

A note on the restricted size Ramsey number for $2K_2$ and all cycles

TOMASZ DZIDO

*Faculty of Computer Science
Gdynia Maritime University
Morska 81-87, 81-225 Gdynia, Poland
t.dzido@wi.umg.edu.pl*

Abstract

We determine the restricted size Ramsey number $\hat{r}^*(2K_2, C_n)$ for all cycles C_n . In particular, we prove that

$$\hat{r}^*(2K_2, C_n) = \left\lfloor \frac{3n+4}{2} \right\rfloor \quad \text{for } n \geq 7,$$

filling a gap in the literature where no such values were previously known beyond $n = 5$. Our approach is constructive and provides explicit graphs achieving the bound.

1 Introduction

Given graphs G and H , the classical Ramsey number $r(G, H)$ is the smallest integer r such that every red–blue coloring of the edges of the complete graph K_r contains either a red copy of G or a blue copy of H .

While Ramsey numbers concern the order of complete graphs, size Ramsey-type problems focus on minimizing the number of edges required to enforce this property. In particular, the *restricted size Ramsey number* $\hat{r}^*(G, H)$ asks for the smallest size of a graph of order $r(G, H)$ that still guarantees a monochromatic copy of G or H under any red–blue edge coloring.

Initial results on size Ramsey numbers were obtained by Harary and Miller [4], and later extended by Faudree and Sheehan [3], who determined all values for graph pairs of order at most four. The restricted variant was introduced subsequently; for instance, Lortz and Mengersen [5] studied restricted size Ramsey numbers for forest pairs of order up to five.

Despite this progress, the values of $\hat{r}^*(2K_2, C_n)$ for cycles C_n of length greater than five have remained open. In this paper, we completely resolve this problem for all cycles by providing an explicit construction.

2 Preliminaries

All graphs considered are finite and simple. For a graph $G = (V(G), E(G))$, we denote by $v(G) = |V(G)|$ and $e(G) = |E(G)|$ its order and size, respectively. For a vertex $v \in V(G)$, its *degree* is denoted by $\deg_G(v)$, and the *minimum degree* of the graph G is denoted by $\delta(G)$. A vertex of degree zero is called an *isolated vertex*. A graph is *connected* if there is a path between any pair of vertices. The complete graph on m vertices is denoted by K_m , and C_m denotes the cycle of length m . The graph $2K_2$ consists of two independent edges. Deleting all edges of a subgraph H from a graph G is denoted by $G - H$, and deleting a vertex v from G is denoted by $G - v$.

For graphs G and H , a graph F *arrows* (G, H) , written $F \rightarrow (G, H)$, if every red–blue coloring of the edges of F contains either a red copy of G or a blue copy of H . The *Ramsey number* $r(G, H)$ is the smallest integer n such that $K_n \rightarrow (G, H)$.

The *restricted size Ramsey number* $\hat{r}^*(G, H)$ is the minimum number of edges in a graph F of order $r(G, H)$ satisfying $F \rightarrow (G, H)$. Clearly,

$$\hat{r}^*(G, H) \leq \binom{r(G, H)}{2}.$$

Chvátal and Harary [1] found the Ramsey number for $2K_2$ and any graph H with no isolated vertices. This result gives the order of F that arrows $(2K_2, H)$ to find $\hat{r}^*(2K_2, H)$.

Theorem 2.1 ([1]). *For any graph H with no isolated vertices,*

$$r(2K_2, H) = \begin{cases} v(H) + 2, & \text{if } H \text{ is complete,} \\ v(H) + 1, & \text{otherwise.} \end{cases}$$

Silaban *et al.* [8, 9] investigated restricted size Ramsey numbers for pairs consisting of a matching on two edges, $2K_2$, and several classes of graphs, including complete graphs and the cycle C_5 .

Theorem 2.2 ([9]). *For $n \geq 4$,*

$$\hat{r}^*(2K_2, K_n) = \binom{n + 2}{2}.$$

Theorem 2.3 ([8]).

$$\hat{r}^*(2K_2, C_5) = 11.$$

The same authors also obtained the following important properties.

