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Zagier polynomials for Appell sequences and their symmetry relations

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Abstract

Zagier's modification of Bernoulli numbers is extended to Appell polynomials. These polynomials, called modified Appell polynomials, have differential formulas and shift relations. Moreover, we generalize modified Appell polynomials using formal power series and prove their symmetric relations. We also provide some relations obtained from our main results.

1 Appell sequences

Let $B_n(x)$ be the n-th Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n.$$

When $x=0$, the number $B_n(0)=B_n$ is called the n-th Bernoulli number. Bernoulli polynomials satisfy a symmetry relation $B_n(1-x)=(-1)^nB_n(x)$ for $n\geq 0$. Zagier [8] introduced a modification of Bernoulli numbers

$$B_n^*=\sum_{r=0}^{n}\binom{n+r}{2r}\frac{B_r}{n+r}\quad (n\geq 1)$$

in connection with the theory of modular forms. These numbers have an expression

$$2\sum_{n=1}^{\infty}B_n^*t^n=\sum_{r=1}^{\infty}\frac{B_r}{r}\left(\frac{t}{(1-t)^2}\right)^r-2\log(1-t),$$

and a remarkable periodicity $B_{n+12}^*=B_n^*$ for any odd $n\geq 1$.

As an analogue of these numbers, Dixit, Moll and Vignat [4] defined modified Bernoulli polynomials $B_n^*(x)$ as

$$B_n^*(x)=\sum_{r=0}^{n}\binom{n+r}{2r}\frac{B_r(x)}{n+r}\quad (n\geq 1). \tag{1}$$

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One of their results is the following symmetry relation:

$$B_n^*(-3-x) = (-1)^n B_n^*(x) \quad (n \geq 1), \tag{2}$$

which is reminiscent of the property $B_n(1-x) = (-1)^n B_n(x)$ of the ordinary Bernoulli polynomials.

A sequence $\{A_n(x)\}_{n\geq 0}$ of polynomials is called an *Appell sequence* if it satisfies $\deg A_n(x) = n$ $(n \geq 0)$ and $\frac{d}{dx}A_n(x) = nA_{n-1}(x)$ $(n \geq 1)$. Since $A_0(x)$ is a non-zero constant, we simply denote $A_0(x)$ by A_0. We call each polynomial $A_n(x)$ an *Appell polynomial* for an Appell sequence $\{A_n(x)\}_{n\geq 0}$ (see e.g. [2] for the basic properties of Appell polynomials and [5], [6], [7], [9] and [10] for recent developments on Appell polynomials and their analogues). Typical examples of Appell polynomials are Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$ and modified Hermite polynomials (sometimes called "*probabilist's Hermite polynomials*") $\mathrm{He}_n(x)$. These polynomials $E_n(x)$ and $\mathrm{He}_n(x)$ will be defined in Section 5. It is known that a generating function of an Appell sequence $\{A_n(x)\}_{n\geq 0}$ is expressed in the following form:

$$\sum_{n=0}^{\infty} \frac{A_n(x)}{n!} t^n = F(t)e^{xt}, \tag{3}$$

where $F(t) \in \mathbb{R}[\![t]\!]$ with $F(0) \neq 0$.

For an Appell sequence $\{A_n(x)\}_{n\geq 0}$, we define *modified Appell polynomials* $A_n^*(x)$ as

$$A_n^*(x) = \sum_{r=0}^{n} \binom{n+r}{2r} \frac{A_r(x)}{n+r} \quad (n \geq 1). \tag{4}$$

A generating function of $\{A_n^*(x)\}_{n\geq 1}$ is also expressed as

$$2\sum_{n=1}^{\infty} A_n^*(x)t^n = \sum_{r=1}^{\infty} \frac{A_r(x)}{r} \left(\frac{t}{(1-t)^2}\right)^r - 2A_0 \log(1-t) \tag{5}$$

because the left side is written $\sum_{r=1}^{\infty} \frac{A_r(x)}{r} \sum_{m=0}^{\infty} \binom{m+2r-1}{m} t^{m+r} + 2A_0 \sum_{n=1}^{\infty} \frac{t^n}{n}$.

In the present paper, we investigate polynomials $A_n^*(x)$ and give their differential formulas and symmetry relations. In Section 2, we give a differential formula for $A_n^*(x)$. As an application, we present an algorithm which constructs other Appell sequences from $A_n^*(x)$. In Section 3, we prove shift formulas for $A_n^*(x)$. In Section 4, we introduce a generalization of modified Appell polynomials and give a symmetry relation for them. In Section 5, some identities derived from the results of the previous section are given.

2 Differential formulas

In this section, we give a differential formula for $A_n^*(x)$. The following theorem is an analogue of [4, Theorem 8.2].

Theorem 2.1. *Let* $\{A_n(x)\}_{n\geq 0}$ *be an Appell sequence. Then we have*

$$\begin{aligned}\frac{d}{dx}A_{2n}^*(x) &= \sum_{j=1}^{n}(2j-1)A_{2j-1}^*(x) \quad (n\geq 1),\\ \frac{d}{dx}A_{2n+1}^*(x) &= \frac{1}{2}A_0+\sum_{j=1}^{n}2jA_{2j}^*(x) \quad (n\geq 0).\end{aligned} \tag{6}$$

Proof. Eq. (6) is equivalent to the identity

$$\frac{d}{dx}(A_{n+2}^*(x)-A_n^*(x)) = (n+1)A_{n+1}^*(x) \quad (n\geq 1) \tag{7}$$

with initial conditions

$$\frac{d}{dx}A_1^*(x)=\frac{1}{2}A_0 \ \text{ and } \ \frac{d}{dx}A_2^*(x)=A_1^*(x).$$

Clearly, these initial conditions follow from equations $A_1^*(x) = A_0+\frac{1}{2}A_1(x)$ and $A_2^*(x)=\frac{1}{2}A_0+A_1(x)+\frac{1}{4}A_2(x)$.

