AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **95(2)** (2026), Pages 257–282

The Bruhat order on symmetric groups via intrinsic coverings of compositions*

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Abstract

Lehmer's code defines a bijection between the symmetric group and the set of staircase compositions. In this paper, we characterize a poset structure on these compositions that is equivalent to the Bruhat order on the symmetric group. This construction is intrinsic and does not require any reference to the associated permutations.

1 Introduction

This paper aims to characterize the Bruhat order on the symmetric group S_n via the associated Lehmer code. The code defines a bijection between S_n and the set of staircase compositions $\mathcal{C}_n$, which naturally leads to the question: how does the Bruhat order manifest within $\mathcal{C}_n$? In this work, we will define an intrinsic partial order in $\mathcal{C}_n$ and prove its equivalence to the Bruhat order on S_n. Of particular interest, this poset is defined without any reference to the associated permutations.

The construction of the intrinsic partial order in $\mathcal{C}_n$ may be summarized as follows. Any composition $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathcal{C}_n$ determines a unique diagram, from which we construct a matrix $c_{i,j}(\alpha)$ that encodes how the rows of the composition are stacked in the diagram. Given two compositions α and α' such that $|\alpha| = |\alpha'| + 1$,

* This work was supported by the FAPERJ (Carlos Chagas Filho Foundation for Supporting Research in the State of Rio de Janeiro) no. 010.002602/2019, and FAPEMIG (Foundation for Supporting Research in the State of Minas Gerais) RED-00133-21.

ISSN: 2202-3518

we define the covering relation by imposing four conditions among their parts (see Equations (a1), (a2),(a3) and (a4)). The simplest case occurs when the difference between α and α' lies in a unique single part, i.e., $\alpha'_i = \alpha_i - 1$. However, a covering relation may also arise from simultaneous changes in two parts. In such cases, there exist indices $i < j$ such that $\alpha'_i < \alpha_i - 1$, and the excess is transferred to another part, giving $\alpha'_j = \alpha_j + (\alpha_i - \alpha'_i - 1)$. This process is not arbitrary: one must ensure that there is enough space to move and to accommodate the excess in the diagram. Specifically, Equation (a3) ensures that the movement along the diagram is possible, and the condition $c_{i,j}(\alpha) = c_{i,j}(\alpha') = \alpha'_i - \alpha_j$, given by Equation (a4), must be satisfied.

This covering relation defines a poset which is denoted by $(\mathcal{PC}_n, \preceq)$. We show that it is isomorphic to the poset on the symmetric group endowed with the Bruhat order (Theorem 4.3). This result relies on an interplay between the permutation matrices and the Extended Lehmer code, as we show it is the same as the matrix $c_{i,j}$ (Proposition 4.1). We also need to relate the existence of the covering pairs through Coxeter moves over the corresponding reduced decompositions (Proposition 4.2). This is illustrated by the ladder moves over the corresponding diagrams (see Subsection 4.2).

Finally, for a given composition $\alpha \in \mathcal{C}_n$, we present a method to identify the elements covered by α and those by which α is covered. This property is obtained by checking if α is (i, z)-removable or (i, z)-insertable, for any pair (i, z) of positive integers representing either the removed or the inserted box of α (Propositions 5.2 and 5.9). We briefly discuss how it may be used to compute the formula of the Monk's rule (see Subsection 5.3).

We should remark that this problem was motivated by a geometric question concerning the Bruhat decomposition on the maximal flag varieties of $\mathrm{Sl}_n(\mathbb{R})$. The Schubert cells are parametrized by S_n in a such way the computation of the incidence coefficients among them requires data coming from the determination of the covering pairs (for details see Rabelo–San Martin[10] and Matszangosz [9]).

A secondary topic explored in this work is the realization of permutations by diagrams. Our presentation resembles the construction of Young's diagrams in the context of the Grassmannian permutations. There exists a variety of types of diagrams when dealing with permutations (for example, see Manivel [8] for the Rothe Diagrams, Coşkun–Taşkın [4] for the Tower diagrams, and Bergeron-Billey [1] for the RC-graphs). We hope to present here some advantages of our choice, for example, the row-reading map that provides a direct reduced decomposition of permutation in terms of simple reflections.

It is worth noting that our interpretation of the covering relations in terms of the composition's diagrams are similar to those obtained by Coşkun–Taşkın in the context of the Tower diagrams (Proposition 4.5 vs. [4], Theorem 4.1). In addition, Denoncourt [5] also establishes a covering equivalence result in the (left) weak order using the extended Lehmer code (see [5], Proposition 2.8).

This work is arranged as follows. In Section 2, we introduce the main definitions of the combinatorics of the symmetric group. In Section 3, we define the poset for the set of staircase compositions $\mathcal{C}_n$ and derive some intrinsic properties. In Section 4, we prove the equivalence between this poset in $\mathcal{C}_n$ and the Bruhat order of S_n. To conclude, in Section 5, we introduce both the removing and inserting algorithms over the compositions.

2 Preliminaries

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}$ be the set of integers. For $n, m \in \mathbb{Z}$, with $n \leqslant m$, define the set $[n, m] = \{n, n+1, \dots, m\}$. For $n \in \mathbb{N}$, denote $[n] = [1, n]$.

The symmetric group S_n, regarded as a Coxeter group of type A, is generated by simple transpositions s_i for $i \in [n-1]$ subject to the relations:

- $s_i^2 = 1$.
- Commutation: $s_i s_j = s_j s_i$ for $|i - j| \geqslant 2$.
- Braid: $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i \in [n-2]$.

Generically, a move refers to either a commutation or a braid relation. Given any permutation $w \in S_n$, the length $\ell(w)$ of w is the minimal number of simple transpositions needed to decompose w as a product $w = s_{i_1} \dots s_{i_{\ell(w)}}$. Such a decomposition is called reduced. The Word Property guarantees that every two reduced decompositions for w can be connected by a sequence of moves.

There is a partial order in S_n which is called the Bruhat order: $u \leqslant w$ if given a reduced decomposition $w = s_{j_1} \cdots s_{j_r}$ then $u = s_{j_{i_1}} \cdots s_{j_{i_k}}$ for some $1 \leqslant i_1 < \cdots < i_k < r$.

If there exists $w, w' \in S_n$ such that $w' \leqslant w$ and $\ell(w) = \ell(w') + 1$ then w covers w' (alternatively, w, w' is a covering pair). Given a reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$, if w covers w' then $w' = s_{i_1} \cdots \widehat{s_{i_I}} \cdots s_{i_\ell}$, for some $I \in [\ell]$. The integer I depends on both w' and the choice of the reduced decomposition of w.

It is also interesting to denote a permutation $w \in S_n$ in the one-line notation by $w = w(1)w(2)\cdots w(n)$. In this permutation model, the simple reflections may be viewed as transpositions $s_i = (i, i+1)$ in such a way that s_i acts at right by swapping $w(i)$ and $w(i+1)$ (the values at positions i and $i+1$) while s_i acts at left by exchanging the values i and $i+1$. An inversion of w is a pair (i, j) such that $i < j$ and $w(i) > w(j)$. The length $\ell(w)$ is precisely the number of the inversions of w.

The following lemma provides a specific characterization of the covering relation using the one-line notation.

Lemma 2.1 ([2], Lemma 2.1.4)**.** *Let $w, w' \in S_n$. Then, w covers w' in the Bruhat order if and only if $w = w' \cdot (i, j)$ for some transposition (i, j) with $i < j$ such that $w'(i) < w'(j)$ and there does not exist any k such that $i < k < j$, $w'(i) < w'(k) < w'(j)$.*

The lemma says that if $w = w(1)w(2)\cdots w(n)$ then w' is covered by w if and only if w' is obtained from w by switching the values in position i and j, for some pair $i < j$, and such that no value between positions i and j lies in $[w(j), w(i)]$.

A finite integer sequence $\alpha = (\alpha_1, \dots, \alpha_m)$ is a (weak) composition if $\alpha_i \geqslant 0$ for all $i \in [m]$. The elements α_i of the sequence are called the parts, the number $\ell(\alpha)$ of parts is the length, and the sum $|\alpha|$ of the parts is the weight of the composition. A partition is a weakly decreasing composition.

Define the set $\mathcal{C}_n$ as the set of compositions α such that $\alpha_i \leqslant n-i$, i.e., the set of elements inside the cartesian product $[0, n-1] \times [0, n-2] \times \cdots \times [0, 1]$.

The Lehmer code (briefly called code) of a permutation $w \in S_n$ is an integer sequence α with $\alpha_i = \#\{k > i \mid w(k) < w(i)\}$ and it will be denoted by $L(w)$. In other words, each entry of the code corresponds to the number of inversions to the right of $w(i)$. Since $0 \leqslant \alpha_i \leqslant n-i$, we have $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) = L(w) \in \mathcal{C}_n$.

Let us describe the permutation matrix of w. Consider an $n \times n$ array of boxes with rows and columns indexed by integers $[n]$ in matrix style. The *permutation matrix* associated to a permutation $w \in S_n$ is obtained by placing dots in positions $(w(i), i)$, for all $1 \leqslant i \leqslant n$, in the array. The code $\alpha = L(w)$ admits the following interpretation: α_i is the number of dots in the region strictly above and to the right of the dot in the i-th column of the permutation matrix.