Lemma 2.4 ([8]). *Let H be a graph. Then $F \rightarrow (2K_2, H)$ holds if and only if both of the following conditions hold:*

1. $H \subseteq F - v$ for every $v \in V(F)$,
2. $H \subseteq F - C_3$ for every $C_3 \subseteq F$, where $F - C_3$ denotes the graph obtained from F by deleting the edges of C_3 .

3 Main results

To begin with, let us prove the following simple fact:

Lemma 3.1.

$$\hat{r}^*(2K_2, C_3) = 10.$$

Proof. Since $r(2K_2, C_3) = 5$, any graph witnessing $\hat{r}^*(2K_2, C_3)$ must have order 5. The complete graph K_5 has 10 edges and clearly satisfies $K_5 \rightarrow (2K_2, C_3)$, which implies $\hat{r}^*(2K_2, C_3) \leq 10$.

For the lower bound, consider graphs of order 5 with 9 edges. Up to isomorphism, the unique such graph is $K_5 - e$. Let T be the triangle induced by the three vertices not incident with the missing edge. Removing the edges of T leaves a graph with no copy of C_3 . Hence, by Lemma 2.4, we conclude that $K_5 - e \not\rightarrow (2K_2, C_3)$. Therefore, $\hat{r}^*(2K_2, C_3) \geq 10$.

Combining both bounds yields $\hat{r}^*(2K_2, C_3) = 10$. □

Now we will prove a theorem that will be crucial in the main proof of the article.

Lemma 3.2. *Let $n \geq 4$ and let H be a connected graph with $v(H) = n$. If $F \rightarrow (2K_2, H)$ and $v(F) = r(2K_2, H)$, then*

$$\delta(F) \geq \delta(H) + 1.$$

Proof. Let H be a connected graph with $v(H) = n \geq 4$ and let F be a graph of order $r(2K_2, H)$ such that $F \rightarrow (2K_2, H)$. If $H \cong K_n$, then Theorem 2.2 implies $\delta(F) \geq \delta(H) + 1$, and the claim follows.

Assume now that $H \not\cong K_n$. By Theorem 2.1, we have $r(2K_2, H) = n + 1$, and hence $v(F) = n + 1$. Suppose, for a contradiction, that $\delta(F) \leq \delta(H)$. Then there exists a vertex $u \in V(F)$ with $\deg_F(u) \leq \delta(H)$. If $\deg_F(u) = 0$, then u is an isolated vertex and $F \rightarrow (2K_2, H)$ is equivalent to $F - u \rightarrow (2K_2, H)$, which is impossible since $v(F - u) = r(2K_2, H) - 1$. Hence, we may assume that $\deg_F(u) \geq 1$. Let v be a neighbor of u . Consider the graph $F - v$. By Lemma 2.4, since $F \rightarrow (2K_2, H)$, the graph $F - v$ must contain a copy of H . However, the degree of u in $F - v$ satisfies

$$\deg_{F-v}(u) \leq \delta(H) - 1.$$

Consequently, $F - v$ cannot contain a copy of H , since u has too few neighbors to realize the minimum degree of H . This contradicts the assumption that $F \rightarrow (2K_2, H)$.

Therefore, no such vertex u exists, and we conclude that $\delta(F) \geq \delta(H) + 1$. □

Lemma 3.3. *For odd $n \geq 7$,*

$$\hat{r}^*(2K_2, C_n) = \frac{3n + 3}{2}.$$

Proof. By Theorem 2.1, $r(2K_2, C_n) = n + 1$.

Case 1: $n = 7$.

Consider the graph F with $V(F) = \{v_0, \dots, v_7\}$, cycle $v_0v_1v_2v_3v_4v_5v_6v_7v_0$ and chords $v_0v_4, v_1v_6, v_2v_5, v_3v_7$. This graph contains no triangles, and after removing any vertex, the remaining graph still contains a cycle of length 7. Thus $F \rightarrow (2K_2, C_7)$ and $\hat{r}^*(2K_2, C_7) \leq e(F) = 12$. For the lower bound, any graph F with $v(F) = 8$ and $e(F) \leq 11$ satisfies

$$\delta(F) < 3 = \delta(C_7) + 1.$$

Therefore, by Lemma 3.2, $F \not\rightarrow (2K_2, C_7)$, and hence $\hat{r}^*(2K_2, C_7) = 12$.