By the expression (4) and the identity $\frac{d}{dx}A_r(x)=rA_{r-1}(x)$, for $r\geq 1$ we have

$$\begin{aligned}\frac{d}{dx}\left(A_{n+2}^*(x)-A_n^*(x)\right) &= \frac{d}{dx}\left(\sum_{r=0}^{n+2}\binom{n+2+r}{2r}\frac{A_r(x)}{n+2+r}-\sum_{r=0}^{n}\binom{n+r}{2r}\frac{A_r(x)}{n+r}\right)\\ &= \sum_{r=1}^{n+2}\binom{n+1+r}{2r-1}\frac{A_{r-1}(x)}{2}-\sum_{r=1}^{n}\binom{n+r-1}{2r-1}\frac{A_{r-1}(x)}{2}.\end{aligned}$$

By replacing r with $r+1$ in the sums, this equals

$$\sum_{r=0}^{n+1}\left(\binom{n+r+2}{2r+1}-\binom{n+r}{2r+1}\right)\frac{A_r(x)}{2},$$

where binomial coefficients $\binom{n}{r}$ are interpreted as zero when $r>n$. We can see the identity

$$\binom{n+r+2}{2r+1}-\binom{n+r}{2r+1}=\binom{n+r+1}{2r}\frac{2(n+1)}{n+r+1} \quad (n,r\geq 0).$$

In fact,

$$\begin{aligned}\binom{n{+}r{+}2}{2r{+}1}-\binom{n{+}r}{2r{+}1} &= \binom{n{+}r{+}1}{2r}\frac{n{+}r{+}2}{2r+1}-\binom{n{+}r{+}1}{2r}\frac{(n{-}r{+}1)(n{-}r)}{(2r{+}1)(n{+}r{+}1)}\\ &= \binom{n+r+1}{2r}\frac{4nr+2n+4r+2}{(2r+1)(n+r+1)}\\ &= \binom{n+r+1}{2r}\frac{2(n+1)}{n+r+1}.\end{aligned}$$

Therefore

$$\frac{d}{dx}\left(A_{n+2}^*(x)-A_n^*(x)\right)=(n+1)\sum_{r=0}^{n+1}\binom{n+r+1}{2r}\frac{A_r(x)}{n+r+1}=(n+1)A_{n+1}^*(x)$$

and this completes the proof. □

Remark 2.2. For $n=0$, the polynomial $A_0^*(x)$ is not defined, but we may interpret "$0A_0^*(x)=\frac{1}{2}A_0$". Then the second equation of (6) can be expressed simply as

$$\frac{d}{dx}A_{2n+1}^*(x)=\sum_{j=0}^{n}2jA_{2j}^*(x)\quad (n\geq 0).$$

This interpretation will play a key role in the subsequent sections.

In [4, Theorems 4.2 and 8.1], the n-th Bernoulli polynomial $B_n(x)$ was written explicitly as a linear combination of 1, $B_1^*(x)$, $B_2^*(x),\ldots,B_n^*(x)$. Applying the same method, we can prove the identity

$$\begin{aligned}A_n(x)=2n\sum_{k=1}^{n}(-1)^{n+k}\left[\binom{2n-1}{n-k}-\binom{2n-1}{n-k-1}\right]A_k^*(x)\\ +2A_0(-1)^n\binom{2n-1}{n}\quad (n\geq 1).\end{aligned}\tag{8}$$

In general, we can construct an Appell sequence as a linear combination of modified Appell polynomials. For an Appell sequence $\{A_n(x)\}_{n\geq 0}$ and real numbers $a_{n,m}$ $(n,m\geq 0)$, we define a sequence of polynomials $\{Q_n(x)\}_{n\geq 0}$ as

$$Q_n(x):=n!\left(\frac{a_{n,0}}{2}A_0+\sum_{i=1}^{n}a_{n,i}\,iA_i^*(x)\right)\quad (n\geq 0).\tag{9}$$

We adopt the convention that an empty sum equals zero. Since $Q_0(x)=a_{0,0}A_0/2$ is a constant, we simply denote it by Q_0. Using the interpretation "$0A_0^*=\frac{1}{2}A_0$" described in Remark 2.2, the definition (9) of $Q_n(x)$ can be expressed as

$$Q_n(x)=n!\sum_{i=0}^{n}a_{n,i}\,iA_i^*(x)\quad (n\geq 0).$$

This equation can be written in the following matrix form:

$$\begin{pmatrix}Q_0\\ Q_1(x)\\ \frac{1}{2!}Q_2(x)\\ \frac{1}{3!}Q_3(x)\\ \vdots\end{pmatrix}=\begin{pmatrix}a_{0,0}&0&0&0&0&\cdots\\ a_{1,0}&a_{1,1}&0&0&0&\cdots\\ a_{2,0}&a_{2,1}&a_{2,2}&0&0&\\ a_{3,0}&a_{3,1}&a_{3,2}&a_{3,3}&0&\\ \vdots&\vdots&&&&\end{pmatrix}\begin{pmatrix}\frac{1}{2}A_0\\ A_1^*(x)\\ 2A_2^*(x)\\ 3A_3^*(x)\\ \vdots\end{pmatrix}.$$