This data allows us to represent any permutation $w \in S_n$ as a diagram inside a staircase shape $(n-1) \times (n-1)$ by piling up the parts α_i of the composition α. In this way, the diagram of w is the collection of left-justified boxes where the i-th row counted from bottom to top contains α_i boxes. A box in row x and column y will be referred to as a box located at (x, y).

Remark 1. A related combinatorial object is the RC-graph (or reduced pipe dream) of a permutation. This is a grid diagram consisting of strands and crossings that encodes a reduced word of a permutation, with the property that each pair of strands crosses at most once. In this sense, the diagram of an element w corresponds to the bottom RC-graph of w, where the boxes represent crossings.

Indeed, the Lehmer code provides a bijection between S_n and $\mathcal{C}_n$.

Lemma 2.2 ([8], Proposition 2.1.2; [3], Corollary 5.3)**.** *A permutation is determined by its code and, therefore, by its diagram.*

For instance, consider $w = 5\,7\,6\,2\,1\,8\,3\,4 \in S_8$. The code $L(w) = (4, 5, 4, 1, 0, 2, 0)$ is represented by its diagram in Figure 1 (left). In the sequence, the corresponding permutation matrix of w in Figure 1(right).

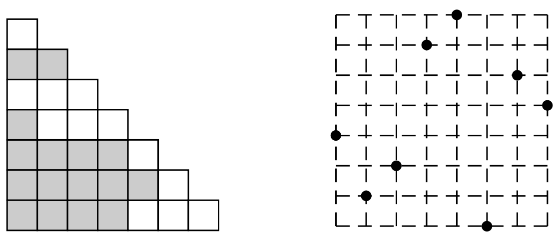


Figure 1: On the left, diagram of $w = 5\,7\,6\,2\,1\,8\,3\,4$. On the right, its permutation matrix.

If one has a composition (the code), there is a nice way to illustrate the procedure described by Manivel ([8], Chapter 2) to determine the permutation in the diagram.

For each row i, beginning from the bottom to the top, write a path along the bottom edges of the boxes starting from the left and finishing with the right side edge at the last box, such that $w(i)$ is the label of the last top arrow of the path assigning the numbers from 1 to n omitting $w(1), \ldots, w(i-1)$. For instance, the Figure 2 shows the first three steps to recover the permutation whose composition is $\alpha = (4, 5, 4, 1, 0, 2, 0)$.

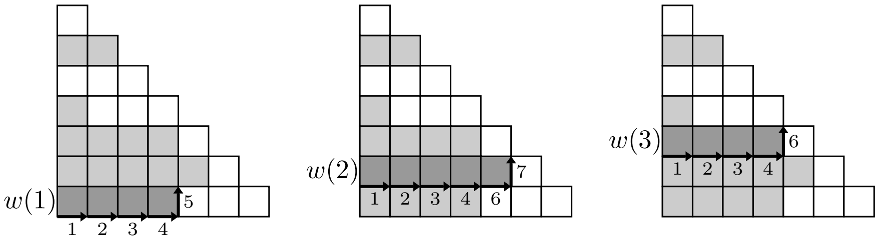


Figure 2: Recovering the permutation from $\alpha = (4, 5, 4, 1, 0, 2, 0)$.

This arrangement of boxes provides an easy way to express the permutation w in terms of simple reflections s_i, which we will call the row-reading expression of w. We begin assigning a simple reflection consecutively to each box from left to right and upwards, starting from s_1 in the bottom leftmost box in the staircase shape. More specifically, to a box located at (x, y), we assign the reflection s_{x+y-1}. Then, we obtain a reduced decomposition by reading each row in the diagram from right to left, and the rows from bottom to top. As a consequence, the reading for the i-th row is given by $\mathbf{w}_i = s_{\alpha_i+i-1} \cdot s_{\alpha_i+i-2} \cdots s_{i+1} \cdot s_i$ if α_i is non-zero, and $\mathbf{w}_i = e$ otherwise. The row-reading expression $\mathbf{w}$ of w is

$$\mathbf{w} = \mathbf{w}_1 \cdots \mathbf{w}_{n-1}.$$

Notice this row-reading resembles that described by Manivel (see [8], Remark 2.1.9). For instance, the row-reading expression of $w = 5\,7\,6\,2\,1\,8\,3\,4$ is $\mathbf{w} = s_4 s_3 s_2 s_1 \cdot s_6 s_5 s_4 s_3 s_2 \cdot s_6 s_5 s_4 s_3 \cdot s_4 \cdot s_7 s_6$ which can be obtained from Figure 3 by reading each row from right to left, beginning from the bottom.

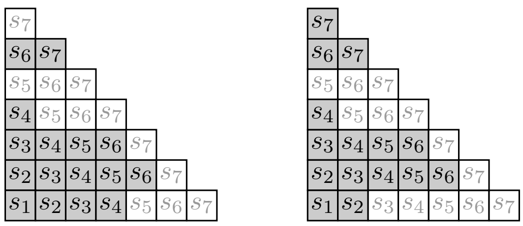


Figure 3: The row-reading of $w = 5\,7\,6\,2\,1\,8\,3\,4$ on the left and of $w' = 3\,7\,6\,2\,1\,8\,5\,4$ on the right.

Recall that if w covers w' such that $w = s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition then there is an integer in $I \in [\ell]$ such that $w' = s_{i_1} \cdots \widehat{s_{i_I}} \cdots s_{i_\ell}$. Let us denote by $\widehat{\mathbf{w}}_I$ this reduced decomposition of w' obtained by removing the I-th simple reflection from the row-reading $\mathbf{w}$.

For instance, by Lemma 2.1, $w = 5\,7\,6\,2\,1\,8\,3\,4$ covers $w' = \mathbf{3}\,7\,6\,2\,1\,8\,\mathbf{5}\,4$. Moreover, $\widehat{\mathbf{w}}_2 = s_4\widehat{s}_3s_2s_1 \cdot s_6s_5s_4s_3s_2 \cdot s_6s_5s_4s_3 \cdot s_4 \cdot s_7s_6$ is a reduced decomposition of w' obtained from $\mathbf{w}$. However, it differs from the row-reading $\mathbf{w}'$ since $L(w') = (2,5,4,1,0,2,1)$ and $\mathbf{w}' = s_2s_1 \cdot s_6s_5s_4s_3s_2 \cdot s_6s_5s_4s_3 \cdot s_4 \cdot s_7s_6 \cdot s_7$ (cf. Figure 3). The Word Property guarantees that there is a sequence of moves that transforms $\widehat{\mathbf{w}}_2$ into $\mathbf{w}'$; in this case, s_4 in the first row turns into s_7 in the seventh row after a sequence of moves.

There is a known characterization of covering relations in terms of the one-line notation of permutations w and w' (see Lemma 2.1). However, our goal is to establish a relationship that also takes into account the reduced decompositions, alongside the permutation structure. Since the Lehmer code of a permutation encodes both the length and information about its reduced decompositions, we aim to characterize covering relations directly in terms of these codes.

As a motivation, consider the following situation. Let $w = 5\,7\,6\,2\,1\,8\,3\,4 \in S_8$ with $L(w) = (4,5,4,1,0,2,0)$. As a direct application of Lemma 2.1, it is immediate that w covers both $w' = 3\,7\,6\,2\,1\,8\,5\,4$ and $w'' = 5\,7\,4\,2\,1\,8\,3\,6$, where $w = w' \cdot (1,7)$ and $w = w'' \cdot (3,8)$. Notice also that $\alpha' = L(w') = (2,5,4,1,0,2,1)$ and $\alpha'' = L(w'') = (4,5,3,1,0,2,0)$. For the pair w, w'', the only distinction between their codes is $\alpha''_3 = \alpha_3 - 1$, while for the pair w, w', the distinction is $\alpha'_1 = \alpha_1 - 2$ and $\alpha''_7 = \alpha_7 + 1$. Hence, for the pair w, w'', their codes differ only at one position with $\alpha''_i = \alpha_i - 1$. However, for the pair w, w', their codes differ at two positions and it is not clear a priori how the values in those positions are related. It reveals that the Bruhat order given in terms of the Lehmer code sometimes is manifested by changes in two positions of the code.

What happens for the pair w, w'' is justified by the following lemma.

Lemma 2.3. *Let $w, w' \in S_n$ and denote by $\alpha = L(w)$ and $\alpha' = L(w')$. If there exists i such that $\alpha'_i = \alpha_i - 1$ and $\alpha'_k = \alpha_k$ for every $k \neq i$ then w covers w'.*

Proof. Let us show that for some $I \in [\ell(w)]$ the reduced decompositions $\widehat{\mathbf{w}}_I$ and $\mathbf{w}'$ of w' are equal. We have that $\mathbf{w}_i = s_{\alpha_i+i-1} \cdot \mathbf{w}'_i$, and $\mathbf{w}_k = \mathbf{w}'_k$ for every $k \neq i$. Then, for $I = \left(\sum_{k=1}^{i-1} \alpha_k\right) + 1$, we have $\mathbf{w}' = \mathbf{w}'_1 \cdots \mathbf{w}'_{i-1} \cdot \widehat{s}_{\alpha_i+i-1} \cdot \mathbf{w}'_i \cdots \mathbf{w}'_{n-1} = \widehat{\mathbf{w}}_I$. ☐

As it was already noticed, the pair w, w' in the above example shows that the converse of Lemma 2.3 is not true. In the next section we will introduce a poset in the set $\mathcal{C}_n$ that will solve this question. In Section 4 we will show that such poset is isomorphic to the Bruhat order in S_n.

