having one representative of each complementary pair and so $F_3, F_4 \not\prec A(3)$. Thus $\text{forb}(m,F_3)=\text{forb}(m,F_4)=2m$ by Theorem 1.1 for $k=3$ and the construction we have now provided. Note that $F_3, F_4 \prec F_{1,3}$. Recall that we do allow row and column permutations after $A(k)$ has been formed.

Proof of Theorem 1.3: For k even, there are $\frac{1}{2}\binom{k}{k/2}$ complementary pairs of columns of sum $k/2$. For k odd there are $\binom{k}{\lceil k/2\rceil}$ complementary pairs of a column of sum $\lceil k/2\rceil$ and a column of sum $\lfloor k/2\rfloor$. If a k-rowed F has one representative of each such complementary pair, then $F \not\prec A(k)$. We use this to consider $A(k)$

For $k=4$, there are $\frac{1}{2}\binom{k}{2}=3$ complementary pairs. We consider the 4-rowed matrices $F_{1,4}$ and $F_{3,4}$.

Let

$$F_5 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, F_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Both $F_5, F_6 \prec F_{1,4}, F_{3,4}$ and each have one representative of each complementary pair of sum 2 and so $F_5, F_6 \not\prec A(4)$. Thus $\text{forb}(m,F_5), \text{forb}(m,F_6) \geq f(m,4)$. Given $F_5, F_6 \prec F_{1,4}, F_{3,4}$, we deduce $\text{forb}(m,F_5)=\text{forb}(m,F_6)=f(m,4)=2\binom{m}{2}+2\binom{m}{0}$ by Theorem 1.1. Also for any F with $F_5 \prec F \prec F_{1,4}$ or $F_6 \prec F \prec F_{1,4}$ or $F_5 \prec F \prec F_{3,4}$ or $F_6 \prec F \prec F_{3,4}$, we have $\text{forb}(m,F)=f(m,4)$.

Consider the matrices $F_{3,k}$ for $k=5,6$. On k rows, these matrices have (at least) one representative of each pair of complementary columns of column sum 2,3 for $k=5$ and column sum 3 for $k=6$.

For $k=5$, for each of the ten columns of sum 2, there is a complementary column of sum 3. We need to verify that $F_{3,5}$ has a representative of each such complementary pair. In particular $F_{3,5}$ has six columns of sum 2 and six columns of sum 3 with a representative of each such complementary pair (there are exactly two complementary pairs with both present among those 12 columns). The matrix $A(5)$ from Lemma 2.3 avoids such a complementary pair on each 5-set of rows and $F_{3,5} \not\prec A(5)$. Thus $\text{forb}(m,F_{3,5})=f(m,5)$. But we can consider the minimal ways to have a representative of each such complementary pairs (of columns of sum 2,3) while remaining in $F_{3,5}$, yielding four 5×10 matrices F for which each has $\text{forb}(m,F)=f(m,5)$.

$$\begin{bmatrix} 0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&1&1&1&0&0&0\\ 0&1&1&1&0&0&1&1&0&0\\ 1&0&1&1&1&1&1&1&1&1\\ 1&1&0&1&0&1&0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&1&1&1&0&0&1\\ 0&1&1&1&0&0&1&1&0&1\\ 1&0&1&1&1&1&1&1&1&0\\ 1&1&0&1&0&1&0&0&0&0 \end{bmatrix},$$

$$\begin{bmatrix} 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&1&0&1&1\\ 0&1&1&1&0&0&1&1&1&1\\ 1&0&1&1&1&1&1&1&0&0\\ 1&1&0&1&0&1&0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&1&1&1&0&1&0\\ 0&1&1&1&0&0&1&1&1&0\\ 1&0&1&1&1&1&1&1&0&1\\ 1&1&0&1&0&1&0&0&1&1 \end{bmatrix}.$$

For $k = 6$, there are $\frac{1}{2}\binom{6}{3} = 10$ complementary pairs of columns of sum 3. In $F_{3,6}$, there are exactly twelve columns of sum 3 with a representative of each of the ten complementary pairs of sum 3. If a matrix avoids a complementary pair of columns of sum 3 on each 6-set of rows, then it avoids $F_{3,6}$. Thus $F_{3,6} \not\prec A(6)$ so $\text{forb}(m, F_{3,6}) = f(m, 6)$. But we can consider the minimal ways to have a representative of each of the ten complementary pairs of columns of sum 3 while remaining in $F_{3,6}$, yielding four 6×10 matrices F for which each has $\text{forb}(m, F) = f(m, 6)$.

$$\begin{bmatrix} 0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&1&1&1&0&0&0\\ 0&1&1&1&0&0&1&1&0&0\\ 1&0&1&1&1&1&1&1&1&1\\ 1&1&1&0&1&0&0&0&1&0\\ 1&1&0&1&0&1&0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&1&1&1&0&0&0\\ 0&1&1&1&0&0&1&1&0&1\\ 1&0&1&1&1&1&1&1&1&0\\ 1&1&1&0&1&0&0&0&1&1\\ 1&1&0&1&0&1&0&0&0&0 \end{bmatrix},$$

$$\begin{bmatrix} 0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&1&1&1&0&0&1\\ 0&1&1&1&0&0&1&1&0&1\\ 1&0&1&1&1&1&1&1&1&0\\ 1&1&1&0&1&0&0&0&0&0\\ 1&1&0&1&0&1&0&0&1&1 \end{bmatrix}, \quad \begin{bmatrix} 0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&1&1&1&0&1&1\\ 0&1&1&1&0&0&1&1&1&1\\ 1&0&1&1&1&1&1&1&0&0\\ 1&1&1&0&1&0&0&0&1&0\\ 1&1&0&1&0&1&0&0&0&1 \end{bmatrix}.$$

□

For $k = 8$, there are representatives of only 34 complementary pairs of columns of sum 4 in $F_{3,8}$ when there are 35 complementary pairs in total. Thus $A(8)$ may not avoid $F_{3,8}$ although $A(8)$ is a simple matrix achieving the bound $\|A(8)\| = f(m, 8)$.

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(Received 26 July 2025; revised 25 Mar 2026)