Furthermore, we assume that the components of the matrix $(a_{n,m})_{n,m\geq 0}$ satisfy the recurrence relation

$$\begin{aligned} &a_{0,0} \neq 0, \\ &a_{n,m} = 0 \quad (0 \leq n < m), \\ &a_{n,m} = \frac{1}{m}\left(a_{n-1,m-1} - a_{n-1,m+1}\right) \quad (0 < m \leq n). \end{aligned} \tag{10}$$

Note that the matrix $(a_{n,m})_{n,m\geq 0}$ is completely determined by the components $a_{0,0}, a_{1,0}, a_{2,0}, \ldots$ of its first column. Then the following proposition holds.

Proposition 2.3. *Let $\{A_n(x)\}_{n\geq 0}$ be an Appell sequence. If a matrix $(a_{n,m})_{n,m\geq 0}$ satisfies a condition* (10)*, then a sequence $\{Q_n(x)\}_{n\geq 0}$ is also an Appell sequence.*

Proof. When $n = 0$, the value $Q_0 = a_{0,0}A_0/2$ is a non-zero constant. By the differential formula (7) and the assumption (10), we have

$$\begin{aligned} \frac{d}{dx}\frac{Q_n(x)}{n!} &= \frac{d}{dx}\left(a_{n,0}\frac{A_0}{2} + \sum_{i=1}^{n} a_{n,i} i A_i^*(x)\right) \\ &= \frac{d}{dx}\left(a_{n,0}\frac{A_0}{2} + \sum_{i=1}^{n} a_{n-1,i-1}(A_i^*(x) - A_{i-2}^*(x))\right) \\ &= a_{n-1,0}\frac{A_0}{2} + \sum_{i=2}^{n} a_{n-1,i-1}(i-1)A_{i-1}^*(x) \\ &= \frac{Q_{n-1}(x)}{(n-1)!} \end{aligned}$$

for $n \geq 1$. In the summation above, we interpret $A_i^*(x) = 0$ when $i \leq 0$. As a result, the equation $\frac{d}{dx}Q_n(x) = nQ_{n-1}(x)$ holds for $n \geq 1$ and the proposition is proved. □

Remark 2.4. The identity (8), which reconstructs $\{A_n(x)\}$ from $\{A_n^*(x)\}$, is expressed as

$$\begin{pmatrix} A_0 \\ A_1(x) \\ \frac{1}{2!}A_2(x) \\ \frac{1}{3!}A_3(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & \cdots \\ -4 & 2 & 0 & 0 & 0 & \cdots \\ 6 & -4 & 1 & 0 & 0 & \\ -\frac{20}{3} & 5 & -2 & \frac{1}{3} & 0 & \\ \vdots & \vdots & & & & \end{pmatrix} \begin{pmatrix} \frac{1}{2}A_0 \\ A_1^*(x) \\ 2A_2^*(x) \\ 3A_3^*(x) \\ \vdots \end{pmatrix}.$$

We can see that the components of matrix $(a_{n,m})_{n,m\geq 0}$ in this equation satisfy the recurrence relation (10) with initial values $a_{0,0} = 2$ and $a_{n,0} = (-1)^n \frac{4}{n!}\binom{2n-1}{n}$ for $n \geq 1$. More explicitly, since $\binom{2n-1}{n-k} - \binom{2n-1}{n-k-1} = \frac{k}{n}\binom{2n}{n-k}$, by Equation (8) the Equation (9) holds with $Q_n(x) = A_n(x)$ and $a_{n,k} = (-1)^{n+k}\frac{2}{n!}\binom{2n}{n-k}$; the identity $\binom{2n-2}{n-k} - \binom{2n-2}{n-k-2} = \frac{k}{n}\binom{2n}{n-k}$, $0 < k \leq n$, then verifies Equation (10).

In [4, Note 10.3], Dixit, Moll and Vignat proved the identity

$$\sum_{r=0}^{n}(-1)^{n+r}\binom{n+r}{2r}\frac{B_{2r}(x)}{n+r} = 2B_{2n}^*(x-2) \quad (n \geq 1). \tag{11}$$

It is surprising that the left-hand side of Eq. (11) is very similar to the definition (1) of $B_n^*(x)$. The following proposition shows that this relation can be extended to $A_n^*(x)$ as well.

Proposition 2.5. *For an Appell sequence* $\{A_n(x)\}_{n\geq 0}$*, the following identities hold:*

$$\begin{aligned} \sum_{r=0}^{n}(-1)^{n+r}\binom{n+r}{2r}\frac{A_{2r}(x)}{n+r} &= 2A_{2n}^*(x-2) \quad (n\geq 1),\\ \sum_{r=0}^{n}(-1)^{n+r}\binom{n+r+1}{2r+1}\frac{A_{2r+1}(x)}{n+r+1} &= 2A_{2n+1}^*(x-2) \quad (n\geq 0). \end{aligned} \tag{12}$$

Proof. Dixit, Moll and Vignat [4] proved Eq. (11) by an umbral calculus method. They used the umbra $\mathfrak{B}$ which obeys the rule that Bernoulli polynomials $B_n(x)$ are obtained via the evaluation map: eval $\{(\mathfrak{B}(x))^n\} = B_n(x)$. We can obtain the first equation of (12) (even case) by the same method that changes $\mathfrak{B}$ to $\mathfrak{A}$ with eval $\{(\mathfrak{A}(x))^n\} = A_n(x)$.