3 An intrinsic covering relation for compositions

In this section, we describe a covering relation for compositions that provides a poset structure for $\mathcal{C}_n$. We highlight that all results obtained in this section do not require any mention of the permutation, using only information available from compositions.

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a composition in $\mathbb{N}^m$. Consider $N(\alpha)$ the integer given by

$$N(\alpha) = \max_{\alpha_i \neq 0}\{\alpha_i + i\}.$$

Notice that α lies in $\mathcal{C}_n$ only for $n \geqslant N(\alpha)$. In fact, for every i, $\alpha_i \leqslant N(\alpha) - i$ and there exists I such that $\alpha_I = N(\alpha) - I$. In particular, $\alpha_{N(\alpha)} = 0$.

Given $i \in [m]$ and $j \in \mathbb{N}$, define $c_{i,j}(\alpha)$ to be the integer defined recursively with respect to j as follows:

- If $j \leqslant i + 1$, then $c_{i,j}(\alpha) = 0$.
- If $j > i + 1$, then $c_{i,j}(\alpha) = c_{i,j-1}(\alpha) + \begin{cases} 1, & \text{if } \alpha_{j-1} < \alpha_i - c_{i,j-1}(\alpha); \\ 0, & \text{if } \alpha_{j-1} \geqslant \alpha_i - c_{i,j-1}(\alpha). \end{cases}$

This defines a matrix $c(\alpha)$ with infinite entries. The quantity $c_{i,j}(\alpha)$ can be interpreted as the horizontal displacement of a polygonal path constructed on the diagram of α. For a fixed row i, the path is generated by the following procedure:

Start Begin at the last box in row i.

Recursive Step To move from row k to row $k + 1$ (where $k \geqslant i$), compare the current horizontal position of the path with the length of the next row, α_k.

- **Vertical Move (North):** If the box directly above is filled (i.e., $\alpha_k \geqslant \alpha_i - c_{i,k}(\alpha)$) or if the path has reached the left edge (outside the diagram), move vertically to the next box directly above.
- **Diagonal Move (North-West):** If the box directly above is empty (i.e., $\alpha_k < \alpha_i - c_{i,k}(\alpha)$), move diagonally up and to the left.

End Repeat this process for each row until the top of the diagram is reached.

Under this construction, $c_{i,j}(\alpha)$ counts exactly the total number of diagonal steps (shifts to the left) taken by the path from row i up to row $j - 1$.

Example 1. Consider the composition $\alpha = (4, 5, 4, 1, 0, 2, 0)$. The corresponding matrix $c(\alpha)$ is given by:

$$c(\alpha) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 2 & 3 & 4 & 4 & \cdots \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 4 & 5 & 5 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 3 & 4 & 4 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}$$

To illustrate the computation, let us detail the path associated with the first row, i.e., the values $c_{1,j}(\alpha)$ for $j \geqslant 1$. The path starts at the last box of the first row (column 4). Following the recursive definition:

- For $j \leqslant 2$: the path begins at row $i = 1$. Since it has not yet moved to any row above $i + 1 = 2$, the displacement is zero. Thus, $c_{1,1} = c_{1,2} = 0$.
- For $j = 3, 4$: the boxes directly above in rows 2 and 3 are filled ($\alpha_2, \alpha_3 \geqslant 4$). Thus, the path moves vertically and $c_{1,3} = c_{1,4} = 0$.

- At $j = 5$: the box directly above in row 4 is empty ($\alpha_4 = 1 < 4 - 0$). The path must take a diagonal step to the left. Hence, $c_{1,5}(\alpha) = c_{1,4}(\alpha) + 1 = 1$.
- At $j = 6$: the box directly above in row 5 is empty ($\alpha_5 = 0 < 4 - 1$). The path takes another diagonal step. Hence, $c_{1,6}(\alpha) = 1 + 1 = 2$.
- At $j = 7$: the box directly above in row 6 is filled ($\alpha_6 = 2 \geqslant 4 - 2$). The path moves vertically. Hence, $c_{1,7}(\alpha) = 2$.
- At $j = 8$: the box directly above in row 7 is empty ($\alpha_7 = 0 < 4 - 2$). The path takes a diagonal step. Hence, $c_{1,8}(\alpha) = 2 + 1 = 3$.
- At $j = 9$: the box directly above in row 8 is empty ($\alpha_8 = 0 < 4 - 3$). The path takes the final diagonal step. Hence, $c_{1,9}(\alpha) = 3 + 1 = 4$.
- For $j \geqslant 10$: the path has reached the left edge of the diagram, since $\alpha_1 - c_{1,9}(\alpha) = 4 - 4 = 0$. From this point on, it moves vertically (treating the "column zero" as filled), and the value stabilizes at 4.

Figure 4 visualizes the polygonal paths for selected entries of $c(\alpha)$. Specifically, notice how the path for $c_{1,j}$ deviates to the left exactly when it encounters the shorter rows.

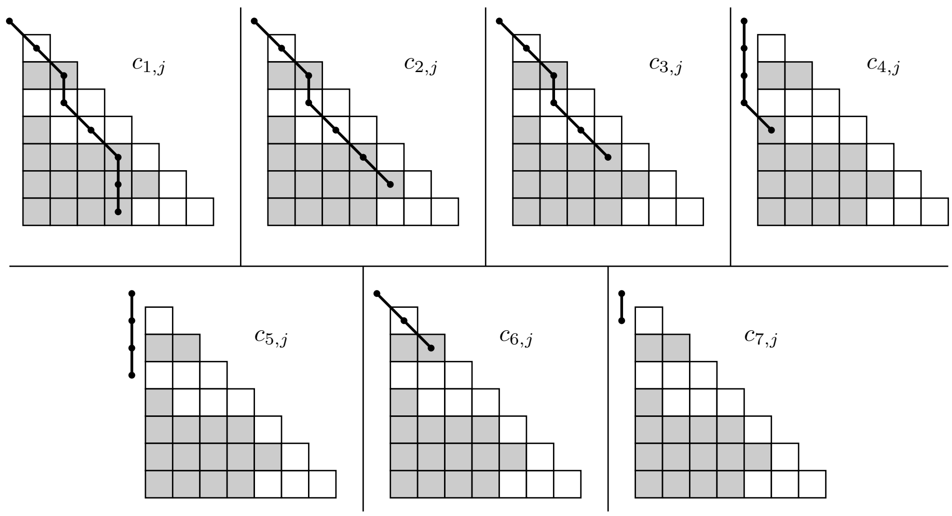


Figure 4: Geometric interpretation of the values $c_{i,j}(\alpha)$ as paths in the diagram of $\alpha = (4, 5, 4, 1, 0, 2, 0)$.

The following lemma determines some bounds for the entries of $c(\alpha)$.

Lemma 3.1. *Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a composition. Then:*

(i) $0 \leqslant c_{i,j}(\alpha) \leqslant c_{i,j+1}(\alpha) \leqslant \alpha_i$ *for every i and j.*

(ii) If $j \leqslant N(\alpha)$ then $\alpha_i + j - N(\alpha) - 1 \leqslant c_{i,j}(\alpha) \leqslant j - i - 1$.

(iii) $N(\alpha) + 1$ *is the smallest integer j such that $c_{i,j}(\alpha) = \alpha_i$ for every i.*

Proof. (i): It follows directly from the definition, which implies $0 \leqslant c_{i,j}(\alpha) \leqslant c_{i,j+1}(\alpha)$. If $c_{i,j}(\alpha) = \alpha_i$, then $\alpha_i - c_{i,j}(\alpha) = 0 \leqslant \alpha_j$. Hence, by definition, $c_{i,j+1}(\alpha) = c_{i,j}(\alpha) + 0 = \alpha_i$. This proves that $c_{i,j}(\alpha) \leqslant \alpha_i$.

(ii): The upper bound $c_{i,j}(\alpha) \leqslant j - i - 1$ is immediate from the definition. For the lower bound, we proceed by induction on j, assuming $j \leqslant N(\alpha)$. If $j = i + 1$, then $\alpha_i + i - N(\alpha) \leqslant 0 = c_{i,i+1}(\alpha)$, since $N(\alpha) \geqslant \alpha_i + i$. Assume that the inequality holds for j. If $\alpha_j < \alpha_i - c_{i,j}(\alpha)$, then by the definition of $c_{i,j+1}$ and the inductive hypothesis, we have:

$$\alpha_i + (j + 1) - N(\alpha) - 1 \leqslant c_{i,j}(\alpha) + 1 = c_{i,j+1}(\alpha).$$

On the other hand, if $\alpha_j \geqslant \alpha_i - c_{i,j}(\alpha)$, then using the fact that $\alpha_j + j \leqslant N(\alpha)$, we obtain:

$$\alpha_i + (j + 1) - N(\alpha) - 1 = \alpha_i - \alpha_j + (\alpha_j + j - N(\alpha)) \leqslant \alpha_i - \alpha_j \leqslant c_{i,j}(\alpha) = c_{i,j+1}(\alpha).$$

(iii): It follows from (ii) that $c_{i,N(\alpha)}(\alpha) \geqslant \alpha_i - 1$ for every i. Let us verify that $c_{i,N(\alpha)+1}(\alpha) = \alpha_i$ for every i. If $c_{i,N(\alpha)}(\alpha) = \alpha_i$, we are done. If $c_{i,N(\alpha)}(\alpha) = \alpha_i - 1$, then $\alpha_{N(\alpha)} = 0 < \alpha_i - c_{i,N(\alpha)}(\alpha)$. Thus, by definition, $c_{i,N(\alpha)+1}(\alpha) = c_{i,N(\alpha)}(\alpha) + 1 = \alpha_i$.