Case 2: $n \geq 9$ is odd.

For odd $n \geq 9$, consider the graph F of order $n + 1$ shown in Figure 1, with vertex set $V(F) = \{v_0, \dots, v_n\}$. The edge set of F consists of the edges of a cycle C_{n+1} and the following additional edges.

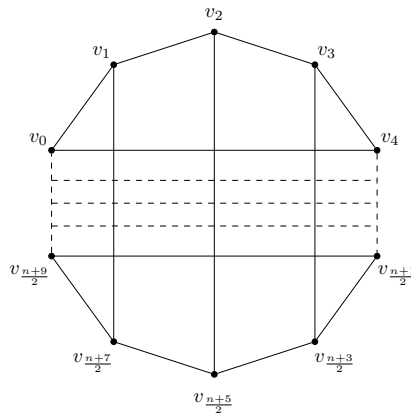


Figure 1: Graph F with $F \rightarrow (2K_2, C_n)$ for odd $n \geq 9$.

- Chords of the first type: $\{v_1, v_{\frac{n+7}{2}}\}, \{v_2, v_{\frac{n+5}{2}}\}, \{v_3, v_{\frac{n+3}{2}}\}$.
- Chords of the second type: $\{v_4, v_0\}, \{v_{\frac{n+1}{2}}, v_{\frac{n+9}{2}}\}$.

Then $|E(F)| = (n + 1) + \frac{n-5}{2} + 3 = \frac{3n+3}{2}$. The graph F contains no C_3 , since each chord connects vertices whose distance along C_{n+1} is at least 3. By Lemma 2.4, it suffices to show that $C_n \subseteq F - v$ for all $v \in V(F)$.

Subcase 2.1: $v = v_1$. A cycle C_n in $F - v_1$ is given by

$$v_0, v_4, \dots, v_{(n+3)/2}, v_3, v_2, v_{(n+5)/2}, \dots, v_n, v_0.$$

Subcase 2.2: $v = v_2$. A cycle C_n in $F - v_2$ is given by

$$v_0, v_1, v_{(n+7)/2}, v_{(n+5)/2}, v_{(n+3)/2}, v_3, v_4, \dots, v_{(n+1)/2}, v_{(n+9)/2}, \dots, v_0.$$