We prove the second equation of (12) (odd case) with the help of the even case. By Eq. (7), we have

$$2A_{2n+1}^*(x-2) = \frac{2}{2n+1}\cdot\frac{d}{dx}\left(A_{2n+2}^*(x-2) - A_{2n}^*(x-2)\right). \tag{13}$$

By using the even case of (12), we have

$$\begin{aligned} \frac{d}{dx}2A_{2n}^*(x-2) &= \sum_{r=1}^{n}(-1)^{n+r}\binom{n+r}{2r}\frac{2rA_{2r-1}(x)}{n+r}\\ &= \sum_{r=1}^{n}(-1)^{n+r}\binom{n+r-1}{2r-1}A_{2r-1}(x). \end{aligned}$$

Therefore the right-hand side of (13) is equal to

$$\begin{aligned} &\frac{1}{2n+1}\left(\sum_{r=1}^{n+1}(-1)^{n+r+1}\binom{n+r}{2r-1}A_{2r-1}(x) - \sum_{r=1}^{n}(-1)^{n+r}\binom{n+r-1}{2r-1}A_{2r-1}(x)\right)\\ &= \frac{1}{2n+1}\sum_{r=0}^{n}(-1)^{n+r}\left(\binom{n+r+1}{2r+1}+\binom{n+r}{2r+1}\right)A_{2r+1}(x). \end{aligned}$$

By the identity

$$\binom{n+r+1}{2r+1}+\binom{n+r}{2r+1} = \binom{n+r+1}{2r+1}\frac{2n+1}{n+r+1},$$

this equals

$$\sum_{r=0}^{n}(-1)^{n+r}\binom{n+r+1}{2r+1}\frac{A_{2r+1}(x)}{n+r+1}$$

and this proves the odd case of (12). □

3 Shift relations

First, we see a condition for an Appell sequence to have a symmetry relation. Let $\{A_n(x)\}_{n\geq 0}$ be an Appell sequence. By Eq. (3), it is easy to see that the identity $A_n(-x) = (-1)^n A_n(x)$ holds for all $n \geq 0$ if and only if

$$\sum_{n=0}^{\infty} \frac{A_n(x)}{n!} t^n = F(t) e^{xt},$$

where $F(t) \in \mathbb{R}[\![t]\!]$ is even with $F(0) \neq 0$. In the case that $\{A_n(x)\}_{n\geq 0}$ has a symmetry relation $A_n(a - x) = (-1)^n A_n(x)$ for $a \neq 0$, we can obtain the following result.

Proposition 3.1. *Let $\{A_n(x)\}_{n\geq 0}$ be an Appell sequence satisfying $A_n(a - x) = (-1)^n A_n(x)$ $(n \geq 0)$ for a non-zero real number a. Then the following identities hold.*

(a)

$$\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \frac{2e^{xt}}{e^{at} - 1} \sum_{\substack{n=1 \\ n:\ \textit{odd}}}^{\infty} A_n(a) \frac{t^n}{n!}. \tag{14}$$

(b)

$$A_n(x) = \frac{2}{n+1} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2i+1} A_{2i+1}(a) a^{n-2i-1} B_{n-2i}\left(\frac{x}{a}\right) \quad (n \geq 0). \tag{15}$$

Proof. (a) We use a generating function (3) of an Appell sequence $\{A_n(x)\}_{n\geq 0}$. By this expression, we have

$$\sum_{n=0}^{\infty} (A_n(a) - A_n(0)) \frac{t^n}{n!} = F(t)(e^{at} - 1).$$

By the symmetry relation $A_n(a - x) = (-1)^n A_n(x)$, we have

$$A_n(a) - A_n(0) = \begin{cases} 2A_n(a) & (n\text{: odd}), \\ 0 & (n\text{: even}). \end{cases}$$

Therefore

$$F(t) = \frac{1}{e^{at} - 1} \sum_{\substack{n=1 \\ n:\ \text{odd}}}^{\infty} 2A_n(a) \frac{t^n}{n!}$$

and this proves Eq. (14).

(b) By Eq. (14), we have

$$\begin{aligned}\sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!} &= \sum_{\substack{n=1\\ n:\ \text{odd}}}^{\infty} 2A_n(a)\frac{t^n}{n!}\cdot\frac{ate^{xt}}{at(e^{at}-1)}\\ &= \sum_{\substack{n=0\\ n:\ \text{even}}}^{\infty} \frac{2A_{n+1}(a)}{a(n+1)}\frac{t^n}{n!}\sum_{m=0}^{\infty} B_m\left(\frac{x}{a}\right)\frac{(at)^m}{m!}\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2i}\frac{2A_{2i+1}(a)}{2i+1}a^{n-2i-1}B_{n-2i}\left(\frac{x}{a}\right)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\frac{2}{n+1}\sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n+1}{2i+1}A_{2i+1}(a)a^{n-2i-1}B_{n-2i}\left(\frac{x}{a}\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients of t^n on both sides, Eq. (15) is obtained. □

Remark 3.2. Bayad and Komatsu [1] gave a result similar to this proposition in a more general setting.