Finally, let I be an index such that $\alpha_I \neq 0$ and $\alpha_I + I = N(\alpha)$ (such an index exists by the definition of $N(\alpha)$). Note that $I < N(\alpha)$. Then, by the upper bound in (ii), $c_{I,N(\alpha)}(\alpha) \leqslant N(\alpha) - I - 1 = \alpha_I - 1$. Combined with the lower bound, we have $c_{I,N(\alpha)}(\alpha) = \alpha_I - 1 \neq \alpha_I$. This proves that $j = N(\alpha) + 1$ is minimal such that $c_{i,j}(\alpha) = \alpha_i$ for all i. □

Lemma 3.1 allows us to describe $c(\alpha)$ as an $(n - 1) \times (n + 1)$ matrix for every $\alpha \in \mathcal{C}_n$, $n \leqslant N(\alpha)$, where we may add a few repeated columns to the right or zero rows below it.

The next lemma establishes some conditions for comparing $c_{i,j}$ for some range of j.

Lemma 3.2. *Suppose that α and $\widetilde{\alpha}$ are compositions satisfying that $\widetilde{\alpha}_i \leqslant \alpha_i$ and there exists $l > i$ such that $\widetilde{\alpha}_k \geqslant \alpha_k$ for every $k \in [i + 1, l]$. Then, $c_{i,k}(\widetilde{\alpha}) \leqslant c_{i,k}(\alpha)$, for any $k \in [i + 1, l + 1]$.*

Proof. We will prove inductively for $k \in [i + 1, l + 1]$. If $k = i + 1$ then it is trivially true by definition. Assume that the claim is true for $k \in [i + 2, l]$. We can consider the following two different cases: either $c_{i,k}(\alpha) - c_{i,k}(\widetilde{\alpha}) > \alpha_i - \widetilde{\alpha}_i$ or $c_{i,k}(\alpha) - c_{i,k}(\widetilde{\alpha}) \leqslant \alpha_i - \widetilde{\alpha}_i$.

Suppose that $c_{i,k}(\alpha) - c_{i,k}(\widetilde{\alpha}) > \alpha_i - \widetilde{\alpha}_i$. Then, $c_{i,k}(\widetilde{\alpha}) < c_{i,k}(\alpha)$ since $\widetilde{\alpha}_i \leqslant \alpha_i$. Hence, $c_{i,k+1}(\widetilde{\alpha}) \leqslant c_{i,k+1}(\alpha)$.

Suppose that $c_{i,k}(\alpha) - c_{i,k}(\widetilde{\alpha}) \leqslant \alpha_i - \widetilde{\alpha}_i$.

If $c_{i,k+1}(\alpha) = c_{i,k}(\alpha) + 1$ then $c_{i,k+1}(\alpha) = c_{i,k}(\alpha) + 1 \geqslant c_{i,k}(\widetilde{\alpha}) + 1 \geqslant c_{i,k+1}(\widetilde{\alpha})$.

If $c_{i,k+1}(\alpha) = c_{i,k}(\alpha)$ then $\widetilde{\alpha}_k \geqslant \alpha_k \geqslant \alpha_i - c_{i,k}(\alpha) \geqslant \widetilde{\alpha}_i - c_{i,k}(\widetilde{\alpha})$. Thus, $c_{i,k+1}(\widetilde{\alpha}) = c_{i,k}(\widetilde{\alpha}) \leqslant c_{i,k}(\alpha) = c_{i,k+1}(\alpha)$. □

Given two compositions α and α' such that $|\alpha| = |\alpha'| + 1$, we say that α covers α' if there exist positive integers $i < j$ satisfying the following four conditions:

$$\alpha'_i \leqslant \alpha_i - 1; \tag{a1}$$

$$\alpha'_j = \alpha_j + \alpha_i - \alpha'_i - 1; \tag{a2}$$

$$\alpha'_k = \alpha_k \text{ for every } k \neq i \text{ and } k \neq j; \tag{a3}$$

$$c_{i,j}(\alpha) = c_{i,j}(\alpha') = \alpha'_i - \alpha_j. \tag{a4}$$

We can emphasize the pair (i, j) saying that α covers α' in positions (i, j).

Example 2. Suppose that $\alpha = (4, 5, 4, 1, 0, 2, 0)$ and $\alpha' = (2, 5, 4, 1, 0, 2, 1)$. Then, we can see that $i = 1$ and $j = 7$ satisfy conditions (a1), (a2), and (a3). We also have that $c_{1,j}(\alpha') = (0, 0, 0, 0, 1, 2, 2, 2, 2)$. Then, $c_{1,7}(\alpha') = c_{1,7}(\alpha) = 2 = \alpha'_1 - \alpha_7$. We conclude that α covers α' in positions $(1, 7)$.

Example 3. We may consider, analogously to Lemma 2.3, compositions α and α' such that $\alpha'_i = \alpha_i - 1$. In this case, Equations (a1), (a2), and (a3) are trivially satisfied. However, there is no direct way to determine $j > i$ such that α covers α' in positions (i, j). For instance, consider the pairs $\alpha = (4, 5, 4, 1, 0, 2, 0)$, $\alpha' = (4, 5, 3, 1, 0, 2, 0)$ and $\beta = (4, 5, 4, 1, 2, 0, 0)$, $\beta' = (4, 5, 3, 1, 2, 0, 0)$. Then α covers α' in positions $(3, 8)$ while β covers β' in positions $(3, 5)$.

Notice that the existence of covering pairs provides a partial order relation for $\mathcal{C}_n$ by transitivity (for details, see Stanley [11], Chapter 3). Denote by $\mathcal{PC}_n$ the poset of $\mathcal{C}_n$ given by the covering relations defined above. We will denote this order relation by $\preceq$ to distinguish it from $\leqslant$ of the product order inherited from $\mathbb{N}^m$.

If α covers α' in positions (i, j), we observe the pair α and α' satisfy the conditions of Lemma 3.2 from which follows that $c_{i,k}(\alpha') \leqslant c_{i,k}(\alpha)$ for any $k > i$. However, conditions (a1) to (a4) provide more information to compare both $c_{i,k}(\alpha')$ and $c_{i,k}(\alpha)$, as we describe in the next proposition.

Proposition 3.3. *Suppose that α covers α' in position (i, j). Then:*

(i) $c_{i,k}(\alpha) = c_{i,k}(\alpha')$ *for every* $k \in [i + 1, j]$.
(ii) $c_{i,j+1}(\alpha) = 1 + c_{i,j}(\alpha)$ *and* $c_{i,j+1}(\alpha') = c_{i,j}(\alpha')$.
(iii) $c_{i,k}(\alpha) > c_{i,k}(\alpha')$, *for every* $k > j$.
(iv) Either $\alpha_k + c_{i,k}(\alpha) \geqslant \alpha_i$ *or* $\alpha_k + c_{i,k}(\alpha) < \alpha'_i$, *for every* $k \in [i + 1, j - 1]$.

Proof. (i): Fix i. Let us show inductively that for any $k \geqslant i + 1$

$$0 \leqslant c_{i,k}(\alpha) - c_{i,k}(\alpha') \leqslant c_{i,k+1}(\alpha) - c_{i,k+1}(\alpha') \leqslant \alpha_i - \alpha'_i. \tag{1}$$

If $k = i + 1$ then, by definition, $0 = c_{i,i+1}(\alpha) - c_{i,i+1}(\alpha') \leqslant c_{i,i+2}(\alpha) - c_{i,i+2}(\alpha') \leqslant 1 \leqslant \alpha_i - \alpha'_i$, where the first inequality follows from the fact that if $c_{i,i+2}(\alpha') = 1$ then $c_{i,i+2}(\alpha) = 1$.

Assume that $k > i + 1$ and the claim is true for k. Suppose that $c_{i,k+2}(\alpha) = c_{i,k+1}(\alpha)$. Notice that (a1), (a2), and (a3) imply that $\alpha'_{k+1} \geqslant \alpha_{k+1}$. It follows by induction that $\alpha'_{k+1} \geqslant \alpha_{k+1} \geqslant \alpha_i - c_{i,k+1}(\alpha) \geqslant \alpha'_i - c_{i,k+1}(\alpha')$. Thus, $c_{i,k+2}(\alpha') = c_{i,k+1}(\alpha')$ and $c_{i,k+1}(\alpha) - c_{i,k+1}(\alpha') = c_{i,k+2}(\alpha) - c_{i,k+2}(\alpha')$.