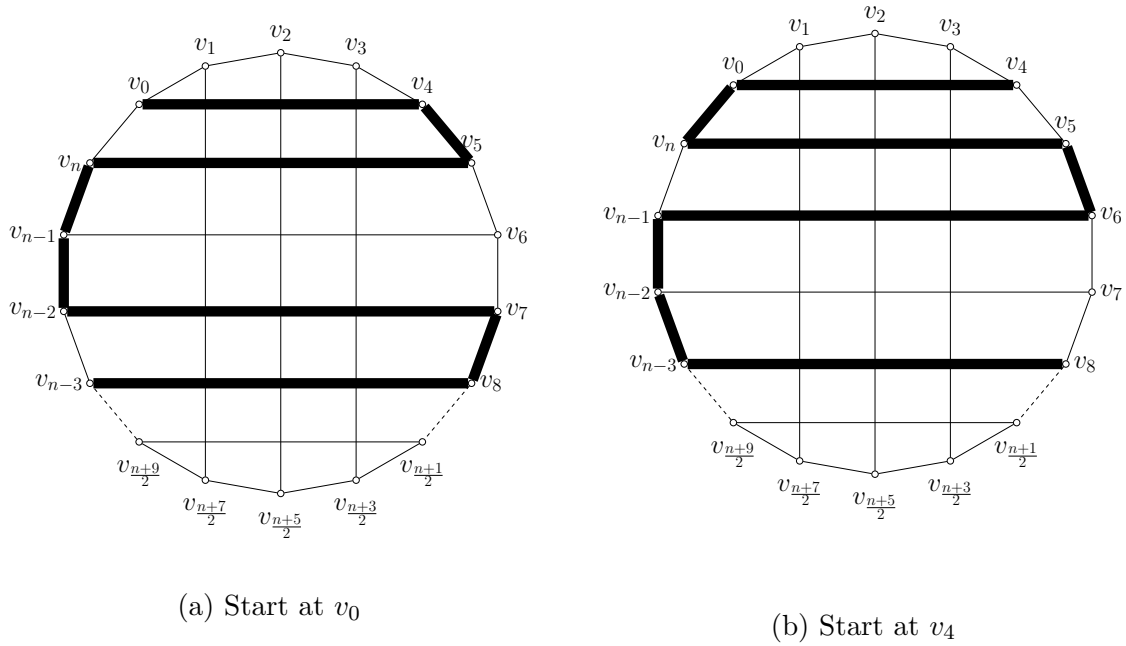


Figure 2: Cycle C_n construction in Subcase 2.3 after removing a vertex v . In the example shown, (a) corresponds to removing v_6 and (b) corresponds to removing v_7 . Thick edges indicate block-by-block traversal to bypass the removed vertex.

Subcase 2.3: Any other $v \in V(F) \setminus \{v_1, v_2, v_3, v_{(n+3)/2}, v_{(n+5)/2}, v_{(n+7)/2}\}$. Depending on the position of the removed vertex v along the cycle C_{n+1} , the construction of a cycle C_n in $F - v$ follows one of two traversal patterns, illustrated in Figure 2. In the first pattern (Figure 2(a)), the traversal starts at v_0 . In the second pattern (Figure 2(b)), the traversal starts at v_4 . In both cases, we traverse the remaining vertices block by block along the original cycle C_{n+1} , bypassing the removed vertex whenever necessary. The construction terminates at either $v_{(n+1)/2}$ or $v_{(n+9)/2}$, depending on the position of v , after which the six special vertices

$$v_1, v_2, v_3, v_{(n+3)/2}, v_{(n+5)/2}, v_{(n+7)/2}$$

close the cycle C_n .

To prove the lower bound, observe that any graph F with $v(F) = n + 1$ and $e(F) \leq \frac{3n+1}{2}$ satisfies

$$\delta(F) < 3 = \delta(C_n) + 1.$$

Therefore, by Lemma 3.2, we conclude that $F \not\rightarrow (2K_2, C_n)$. □

Lemma 3.4. For even $n \geq 14$,

$$\hat{r}^*(2K_2, C_n) = \frac{3n + 4}{2}.$$

Proof. By Theorem 2.1, $r(2K_2, C_n) = n + 1$ for even $n \geq 6$. For even $n \geq 14$, consider the graph F of order $n + 1$ shown in Figure 3, with vertex set $V(F) =$

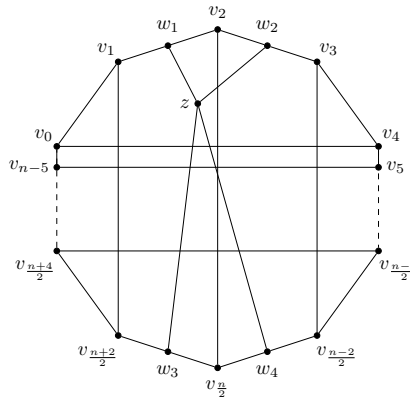


Figure 3: Graph F with $F \rightarrow (2K_2, C_n)$ for even $n \geq 14$.

$\{v_0, v_1, \dots, v_{n-5}\} \cup \{w_1, w_2, w_3, w_4\} \cup \{z\}$. Almost all vertices of the graph F , except z , lie on the following cycle:

$$v_0, v_1, w_1, v_2, w_2, v_3, v_4, \dots, v_{\frac{n-4}{2}}, v_{\frac{n-2}{2}}, w_4, v_{\frac{n}{2}}, w_3, v_{\frac{n+2}{2}}, v_{\frac{n+4}{2}}, \dots, v_{n-5}, v_0.$$

The remaining edges of F are as follows.