The *Chebyshev polynomials of the second kind* $U_m(x)$ $(m = 0, 1, 2, \ldots)$ are defined by $U_m(\cos t) = \sin((m+1)t)/\sin t$. It is known that the generating function of these polynomials is

$$\sum_{m=0}^{\infty} U_m(x)t^m = \frac{1}{1-2xt+t^2}.$$

Dixit, Moll and Vignat [4, Lemma 10.2] proved the following relation between modified Bernoulli polynomials and Chebyshev polynomials:

$$B_n^*(x+1) - B_n^*(x) = \frac{1}{2}U_{n-1}\left(\frac{x}{2}+1\right) \quad (n \geq 1). \tag{16}$$

This formula can be considered as an analogue of the shift formula $B_n(x+1)-B_n(x) = nx^{n-1}$ $(n \geq 0)$ for the ordinary Bernoulli polynomials.

For $\alpha \in \mathbb{C}$, the *Gegenbauer polynomials* $C_n^{(\alpha)}(x)$ are defined by the following generating function:

$$\frac{1}{(1-2xt+t^2)^{\alpha}} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x)t^n.$$

When $\alpha = 1$, $C_n^{(1)}(x)$ is the n-th Chebyshev polynomial of the second kind $U_n(x)$. As a generalization of Eq. (16), we can prove the following shift relations.

Proposition 3.3. *Assume that an Appell sequence* $\{A_n(x)\}_{n\geq 0}$ *satisfies a symmetry relation* $A_n(a-x) = (-1)^n A_n(x)$ $(n \geq 0)$ *for a non-zero real number* a. *Then it holds that*

$$A_n^*(x+a) - A_n^*(x) = \sum_{\substack{1\leq i\leq n\\ i:\ odd}} \frac{A_i(a)}{i}C_{n-i}^{(i)}\left(\frac{x}{2}+1\right).$$

Proof. By Proposition 3.1, the generating function of $\{A_n(x)\}_{n\geq 0}$ is of the form of Eq. (14) and

$$\sum_{r=0}^{\infty}\left(A_r(x+a)-A_r(x)\right)\frac{t^r}{r!}=2e^{xt}\sum_{\substack{r=1\\ r:\text{ odd}}}^{\infty}A_r(a)\frac{t^r}{r!}.$$

Hence we have

$$A_r(x+a)-A_r(x)=2\sum_{\substack{1\leq i\leq r\\ i:\text{ odd}}}\binom{r}{i}A_i(a)x^{r-i}.$$

Therefore, by the generating function (5) in Section 1, we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(A_n^*(x+a)-A_n^*(x)\right)t^n &= \frac{1}{2}\sum_{r=1}^{\infty}\frac{A_r(x+a)-A_r(x)}{r}\left(\frac{t}{(1-t)^2}\right)^r\\ &=\sum_{r=1}^{\infty}\frac{1}{r}\sum_{\substack{1\leq i\leq r\\ i:\text{ odd}}}\binom{r}{i}A_i(a)x^{r-i}\left(\frac{t}{(1-t)^2}\right)^r\\ &=\sum_{\substack{i\geq 1\\ i:\text{ odd}}}\frac{A_i(a)}{i}\left(\frac{t}{(1-t)^2}\right)^i\sum_{r=i}^{\infty}\binom{r-1}{i-1}\left(\frac{xt}{(1-t)^2}\right)^{r-i}.\end{aligned}$$

By using the well-known identity

$$\sum_{r=i}^{\infty}\binom{r-1}{i-1}t^{r-i}=\frac{1}{(1-t)^i}\quad (i\geq 1), \tag{17}$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(A_n^*(x+a)-A_n^*(x)\right)t^n &= \sum_{\substack{i\geq 1\\ i:\text{ odd}}}\frac{A_i(a)}{i}\left(\frac{\frac{t}{(1-t)^2}}{1-\frac{xt}{(1-t)^2}}\right)^i\\ &=\sum_{\substack{i\geq 1\\ i:\text{ odd}}}\frac{A_i(a)}{i}\left(\frac{t}{1-(x+2)t+t^2}\right)^i\\ &=\sum_{\substack{i\geq 1\\ i:\text{ odd}}}\frac{A_i(a)}{i}\sum_{m=0}^{\infty}C_m^{(i)}\left(\frac{x}{2}+1\right)t^{m+i}\end{aligned}$$

By comparing the coefficients of t^n on both sides, we get the desired equation. □

4 Symmetry relations for modified Appell polynomials

In this section, we introduce a generalization of modified Appell polynomials and give its symmetry relations. We start with the following lemma.

Lemma 4.1. *Let* $\{A_n(x)\}_{n\geq 0}$ *be an Appell sequence. For* $b \in \mathbb{R}$, *we have*

$$\sum_{r=1}^{\infty} \frac{A_r(x+b)}{r} X^r = \sum_{i=1}^{\infty} \frac{A_i(x)}{i} \left(\frac{X}{1-bX} \right)^i - A_0 \log(1-bX).$$

Proof. Since $A_r(x+b) = \sum_{i=0}^{r} \binom{r}{i} A_i(x) b^{r-i}$, we have

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{A_r(x+b)}{r} X^r &= \sum_{r=1}^{\infty} \frac{1}{r} \sum_{i=0}^{r} \binom{r}{i} A_i(x) b^{r-i} X^r \\
&= \sum_{r=1}^{\infty} \frac{1}{r} \left(\sum_{i=1}^{r} \binom{r}{i} A_i(x) b^{r-i} + A_0 b^r \right) X^r \\
&= \sum_{i=1}^{\infty} \frac{A_i(x)}{i} X^i \sum_{r=i}^{\infty} \binom{r-1}{i-1} (bX)^{r-i} + A_0 \sum_{r=1}^{\infty} \frac{(bX)^r}{r}.
\end{aligned}$$