Now, suppose that $c_{i,k+2}(\alpha) = c_{i,k+1}(\alpha) + 1$. Since either $c_{i,k+2}(\alpha') = c_{i,k+1}(\alpha')$ or $c_{i,k+2}(\alpha') = c_{i,k+1}(\alpha') + 1$, we have $c_{i,k+1}(\alpha) - c_{i,k+1}(\alpha') \leqslant c_{i,k+2}(\alpha) - c_{i,k+2}(\alpha')$.

For the second inequality, notice that the difference increases at most one. Thus, if $c_{i,k+1}(\alpha) - c_{i,k+1}(\alpha') < \alpha_i - \alpha'_i$ then $c_{i,k+2}(\alpha) - c_{i,k+2}(\alpha') \leqslant \alpha_i - \alpha'_i$. It remains to prove that if $c_{i,k+1}(\alpha) - c_{i,k+1}(\alpha') = \alpha_i - \alpha'_i$ then $c_{i,k+2}(\alpha) - c_{i,k+2}(\alpha') = \alpha_i - \alpha'_i$. Since $c_{i,j}(\alpha) - c_{i,j}(\alpha') = 0$, it follows that $k + 1 \neq j$. Thus, $\alpha_{k+1} < \alpha_i - c_{i,k+1}(\alpha)$ if and only if $\alpha'_{k+1} < \alpha'_i - c_{i,k+1}(\alpha')$. Hence, $c_{i,k+2}(\alpha) - c_{i,k+2}(\alpha') = \alpha_i - \alpha'_i$.

Finally, since $c_{i,j}(\alpha) - c_{i,j}(\alpha') = 0$, by Equation (1), we have $c_{i,k}(\alpha) - c_{i,k}(\alpha') = 0$ for every $k \in [i + 1, j - 1]$.

(ii): It follows by conditions (a1) and (a4) that $\alpha_i - c_{i,j}(\alpha) = \alpha_i - \alpha'_i + \alpha_j > \alpha_j$. Thus, $c_{i,j+1}(\alpha) = c_{i,j}(\alpha) + 1$. On the other hand, by conditions (a1), (a2) and (a4), $\alpha'_i - c_{i,j}(\alpha') = \alpha_j \leqslant \alpha'_j$, by which $c_{i,j+1}(\alpha') = c_{i,j}(\alpha')$.

(iii): By (ii) and Equation (1), we have $c_{i,m}(\alpha) - c_{i,m}(\alpha') \geqslant 1$ for every $m > j$.

(iv): Suppose there exists $k \in [i + 1, j - 1]$ such that $\alpha_k + c_{i,k}(\alpha) < \alpha_i$ and $\alpha_k + c_{i,k}(\alpha) \geqslant \alpha'_i$. It follows by definition that $c_{i,k+1}(\alpha) = c_{i,k}(\alpha) + 1$ and $c_{i,k+1}(\alpha') = c_{i,k}(\alpha')$ which contradicts (i). □

Example 4 (Example 2 continued)**.** Suppose that $\alpha = (4, 5, 4, 1, 0, 2, 0)$ and $\alpha' = (2, 5, 4, 1, 0, 2, 1)$ such that α covers α' in positions $(1, 7)$. The Figure 5 illustrates the paths of $c_{1,j}(\alpha)$ and $c_{1,j}(\alpha')$ which are parallel to each other until they reach the 7-th row, reflecting the equations of Proposition 3.3(i),(ii). Besides, the "distance" between the paralell paths is related to the condition (a4).

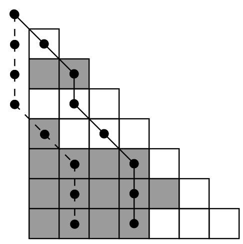


Figure 5: The paths for $c_{1,j}(\alpha)$ and $c_{1,j}(\alpha')$.

4 The equivalence with the Bruhat order

This section establishes our main result relating the partial order in the poset $\mathcal{PC}_n$ of compositions with the Bruhat order in S_n. Recall that the definition of $c(\alpha)$ depends only on the composition α. However, given a permutation $w \in S_n$ we are able to compute $c(\alpha)$ in a different way when α is the code of w.

4.1 The poset isomorphism

We begin establishing the meaning of the $c_{i,j}$'s in terms of the permutation model. In the interpretation of the coefficients $c_{ij}(\alpha)$ obtained by relating a composition α to the code of a permutation, we recover a definition previously introduced by Denoncourt, who refers to it as the extended Lehmer code of a permutation (for details, see [5], Definition 2.1).

Proposition 4.1. *Let $w \in S_n$ and denote by $\alpha = L(w)$. Then, for $1 \leqslant i < j \leqslant n$,*

$$c_{i,j}(\alpha) = \#\{k\colon i < k < j \text{ and } w(k) < w(i)\}. \tag{2}$$

Moreover, $c_{i,n}(\alpha) = \alpha_i$ for $i \in [n]$.

Proof. Consider the regions in the permutation matrix as in Figure 6. Denote by $d(X)$ the number of dots in the respective region X.

Denote by $d(A_j) = \#\{k\colon i < k < j \text{ and } w(k) < w(i)\}$. We will prove that $d(A_j) = c_{i,j}(\alpha)$ by induction on j. If $j = i+1$ then $d(A_{i+1}) = 0 = c_{i,i+1}(\alpha)$. Assume that $d(A_j) = c_{i,j}(\alpha)$.

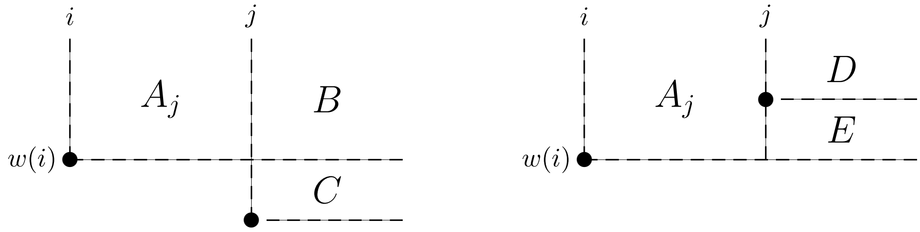


Figure 6: Regions in the permutation matrix of w. The dots in columns i and j do not belong to any region. On the left $w(j) > w(i)$. On the right $w(j) < w(i)$.

If $w(j) > w(i)$ then $\alpha_i = d(A_j) + d(B)$ and $\alpha_j = d(B) + d(C)$. In this case, $\alpha_j \geqslant \alpha_i - d(A_j) = \alpha_i - c_{i,j}(\alpha)$ and, by definition and induction, $c_{i,j+1}(\alpha) = c_{i,j}(\alpha) = d(A_j) = d(A_{j+1})$.

If $w(j) < w(i)$ then $\alpha_i = d(A_j) + 1 + d(D) + d(E)$ and $\alpha_j = d(D)$. In this case, $\alpha_j < \alpha_i - d(A_j) = \alpha_i - c_{i,j}(\alpha)$ and, by definition and induction, $c_{i,j+1}(\alpha) = c_{i,j}(\alpha) + 1 = d(A_j) + 1 = d(A_{j+1})$. □

Proposition 4.2. *Let $w, w' \in S_n$ and denote by $\alpha = L(w)$ and $\alpha' = L(w')$. Suppose that α covers α' in positions (i, j) such that $\alpha_i > \alpha'_i + 1$. Then, for every $k \in [i+1, j-1]$ and $m \in [\alpha'_i + 1, \alpha_i - 1]$,*

$$s_{m+i} \cdot \mathbf{w}'_i = \mathbf{w}'_i \cdot s_{m+i}; \tag{3}$$

$$s_{m+k-1-c_{i,k}(\alpha)} \cdot \mathbf{w}_k = \mathbf{w}_k \cdot s_{m+k-c_{i,k+1}(\alpha)}. \tag{4}$$

Proof. Since $m > \alpha'_i$, we have that s_{m+i} commutes with each simple reflection in $\mathbf{w}'_i = s_{\alpha'_i+i-1} \cdots s_i$. This proves Equation (3).

Let us prove Equation (4). By item (iv) of Proposition 3.3, consider the two cases.

Suppose that $\alpha_k + c_{i,k}(\alpha) < \alpha'_i$. Then $c_{i,k+1}(\alpha) = c_{i,k}(\alpha)+1$ and $m+k-1-c_{i,k}(\alpha) \geqslant \alpha'_i + k - c_{i,k}(\alpha) > \alpha_k + k$. The simple reflection $s_{m+k-1-c_{i,k}(\alpha)}$ commutes with each simple reflection in $\mathbf{w}_k$, i.e., $s_{m+k-1-c_{i,k}(\alpha)} \cdot \mathbf{w}_k = \mathbf{w}_k \cdot s_{m+k-1-c_{i,k}(\alpha)} = \mathbf{w}_k \cdot s_{m+k-c_{i,k+1}(\alpha)}$.