- Chords of the first type: $\{v_1, v_{\frac{n+2}{2}}\}, \{v_2, v_{\frac{n}{2}}\}, \{v_3, v_{\frac{n-2}{2}}\}$.
- Chords of the second type: $\{v_4, v_0\}, \{v_5, v_{n-5}\}, \dots, \{v_{\frac{n-4}{2}}, v_{\frac{n+4}{2}}\}$.
- Edges incident with the vertex z : $\{z, w_1\}, \{z, w_2\}, \{z, w_3\}, \{z, w_4\}$.

The number of edges in F is $|E(F)| = n + 3 + \frac{n-10}{2} + 4 = \frac{3n+4}{2}$, and F contains no triangle. By Lemma 2.4, it suffices to show that $C_n \subseteq F - v$ for every $v \in V(F)$. By the symmetries of F , several vertex-deletion configurations are equivalent. Therefore, in the following case analysis, it suffices to consider one representative from each symmetry class of vertices.

Case 1: $v = z$.

After removing the vertex z , the remaining graph contains the cycle C_n described above.

Case 2: $v \in \{v_1, w_1, v_2, w_2, v_3, v_{\frac{n-2}{2}}, w_4, v_{\frac{n}{2}}, w_3, v_{\frac{n+2}{2}}\}$.

Representative vertices are v_1, w_1 , and v_2 .

Subcase 2.1: $v = v_1$.

A cycle C_n in $F - v_1$ is given by

$$v_0, v_{n-5}, \dots, w_4, z, w_1, \dots, v_3, v_{\frac{n-2}{2}}, \dots, v_4, v_0.$$

Subcase 2.2: $v = w_1$.

A cycle C_n in $F - w_1$ is given by

$$v_0, v_1, v_{\frac{n+2}{2}}, w_3, v_{\frac{n}{2}}, v_2, w_2, z, w_4, v_{\frac{n-2}{2}}, v_3, \dots, v_{\frac{n-4}{2}}, v_{\frac{n+4}{2}}, \dots, v_0.$$

Subcase 2.3: $v = v_2$.

A cycle C_n in $F - v_2$ is given by

$$v_0, v_1, w_1, z, w_2, \dots, v_0.$$

Case 3: $v \in V(F) \setminus \{z, v_1, w_1, v_2, w_2, v_3, v_{\frac{n-2}{2}}, w_4, v_{\frac{n}{2}}, w_3, v_{\frac{n+2}{2}}\}$.

As in the case of odd cycles, we start the construction at the vertex v_0 or v_4 , depending on the position of the removed vertex v . We then traverse the remaining vertices block by block along the original cycle C_n , bypassing the removed vertex if necessary. During this traversal, the construction terminates at one of the vertices $v_{\frac{n-4}{2}}$ or $v_{\frac{n+4}{2}}$. The final step is to close the cycle through the special vertices v_0 or v_4 . The traversal may terminate at either $v_{\frac{n-4}{2}}$ or $v_{\frac{n+4}{2}}$. By the symmetry described above, it suffices to present the construction when the cycle is closed at v_0 ; the analogous construction for v_4 follows symmetrically.

Subcase 3.1: Ending at $v_{\frac{n-4}{2}}$.

A spanning path in $F - v$ is given by

$$v_{\frac{n-4}{2}}, v_{\frac{n-2}{2}}, v_3, \dots, w_1, z, w_4, \dots, v_{\frac{n+2}{2}}, v_1, v_0.$$

Subcase 3.2: Ending at $v_{\frac{n+4}{2}}$.

A spanning path in $F - v$ is given by

$$v_{\frac{n+4}{2}}, v_{\frac{n+2}{2}}, w_3, v_{\frac{n}{2}}, v_2, w_2, v_3, v_{\frac{n-2}{2}}, w_4, z, w_1, v_1, v_0.$$

In both subcases, all vertices of $F - v$ are visited exactly once, and the last vertex is adjacent to v_0 . Hence, by adding the edge joining the endpoints, we obtain a cycle of length n in $F - v$. This completes the proof of the upper bound.