Using the identity (17), this can be written as

$$\sum_{i=1}^{\infty} \frac{A_i(x)}{i} \left(\frac{X}{1-bX} \right)^i - A_0 \log(1-bX)$$

and the desired identity is obtained. □

Let $Y(u)$ and $Z(u)$ be elements of $\mathbb{R}[\![u]\!]$ with $Y(0) = 0$ and $Z(0) = 1$. For an Appell sequence $\{A_n(x)\}_{n\geq 0}$, we define modified Appell polynomials $A_n^*(x; Y, Z)$, $n \geq 1$, associated with (Y, Z) by the following generating function:

$$2 \sum_{n=1}^{\infty} A_n^*(x; Y, Z) t^n = \sum_{r=1}^{\infty} \frac{A_r(x)}{r} (Y(t))^r - A_0 \log Z(t).$$

When $Y(u) = u/(1-u)^2$ and $Z(u) = (1-u)^2$, the polynomial $A_n^*(x; Y, Z)$ is nothing but the original modified Appell polynomial $A_n^*(x)$. For $\overline{Y}(u) := Y(-u)$ and $\overline{Z}(u) := Z(-u)$, the identity $A_n^*(x; Y, Z) = (-1)^n A_n^*(x; \overline{Y}, \overline{Z})$ holds.

The following lemma, in some sense, guarantees the uniqueness of the polynomial $A_n^*(x; Y, Z)$. This uniqueness result will be used later.

Lemma 4.2. *Let* $Y, \tilde{Y}, Z, \tilde{Z} \in \mathbb{R}[\![u]\!]$ *with* $Y(0) = \tilde{Y}(0) = 0$ *and* $Z(0) = \tilde{Z}(0) = 1$. *If the identity*

$$A_n^*(x; Y, Z) = A_n^*(x; \tilde{Y}, \tilde{Z}) \tag{18}$$

holds for all $n \geq 1$, *then* $Y = \tilde{Y}$ *and* $Z = \tilde{Z}$.

Proof. Set Y, Z, $\tilde{Y}$ and $\tilde{Z}$ as

$$\begin{aligned}
Y(u) &= \sum_{m=1}^{\infty} a_m u^m, \quad & Z(u) &= 1 - \sum_{m=1}^{\infty} b_m u^m, \\
\tilde{Y}(u) &= \sum_{m=1}^{\infty} \tilde{a}_m u^m, \quad & \tilde{Z}(u) &= 1 - \sum_{m=1}^{\infty} \tilde{b}_m u^m
\end{aligned}$$

with all $a_m, \tilde{a}_m, b_m, \tilde{b}_m \in \mathbb{R}$. We prove that $a_m = \tilde{a}_m$ and $b_m = \tilde{b}_m$ $(m \geq 1)$ by induction on m. The generating function of $A_n^*(x; Y, Z)$ can be written as

$$\begin{aligned}
&\sum_{r=1}^{\infty} \frac{A_r(x)}{r} (Y(t))^r - A_0 \log Z(t) \\
&= \sum_{k=1}^{\infty} \left(\sum_{r=1}^{k} \frac{A_r(x)}{r} \sum_{i_1+\cdots+i_r=k} a_{i_1} \cdots a_{i_r} \right) t^k + \sum_{k=1}^{\infty} \left(\sum_{r=1}^{k} \frac{A_0}{r} \sum_{i_1+\cdots+i_r=k} b_{i_1} \cdots b_{i_r} \right) t^k \\
&= \sum_{k=1}^{\infty} t^k \sum_{r=1}^{k} \frac{1}{r} \left(A_r(x) \sum_{i_1+\cdots+i_r=k} a_{i_1} \cdots a_{i_r} + A_0 \sum_{i_1+\cdots+i_r=k} b_{i_1} \cdots b_{i_r} \right).
\end{aligned}$$

Thus, if Eq. (18) holds, then

$$\begin{aligned}
&\sum_{r=1}^{k} \frac{1}{r} \left(A_r(x) \sum_{i_1+\cdots+i_r=k} a_{i_1} \cdots a_{i_r} + A_0 \sum_{i_1+\cdots+i_r=k} b_{i_1} \cdots b_{i_r} \right) \\
&\quad = \sum_{r=1}^{k} \frac{1}{r} \left(A_r(x) \sum_{i_1+\cdots+i_r=k} \tilde{a}_{i_1} \cdots \tilde{a}_{i_r} + A_0 \sum_{i_1+\cdots+i_r=k} \tilde{b}_{i_1} \cdots \tilde{b}_{i_r} \right)
\end{aligned} \tag{19}$$

holds for any $k \geq 1$. When $k = 1$, this equation gives $A_1(x)a_1 + A_0 b_1 = A_1(x)\tilde{a}_1 + A_0 \tilde{b}_1$. Because $A_1(x)$ is a polynomial of degree 1, we have $a_1 = \tilde{a}_1$ and $b_1 = \tilde{b}_1$. Assume that $a_k = \tilde{a}_k$ and $b_k = \tilde{b}_k$ hold for all $1 \leq k \leq m$. Then by Eq. (19) for $k = m+1$ and the inductive assumption, we have $A_1(x)a_{m+1} + A_0 b_{m+1} = A_1(x)\tilde{a}_{m+1} + A_0 \tilde{b}_{m+1}$ By the same reason above, we also obtain that $a_{m+1} = \tilde{a}_{m+1}$ and $b_{m+1} = \tilde{b}_{m+1}$ and this completes the proof. □