Suppose that $\alpha_k + c_{i,k}(\alpha) \geqslant \alpha_i$. Then $c_{i,k+1}(\alpha) = c_{i,k}(\alpha)$ and $m+k-1-c_{i,k}(\alpha) \leqslant \alpha_i + k - 2 - c_{i,k}(\alpha) \leqslant \alpha_k + k - 2$. Also notice that $c_{i,k}(\alpha) \leqslant c_{i,j}(\alpha) = \alpha'_i - \alpha_j \leqslant \alpha'_i$ and $m+k-1-c_{i,k}(\alpha) \geqslant \alpha'_i + k - c_{i,k}(\alpha) \geqslant k$. Thus, $k \leqslant m+k-1-c_{i,k}(\alpha) \leqslant \alpha_k + k - 2$. We get the following sequence of moves: $s_{m+k-1-c_{i,k}(\alpha)}$ commutes with each simple reflection in $s_{\alpha_k+k-1} \cdots s_{m+k-1-c_{i,k}(\alpha)+2}$; then we apply a braid move to get

$$s_{m+k-1-c_{i,k}(\alpha)} \cdot s_{m+k-c_{i,k}(\alpha)} \cdot s_{m+k-1-c_{i,k}(\alpha)} = s_{m+k-c_{i,k}(\alpha)} \cdot s_{m+k-1-c_{i,k}(\alpha)} \cdot s_{m+k-c_{i,k}(\alpha)}$$

and we continue to commute $s_{m+k-c_{i,k}(\alpha)}$ with $s_{m+k-2-c_{i,k}(\alpha)} \cdots s_k$. Hence, we have that $s_{m+k-1-c_{i,k}(\alpha)} \cdot \mathbf{w}_k = \mathbf{w}_k \cdot s_{m+k-c_{i,k}(\alpha)} = \mathbf{w}_k \cdot s_{m+k-c_{i,k+1}(\alpha)}$. □

We are now ready to prove our main theorem, which states that covering relations for permutations in Bruhat order are equivalent to coverings for compositions in $\mathcal{PC}_n$.

Theorem 4.3. *Let $w, w' \in S_n$ and denote by $\alpha = L(w)$ and $\alpha' = L(w')$. Then, w covers w' with $w' = w \cdot (i,j)$ if and only if α covers α' in positions (i,j).*

Proof. Suppose that w covers w' such that $w = w' \cdot (i,j)$, i.e., we can write $w = w(1) \cdots w(i) \cdots w(j) \cdots w(n)$ and $w' = w(1) \cdots w(j) \cdots w(i) \cdots w(n)$ with $w(i) = w'(j) > w(j) = w'(i)$. Recall that the code of a permutation counts the number of inversions to the right of the position.

Consider the regions in the permutation matrix as in Figure 7. Denote by $d(X)$ the number of dots in the respective region X.

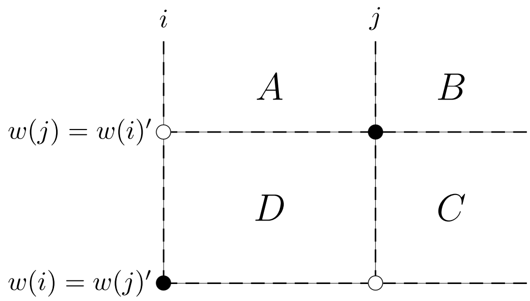


Figure 7: Regions in the permutation matrix of w (black dots) and w' (white dots). The dots in columns i and j do not belong to any region.

It follows from Lemma 2.1 that $d(D) = 0$. Then, $\alpha_i = d(A) + d(B) + d(C) + 1$ and $\alpha'_i = d(A) + d(B)$. Hence, $d(C) = \alpha_i - \alpha'_i - 1 \geqslant 0$ and we have that $\alpha_i \geqslant \alpha'_i + 1$, which proves condition (a1). Observe that $\alpha'_j = d(B) + d(C) = \alpha_j + \alpha_i - \alpha'_i - 1$, which proves condition (a2).

Clearly, $\alpha_k = \alpha'_k$ for $k < i$ or $j < k$. If $i < k < j$ then it follows from Lemma 2.1 that $\alpha_k = \alpha'_k$, which proves condition (a3).

Notice that $d(A) = \alpha'_i - \alpha_j$. By Proposition 4.1, $c_{i,j}(\alpha) = d(A) + d(D) = d(A) = c_{i,j}(\alpha')$, which gives condition (a4).

Now, assume that α covers α' in positions (i, j), where $i < j$. We will show that for some $I \in [\ell(w)]$ there is a sequence of moves from $\widehat{\mathbf{w}}_I$ to $\mathbf{w}'$, i.e., $\widehat{\mathbf{w}}_I$ is a reduced decomposition of w'.

If $\alpha_i = \alpha'_i + 1$ then w covers w' by Lemma 2.3.

Now, suppose that $\alpha_i > \alpha'_i + 1$. As a consequence of conditions (a1), (a2), and (a3) we have that

$$\begin{aligned}
\mathbf{w}_i &= s_{\alpha_i+i-1} s_{\alpha_i+i-2} \cdots s_{\alpha'_i+i} \cdot \mathbf{w}'_i \\
\mathbf{w}'_j &= s_{\alpha'_j+j-1} s_{\alpha'_j+j-2} \cdots s_{\alpha_j+j} \cdot \mathbf{w}_j \\
\mathbf{w}_k &= \mathbf{w}'_k, \text{ for } k \neq i \text{ and } k \neq j.
\end{aligned}$$

Moreover, condition (a2) also says that

$$\ell(s_{\alpha'_j+j-1} s_{\alpha'_j+j-2} \cdots s_{\alpha_j+j}) = \ell(s_{\alpha_i+i-1} s_{\alpha_i+i-2} \cdots s_{\alpha'_i+i+1}).$$

Choose $I = \left(\sum_{k=1}^{i} \alpha_k\right) - \alpha'_i$. We have that $\widehat{\mathbf{w}}_I$ and $\mathbf{w}'$ are written as follows:

$$\begin{aligned}
\widehat{\mathbf{w}}_I &= \mathbf{w}_1 \cdots \mathbf{w}_{i-1} \cdot (s_{\alpha_i+i-1} s_{\alpha_i+i-2} \cdots s_{\alpha'_i+i+1}) \cdot \mathbf{w}'_i \cdot \mathbf{w}_{i+1} \cdots \mathbf{w}_{n-1}, \\
\mathbf{w}' &= \mathbf{w}_1 \cdots \mathbf{w}_{i-1} \cdot \mathbf{w}'_i \cdots \mathbf{w}'_{j-1} \cdot (s_{\alpha'_j+j-1} s_{\alpha'_j+j-2} \cdots s_{\alpha_j+j}) \cdot \mathbf{w}_j \cdots \mathbf{w}_{n-1}.
\end{aligned}$$

Now, let us describe a sequence of moves that transforms $\widehat{\mathbf{w}}_I$ into $\mathbf{w}'$. For each $m \in [\alpha'_i + 1, \alpha_i - 1]$, we can apply Proposition 4.2 to describe the sequence of moves as follows:

$$\begin{aligned}
s_{m+i} \cdot \mathbf{w}'_i \cdots \mathbf{w}'_{j-1} &= \mathbf{w}'_i \cdot s_{m+i} \cdot \mathbf{w}'_{i+1} \cdot \mathbf{w}'_{i+2} \cdots \mathbf{w}'_{j-1} \\
&= \mathbf{w}'_i \cdot \mathbf{w}'_{i+1} \cdot s_{m+i+1-c_{i,i+2}(\alpha)} \cdot \mathbf{w}'_{i+2} \cdots \mathbf{w}'_{j-1} \\
&= \mathbf{w}'_i \cdot \mathbf{w}'_{i+1} \cdot \mathbf{w}'_{i+2} \cdot s_{m+i+2-c_{i,i+3}(\alpha)} \cdots \mathbf{w}'_{j-1} \\
&\ \ \vdots \\
&= \mathbf{w}'_i \cdots \mathbf{w}'_{j-1} \cdot s_{m+j-1-c_{i,j}(\alpha)} = \mathbf{w}'_i \cdots \mathbf{w}'_{j-1} \cdot s_{m+j-1+\alpha_j-\alpha'_i}
\end{aligned}$$

where the last equality is due to condition (a4). Hence, using condition (a2), we have

$$\begin{aligned}
\widehat{\mathbf{w}}_I &= \mathbf{w}_1 \cdots \mathbf{w}_{i-1} \cdot (s_{\alpha_i+i-1} s_{\alpha_i+i-2} \cdots s_{\alpha'_i+i+1}) \cdot \mathbf{w}'_i \cdots \mathbf{w}'_{j-1} \cdot \mathbf{w}_j \cdots \mathbf{w}_{n-1} \\
&= \mathbf{w}_1 \cdots \mathbf{w}_{i-1} \cdot \mathbf{w}'_i \cdots \mathbf{w}'_{j-1} \cdot (s_{\alpha'_j+j-1} s_{\alpha'_j+j-2} \cdots s_{\alpha_j+j}) \cdot \mathbf{w}_j \cdots \mathbf{w}_{n-1} = \mathbf{w}'.
\end{aligned}$$

Hence, the reduced decompositions of $\widehat{\mathbf{w}}_I$ and $\mathbf{w}'$ represent the same permutation w'. By the bijection between S_n and $\mathcal{C}_n$ (see Lemma 2.2 and Figure 2) along with the uniqueness of (i, j) for compositions we have $w = w' \cdot (i, j)$. □

Corollary 4.4. *The poset in S_n defined by the Bruhat order is isomorphic to the poset $\mathcal{PC}_n$.*

4.2 Interpretation of covering relations in the diagram

Given two compositions $\alpha = L(w)$ and $\alpha' = L(w')$ such that α covers α' in positions (i, j), let D' be the collection of boxes obtained from the diagram of α by removing the box located at $(i, \alpha'_i + 1)$.