To prove the lower bound, let F be a graph with $v(F) = n + 1$ and $e(F) \leq \frac{3n+2}{2}$. Since

$$\frac{2e(F)}{v(F)} \leq \frac{3n + 2}{n + 1} < 3,$$

the average degree of F is smaller than 3, and thus

$$\delta(F) < 3 = \delta(C_n) + 1.$$

Therefore, by Lemma 3.2, we conclude that $F \not\rightarrow (2K_2, C_n)$.

Combining the upper and lower bounds, we conclude that

$$\hat{r}^*(2K_2, C_n) = \frac{3n + 4}{2} \text{ for even } n \geq 14.$$

□

Cycle C_n	$\hat{r}^*(2K_2, C_n)$
C_4	9
C_6	12
C_8	14
C_{10}	17
C_{12}	20

Table 1: Values of $\hat{r}^*(2K_2, C_n)$ for small even cycles obtained computationally, as presented in [2].

The above construction and proof establish the exact values of $\hat{r}^*(2K_2, C_n)$ for even cycles with $n \geq 14$. For smaller even n , the values were obtained computationally using reinforcement learning methods, as presented in [2]. The results are presented in Table 1. Combining the above theorem for $n \geq 14$ with these computational results, we obtain the complete formula for all even cycles with $n \geq 8$:

$$\hat{r}^*(2K_2, C_n) = \frac{3n+4}{2}, \quad n \geq 8.$$

Combining all the results above, we obtain the complete formula for the restricted size Ramsey number $\hat{r}^*(2K_2, C_n)$.

Theorem 3.5. *For all integers $n \geq 3$,*

$$\hat{r}^*(2K_2, C_n) = \begin{cases} 10, & n = 3, \\ 9, & n = 4, \\ 11, & n = 5, \\ 12, & n = 6, \\ \lfloor \frac{3n+4}{2} \rfloor, & n \geq 7. \end{cases}$$

Acknowledgements

The author would like to thank the anonymous reviewers for their careful reading of the manuscript and their valuable comments, which helped to improve the quality of this paper.

References

- [1] V. Chvátal and F. Harary, Generalized Ramsey theory for graphs III: small off-diagonal numbers, *Pacific J. Math.* **41** (1972), 335–345.
- [2] T. Dzido, Computational approaches to restricted and connected size Ramsey numbers for $2K_2$ and cycles, *Procedia Comput. Sci.* **270** (2025), 2799–2807.

- [3] R. J. Faudree and J. Sheehan, Size Ramsey numbers for small-order graphs, *J. Graph Theory* **7** (1983), 53–55.
- [4] F. Harary and Z. Miller, Generalized Ramsey theory VIII, The size Ramsey number of small graphs, *Stud. Pure Math.* (1983), 271–283.
- [5] R. Lortz and I. Mengersen, Size Ramsey results for paths versus stars, *Australas. J. Combin.* **18** (1998), 3–12.
- [6] E. Safitri, P. John and D. R. Silaban, Restricted size Ramsey number for $2K_2$ versus disconnected graphs of order six, *J. Phys. Conf. Ser.* **1722** (2021), 012048.
- [7] D. R. Silaban, E. T. Baskoro and S. Uttunggadewa, Restricted size Ramsey number for $2K_2$ versus dense connected graphs of order six, *J. Phys. Conf. Ser.* **1008** (2018), 012034.
- [8] D. R. Silaban, E. T. Baskoro and S. Uttunggadewa, Restricted size Ramsey number involving matching and graph of order five, *J. Math. Fundam. Sci.* **52** (2020), 133–142.
- [9] D. R. Silaban, E. T. Baskoro and S. Uttunggadewa, On the restricted size Ramsey number involving matchings, unpublished manuscript.

(Received 19 July 2025; revised 13 Jan 2026, 3 May 2026)