Theorem 4.3. *Let* $\{A_n(x)\}_{n \geq 0}$ *be an Appell sequence satisfying* $A_n(a - x) = (-1)^n A_n(x)$ $(n \geq 0)$ *for some* $a \in \mathbb{R}$. *For any* $b \in \mathbb{R}$, *we have*

$$A_n^*(a + b - x; Y, Z) = A_n^* \left(x; \frac{-Y}{1 - bY}, (1 - bY)Z \right) \quad (n \geq 1). \tag{20}$$

Proof. By Lemma 4.1, we have

$$\begin{aligned}
2 \sum_{n=1}^{\infty} A_n^*(a + b - x; Y, Z) t^n &= \sum_{r=1}^{\infty} \frac{A_r(a + b - x)}{r} (Y(t))^r - A_0 \log Z(t) \\
&= \sum_{i=1}^{\infty} \frac{A_i(a - x)}{i} \left(\frac{Y(t)}{1 - bY(t)} \right)^i - A_0 \log (1 - bY(t)) \\
&\quad - A_0 \log Z(t) \\
&= \sum_{i=1}^{\infty} \frac{A_i(x)}{i} \left(\frac{-Y(t)}{1 - bY(t)} \right)^i - A_0 \log ((1 - bY(t))Z(t)) \\
&= 2 \sum_{n=1}^{\infty} A_n^* \left(x; \frac{-Y}{1 - bY}, (1 - bY)Z \right) t^n.
\end{aligned}$$

We obtain Eq. (20) by comparing the coefficients of both sides. □

In special cases, Eq. (20) gives a symmetry relation for $A_n^*(x;Y,Z)$. The following theorem gives a necessary and sufficient condition for a polynomial $A_n^*(x;Y,Z)$ to have such a relation.

Theorem 4.4. *Let $\{A_n(x)\}_{n\geq 0}$ be an Appell sequence satisfying $A_n(a-x) = (-1)^n A_n(x)$ $(n \geq 0)$ for some $a \in \mathbb{R}$.*

- (i) *The symmetry relation $A_n^*(a-x;Y,Z) = (-1)^n A_n^*(x;Y,Z)$ $(n \geq 0)$ holds if and only if Y is odd and Z is even.*
- (ii) *For $b \neq 0$, a symmetry relation $A_n^*(a+b-x;Y,Z) = (-1)^n A_n^*(x;Y,Z)$ $(n \geq 1)$ holds if and only if*

$$Y = \frac{1}{b}\left(1 - \frac{\overline{Z}}{Z}\right). \tag{21}$$

Proof. By Theorem 4.3, we have

$$A_n^*(a+b-x;Y,Z) = A_n^*\left(x; \frac{-Y}{1-bY}, (1-bY)Z\right) \quad (n \geq 1).$$

By the identity $(-1)^n A_n^*(x;Y,Z) = A_n^*(x;\overline{Y},\overline{Z})$ and Lemma 4.2, the equation $A_n^*(a+b-x;Y,Z) = (-1)^n A_n^*(x;Y,Z)$ $(n \geq 1)$ holds if and only if

$$\frac{-Y}{1-bY} = \overline{Y} \quad \text{and} \quad (1-bY)Z = \overline{Z}.$$

- (i) When $b = 0$, this condition can be written as $Y = -\overline{Y}$ and $Z = \overline{Z}$. This means that Y is odd and Z is even.
- (ii) When $b \neq 0$, this condition can be written as $Y = \frac{1}{b}\left(1 - \frac{\overline{Z}}{Z}\right)$.

□

Consider the case $b = -4$, $Y(u) = u/(1-u)^2$ and $Z(u) = (1-u)^2$ in Theorem 4.4. Then the condition (21) is satisfied and the function $A_n^*(x;Y,Z)$ coincides with $A_n^*(x)$. Hence we get the following corollary, which is a generalization of the symmetry relation (2).

Corollary 4.5. *Let $\{A_n(x)\}_{n\geq 0}$ be an Appell sequence satisfying $A_n(a-x) = (-1)^n A_n(x)$ $(n \geq 0)$ for some $a \in \mathbb{R}$. Then*

$$A_n^*(a-4-x) = (-1)^n A_n^*(x) \quad (n \geq 1).$$

5 Examples

In this last section, we present some examples derived from Theorem 4.4 and Corollary 4.5.

(A) When $A_n(x) = B_n(x)$ (then we can take $a = 1$), Corollary 4.5 gives the symmetry relation (2). Also, we obtain similar relations for Euler and Hermite polynomials. Euler polynomials $E_n(x)$ and (modified) Hermite polynomials $\mathrm{He}_n(x)$ are defined by

$$\frac{2e^{xt}}{e^t+1} = \frac{2\tanh(e^{t/2})}{e^t-1}e^{xt} = \sum_{n=0}^{\infty}\frac{E_n(x)}{n!}t^n,$$
$$e^{-t^2/2}e^{xt} = \sum_{n=0}^{\infty}\frac{\mathrm{He}_n(x)}{n!}t^n.$$

Two sequences $\{E_n(x)\}_{n\geq 0}$ and $\{\mathrm{He}_n(x)\}_{n\geq 0}$ are both Appell sequences and satisfy symmetry relations

$$E_n(1-x) = (-1)^n E_n(x), \quad \mathrm{He}_n(-x) = (-1)^n\mathrm{He}_n(x) \quad (n \geq 0).$$