If $\alpha'_i = \alpha_i - 1$ then D' is already the diagram of w'. If $\alpha_i > \alpha'_i + 1$, then the remaining block of boxes in the i-th row to the right of the removed position must be displaced to obtain the diagram of w'. This displacement is performed via a specific geometric operation called a ladder move.

A ladder move shifts a box located at (y, x) to $(r, x - 1)$, provided that the following conditions hold:

(A) The coordinate $(y, x - 1)$ is empty.
(B) There are boxes in columns $x - 1$ and x in all intermediate rows.
(C) The coordinates $(r, x - 1)$ and (r, x) are both empty.

This operation corresponds to the ladder moves on RC-graphs described by Bergeron and Billey [1], as illustrated in Figure 8.

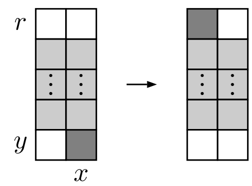


Figure 8: A ladder move shifting a box located at (y, x) to $(r, x - 1)$.

Proposition 4.5. *Let $\alpha, \alpha' \in \mathcal{C}_n$ be compositions. The composition α covers α' in positions (i, j) with $\alpha_i > \alpha'_i + 1$ if and only if there exists a sequence of ladder moves on D' that transports the block of remaining boxes in row i to row j, giving the diagram of α', where D' is the collection of boxes obtained by removing the box located at $(i, \alpha'_i + 1)$ from the diagram of α.*

Proof. Assume that α covers α' in positions (i, j) with $\alpha_i > \alpha'_i + 1$. Let D' be the collection of the boxes obtained from the diagram of α by removing the box in row i at position $\alpha'_i + 1$. Then there remains a block of $\alpha_i - \alpha'_i - 1$ boxes in row i to the right of the removed position which occupies columns from $\alpha'_i + 2$ to α_i. We will process this block sequentially from left to right, for $m = \alpha'_i + 2, \ldots, \alpha_i$.

Algebraically, moving this block to the row j corresponds to the transport of the simple reflections s_{m+i} across the intermediate rows $k \in [i+1, j-1]$. This trajectory is directed by the sequence $c_{i,k}(\alpha)$, which starts at 0 and, by condition (a4), ends at $c_{i,j}(\alpha) = \alpha'_i - \alpha_j$.

Let us partition the interval $[i, j]$ into contiguous subintervals

$$I_p = \{k \in [i, j] \colon c_{i,k+1}(\alpha) = p\}$$

and denote $k_p = \min I_p$. Note that $p \in [0, \alpha'_i - \alpha_j + 1]$, $k_0 = i$, and $k_{\alpha'_i - \alpha_j + 1} = j$ (by Proposition 3.3(ii)).

We claim that each subinterval I_p, $p \in [0, \alpha'_i - \alpha_j]$, bounds exactly one ladder move, which transports the box located at $(k_p, m-p)$ to $(k_{p+1}, m-p-1)$. This follows from a direct verification that all three defining conditions of a ladder move are satisfied.

For $0 < p \leqslant \alpha'_i - \alpha_j$, since $c_{i,k_p+1}(\alpha) = p = c_{i,k_p}(\alpha) + 1$, by definition we obtain $\alpha_{k_p} < \alpha_i - p + 1$. Moreover, since $k_p < j$, Proposition 3.3(iv) implies that

$$\alpha_{k_p} < \alpha'_i - (p-1) \leqslant (m-2) - p + 1 = m - p - 1. \tag{5}$$

Recall that a simple reflection s_x in row y geometrically aligns with column $x - y + 1$. Since

$$(m + k_p - 1 - p) - k_p + 1 = m - p > k_p,$$

it follows that, in Equation (4), the incoming reflection s_{m+k_p-1-p} must be placed in row k_p and column $m - p$.

Geometrically, there is a box located at $(k_p, m-p)$, as it corresponds to the current position of the tracking box. The adjacent coordinate $(k_p, m-p-1)$ is empty. Indeed, for the initial box of the block ($m = \alpha'_i + 2$), this space is empty due to the original removal of the box at column $\alpha'_i + 1$ (for $p = 0$) and Equation (5) (for $p > 0$). For any subsequent box ($m > \alpha'_i + 2$), this space was strictly vacated by the preceding box (originating at $m - 1$) during its own sequence of ladder moves. This satisfies Condition (A) of a ladder move.

For every intermediate row $k \in I_p \backslash \{k_p\}$, $c_{i,k+1}(\alpha) = c_{i,k}(\alpha) = p$ ensures that $\alpha_k \geqslant \alpha_i - p \geqslant m - p$. Geometrically, this means row k extends at least up to column $m-p$, then there are the boxes located at $(k, m-p-1)$ and $(k, m-p)$. This satisfies Condition (B) of a ladder move.

If $k_{p+1} < j$, the transition to the next subinterval imposes $\alpha_{k_{p+1}} < m - p - 2$ (by substituting $p + 1$ into Equation (5)). Geometrically, this indicates that row k_{p+1} is strictly shorter, leaving both columns $m - p - 1$ and $m - p$ empty. This satisfies Condition (C) of a ladder move when $k_{p+1} < j$.

Finally, if $k_{p+1} = j$ then $p = c_{i,j}(\alpha) = \alpha'_i - \alpha_j$ and it remains to show that it reaches the end of this construction. In Equation (4), the outgoing reflection for $k = j - 1$ is $s_{m+j-1-c_{i,j}(\alpha)} = s_{m+j-2-\alpha'_i+\alpha_j}$ and its column at row j is:

$$(m + j - 2 - \alpha'_i + \alpha_j) - j + 1 = m - \alpha'_i + \alpha_j - 1.$$

Upon reaching the final row j, the block occupies the columns $\alpha_j + 1, \ldots, \alpha_i - \alpha'_i + \alpha_j - 1$. Since α_j is the length of row j and, by Equation (a2), $\alpha'_j = \alpha_i - \alpha'_i + \alpha_j - 1$, these columns are precisely the empty positions immediately to the right of the existing boxes in row j, which is exactly the diagram of α'.

Conversely, assume that there exists a sequence of ladder moves on D' that transports the remaining contiguous block of boxes in row i to row j, resulting precisely in the diagram of α'. It remains to verify that α covers α' by checking that the four covering conditions are satisfied.

Deleting the box in column $\alpha'_i + 1$ and shifting the following block of boxes left results in exactly α'_i boxes remaining in row i. Hence, the updated length of row i is α'_i. Moreover, because the hypothesis assumes that there is a contiguous block of boxes to the right of the removed position, it follows that $\alpha_i > \alpha'_i + 1$.

The transported block has length $\alpha_i - \alpha'_i - 1$. This block must be placed contiguously, directly following the existing α_j boxes in row j. Consequently, the new length of row j is determined by $\alpha'_j = \alpha_j + \alpha_i - \alpha'_i - 1$.

Because ladder moves do not append or remove boxes from intermediate rows $k \in [i+1, j-1]$, we have $\alpha'_k = \alpha_k$ for all $k \neq i, j$.

Finally, it remains to verify condition (a4), namely that $c_{i,j}(\alpha) = \alpha'_i - \alpha_j$. To this end, we track the column index of the first box of the transported block, which is given by $m = \alpha'_i + 2$. We observe that each leftward shift of the column by a ladder move corresponds to an increase in the sequence $c_{i,k}(\alpha)$. Upon reaching row j, the final column of this tracking box is given by the starting index minus the total number of shifts, namely $m - (c_{i,j}(\alpha) + 1) = \alpha'_i + 1 - c_{i,j}(\alpha)$. This initial box must be placed exactly in the first unoccupied position of row j, namely in column $\alpha_j + 1$. Consequently, we have $\alpha'_i + 1 - c_{i,j}(\alpha) = \alpha_j + 1$, which implies $c_{i,j}(\alpha) = \alpha'_i - \alpha_j$. □

Example 5 (Ex. 2 continued)**.** If $\alpha = (4, 5, 4, 1, 0, 2, 0)$ and $\alpha' = (2, 5, 4, 1, 0, 2, 1)$, then the collection of boxes D' is obtained from the diagram of α by removing the third box in the bottom row from the diagram of w. There remains a block containing a unique box. The set D' can be modified into α' through consecutive ladder moves in the diagram, as we can see in Figure 9. Since α covers α' in positions $(1, 7)$, it follows that $[1, 7]$ is partitioned into intervals I_p, $p \in [0, 3]$ such that $k_0 = 1$, $k_1 = 4$, $k_2 = 5$ and $k_3 = 7$. This provides the sequence of ladder moves for the remaining box ($m = 4$) given by $(1, 4) \mapsto (4, 3) \mapsto (5, 2) \mapsto (7, 1)$.