Therefore, by Corollary 4.5, we have

$$E_n^*(-3-x) = (-1)^n E_n^*(x), \quad \mathrm{He}_n^*(-4-x) = (-1)^n\mathrm{He}_n^*(x) \quad (n \geq 1).$$

(B) Let us consider the case $A_n(x) = B_n(x)$, $b = -1$, $Z(u) = e^{-u/2}$ and $Y(u) = \frac{1}{b}\left(1 - \frac{Z(-u)}{Z(u)}\right) = e^u - 1$. Then it follows from Theorem 4.4 that polynomials $B_n^\sharp(x)$ $(n \geq 1)$ defined as

$$\sum_{r=1}^{\infty}\frac{B_r(x)}{r}(e^t-1)^r + \frac{t}{2} = 2\sum_{n=1}^{\infty}B_n^\sharp(x)t^n$$

satisify the relation $B_n^\sharp(-x) = (-1)^n B_n^\sharp(x)$ $(n \geq 1)$. In other words, the identity $M(x,t) = M(-x,-t)$ holds for $M(x,t) := \sum_{r=1}^{\infty}\frac{B_r(x)}{r}(e^t-1)^r + \frac{t}{2}$.

(C) For $\alpha \in \mathbb{C}$, the *Nörlund polynomials* $B_n^{(\alpha)}$ (see [3, (1.2)–(1.3)]) are defined by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} = \sum_{n=0}^{\infty}B_n^{(\alpha)}\frac{t^n}{n!}.$$

Dixit et al. [3] studied a modification $B_n^{(\alpha)*}$ of $B_n^{(\alpha)}$ and its generating function. For a positive integer k, we define $B_n^{(k)}(x)$ as

$$\left(\frac{t}{e^t-1}\right)^{k}e^{tx} = \sum_{n=0}^{\infty}\frac{B_n^{(k)}(x)}{n!}t^n.$$

It is clear that $B_n^{(1)}(x)$ coincides with the ordinary Bernoulli polynomials $B_n(x)$. When $x = 0$, the numbers $B_n^{(k)}(0)$ are special cases of Nörlund polynomials and are sometimes called Bernoulli numbers of order k.

It can be easily seen that $\{B_n^{(k)}(x)\}_{n\geq 0}$ is an Appell sequence and satisfies a relation $B_n^{(k)}(k-x) = (-1)^n B_n^{(k)}(x)$ $(n \geq 0)$. Therefore, by Corollary 4.5, we obtain the symmetry relation

$$B_n^{(k)*}(k-4-x) = (-1)^n B_n^{(k)*}(x) \quad (n \geq 1).$$

(D) The *Lucas sequence of the first kind* $\mathcal{U}_n(p,q)$ are defined by the recurrence equation $\mathcal{U}_{n+2}(p,q) = p\mathcal{U}_{n+1}(p,q) - q\mathcal{U}_n(p,q)$ $(n \geq 0)$ with $\mathcal{U}_0(p,q) = 0$ and $\mathcal{U}_1(p,q) = 1$ where p and q are fixed integers. Note that the numbers $\mathcal{U}_n(1,-1)$ are the classical Fibonacci numbers F_n. It is well-known that the generating function of $\mathcal{U}_n(p,q)$ can be expressed as

$$\sum_{n=0}^{\infty} \mathcal{U}_n(p,q)t^n = \frac{t}{1-pt+qt^2}.$$

Let us consider the case $A_n(x) = B_n(x)$, $b = -2p$, $Z(u) = 1 - pu + qu^2$ and

$$Y(u) = \frac{1}{b}\left(1 - \frac{Z(-u)}{Z(u)}\right) = \frac{u}{1-pu+qu^2} = \sum_{n=0}^{\infty} \mathcal{U}_n(p,q)u^n.$$

For such series Y and Z, define $B_n^{\mathcal{U}}(x) := B_n^*(x; Y, Z)$ $(n \geq 1)$. That is, polynomials $B_n^{\mathcal{U}}(x)$ are defined by the generating function

$$2\sum_{n=1}^{\infty} B_n^{\mathcal{U}}(x)t^n = \sum_{r=1}^{\infty} \frac{B_r(x)}{r}\left(\sum_{m=0}^{\infty} \mathcal{U}_m(p,q)t^m\right)^r - \log(1-pt+qt^2).$$

Then Theorem 4.4 gives a symmetry relation $B_n^{\mathcal{U}}(1-2p-x) = (-1)^n B_n^{\mathcal{U}}(x)$ $(n \geq 1)$.

We note that this identity is a generalization of (2). In fact, when $p = 2$ and $q = 1$, we have $\mathcal{U}_n(2,1) = n$ and the polynomials $B_n^{\mathcal{U}}(x)$ coincide with $B_n^*(x)$.

6 Conclusion

In this paper we have extended Zagier's modification of Bernoulli numbers to general Appell polynomials $A_n(x)$. These modified polynomials, denoted by $A_n^*(x)$, satisfy differential formulas (Theorem 2.1) and shift relations (Proposition 3.3). We have also introduced modified Appell polynomials $\{A_n^*(x; Y, Z)\}$ with formal power series parameters and proved their symmetry relations (Theorem 4.3), together with a necessary and sufficient condition for these relations to take a simple form (Theorem 4.4). Several examples derived from our results were provided in Section 5, and they suggest further developments in the study of Appell polynomials and related areas.

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