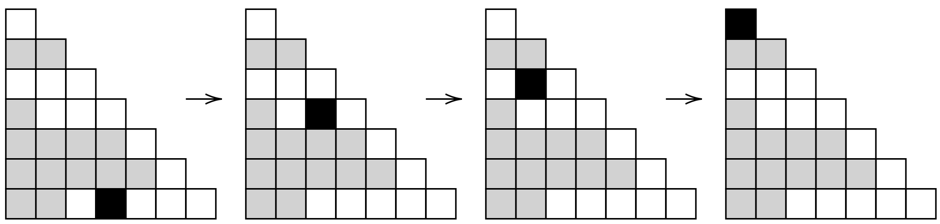


Figure 9: Ladder moves applied to D' result in the diagram of $\alpha' = (2, 5, 4, 1, 0, 2, 1)$.

Proposition 4.5 also resembles the excitations of boxes in the diagrams as introduced by Ikeda-Naruse [6] in the context of the Grassmannian permutations.

5 Removable and insertable compositions

As a byproduct of the partial order in $\mathcal{C}_n$, this section presents an algorithm based on a removal-insertion process to determine the covering relations in the poset.

5.1 Removable compositions

Let $\alpha \in \mathcal{C}_n$ be a composition. Given $i \in [n-1]$ and $z \in [\alpha_i]$, we say that α is (i,z)-removable if there exists α' such that α covers α' and $\alpha'_i = \alpha_i - z$. We will show in the next lemma that such α' is unique. Since the covering relation among α and α' requires a pair $i < j$, we now address the question of finding the corresponding j for each i and z.

First of all, set the composition $\widetilde{\alpha}$ as $\widetilde{\alpha}_i = \alpha_i - z$ and $\widetilde{\alpha}_m = \alpha_m$ for $m \neq i$. Denote by $\widetilde{J}_\alpha(i,z)$ the greatest $j > i$ such that $c_{i,j}(\alpha) = c_{i,j}(\widetilde{\alpha})$, i.e.,

$$\widetilde{J}_\alpha(i,z) = \max\{j > i \colon c_{i,j}(\alpha) = c_{i,j}(\widetilde{\alpha})\}. \tag{6}$$

Such $\widetilde{J}_\alpha(i,z)$ exists due to Lemma 3.1. In fact, by Lemma 3.1(iii), there exists j_1 and j_2 such that $c_{i,j_1}(\alpha) = \alpha_i$ and $c_{i,j_2}(\widetilde{\alpha}) = \widetilde{\alpha}_i$. Since $\widetilde{\alpha}_i < \alpha_i$, then there is a greatest j such that $c_{i,j}(\alpha) = c_{i,j}(\widetilde{\alpha})$. If it is clear, we will denote $\widetilde{J}(i,z) = \widetilde{J}_\alpha(i,z)$.

Lemma 5.1. *If α is (i,z)-removable then there exists a unique composition α' such that α covers α' and $\alpha'_i = \alpha_i - z$. In this case, α covers the composition α' defined by $\alpha'_i = \alpha_i - z$, $\alpha'_{\widetilde{J}(i,z)} = \alpha_{\widetilde{J}(i,z)} + z - 1$, and $\alpha'_m = \alpha_m$ for $m \notin \{i, \widetilde{J}(i,z)\}$.*

Proof. Assume that α covers α' in positions (i,j) such that $\alpha'_i = \alpha_i - z$. We want to prove that $j = \widetilde{J}(i,z)$.

Initially, we claim that $c_{i,k}(\alpha') = c_{i,k}(\widetilde{\alpha})$ for any $k \in [i+1,j]$. In fact, since α' and $\widetilde{\alpha}$ only differ for $\alpha'_j \geqslant \widetilde{\alpha}_j$, it follows from Lemma 3.2 that $c_{i,k}(\alpha') \leqslant c_{i,k}(\widetilde{\alpha}) \leqslant c_{i,k}(\alpha)$ for any $k > i$ and $c_{i,k}(\widetilde{\alpha}) \leqslant c_{i,k}(\alpha')$ for any $k \in [i+1,j]$.

Suppose that $\widetilde{J}(i,z) < j$. It follows that $c_{i,\widetilde{J}(i,z)+1}(\alpha) \neq c_{i,\widetilde{J}(i,z)+1}(\widetilde{\alpha}) = c_{i,\widetilde{J}(i,z)+1}(\alpha')$ which contradicts Proposition 3.3(i). Hence $\widetilde{J}(i,z) \geqslant j$. Notice that, by condition (a4), $\widetilde{\alpha}_j = \alpha_j = \alpha'_i - c_{i,j}(\alpha') = \widetilde{\alpha}_i - c_{i,j}(\widetilde{\alpha})$. Then, $c_{i,j+1}(\widetilde{\alpha}) = c_{i,j}(\widetilde{\alpha}) = c_{i,j}(\alpha') = c_{i,j}(\alpha) \neq c_{i,j+1}(\alpha)$, which implies $j = \widetilde{J}(i,z)$. ☐

Since α' in Lemma 5.1 is unique then we will call α' the (i,z)-removal of α.

The next proposition gives an equivalent characterization for α be (i,z)-removable.

We define $k_\alpha(i)$ to be the first k greater than i such that $\alpha_k < \alpha_i$, i.e.,

$$k_\alpha(i) = \min\{k > i \colon \alpha_k < \alpha_i\}. \tag{7}$$

Proposition 5.2. *Let α be a composition, $i \in [n-1]$ and $z \in [\alpha_i]$. We have the following:*

(i) α is (i,z)-removable if and only if $c_{i,\widetilde{J}(i,z)}(\alpha) = \alpha_i - \alpha_{\widetilde{J}(i,z)} - z$.

(ii) If α is (i,z)-removable then $z \leqslant \alpha_i - \alpha_{k_\alpha(i)}$.

(iii) If $\alpha_i > 0$ then α is always $(i,1)$-removable.

Proof. (i): Suppose that α is (i,z)-removable. Then, α' as defined in Lemma 5.1 is covered by α. Hence, condition (a4) gives $c_{i,\widetilde{J}(i,z)}(\alpha) = \alpha_i - \alpha_{\widetilde{J}(i,z)} - z$.

Conversely, α' defined by $\alpha'_i = \alpha_i - z$, $\alpha'_{\widetilde{J}(i,z)} = \alpha_{\widetilde{J}(i,z)} + z - 1$, and $\alpha'_m = \alpha_m$ for $m \neq i$ or $\widetilde{J}(i,z)$ satisfies conditions (a1), (a2), (a3), and partially (a4). It remains to prove that $c_{i,\widetilde{J}(i,z)}(\alpha) = c_{i,\widetilde{J}(i,z)}(\alpha')$. By Lemma 3.2, we have that $c_{i,\widetilde{J}(i,z)}(\alpha') = c_{i,\widetilde{J}(i,z)}(\widetilde{\alpha}) = c_{i,\widetilde{J}(i,z)}(\alpha)$.

(ii): By definition of $k_\alpha(i)$, $c_{i,k}(\alpha) = 0$ for $k \in [k_\alpha(i)]$, $c_{i,k_\alpha(i)+1}(\alpha) = 1$, and $\widetilde{J}(i,z) \geqslant k_\alpha(i)$. If $\widetilde{J}(i,z) = k_\alpha(i)$ then, by assertion (i), $z = \alpha_i - \alpha_{k_\alpha(i)} - c_{i,k_\alpha(i)}(\alpha) = \alpha_i - \alpha_{k_\alpha(i)}$. If $\widetilde{J}(i,z) > k_\alpha(i)$ then, by Proposition 3.3(i), $c_{i,k_\alpha(i)+1}(\alpha') = c_{i,k_\alpha(i)+1}(\alpha) = 1$ and, hence, by definition and condition (a3), $\alpha_{k_\alpha(i)} = \alpha'_{k_\alpha(i)} < \alpha'_i - c_{i,\alpha_{k_\alpha(i)}}(\alpha') = \alpha'_i$. Then, $z = \alpha_i - \alpha'_i < \alpha_i - \alpha_{k_\alpha(i)}$.

(iii): Let α' be defined by $\alpha'_i = \alpha_i - 1$ and $\alpha'_k = \alpha_k$ for $k \neq i$. By definition of $\widetilde{J}(i,z)$, $c_{i,\widetilde{J}(i,z)+1}(\alpha') < c_{i,\widetilde{J}(i,z)+1}(\alpha)$ and, hence, $c_{i,\widetilde{J}(i,z)+1}(\alpha') = c_{i,\widetilde{J}(i,z)}(\alpha')$ and $c_{i,\widetilde{J}(i,z)+1}(\alpha) = 1 + c_{i,\widetilde{J}(i,z)}(\alpha)$. By definition, $\alpha_{\widetilde{J}(i,z)} < \alpha_i - c_{i,\widetilde{J}(i,z)}(\alpha) = \alpha'_i + 1 - c_{i,\widetilde{J}(i,z)}(\alpha)$ and $\alpha_{\widetilde{J}(i,z)} = \alpha'_{\widetilde{J}(i,z)} \geqslant \alpha'_i - c_{i,\widetilde{J}(i,z)}(\alpha') = \alpha'_i - c_{i,\widetilde{J}(i,z)}(\alpha)$. Reordering both inequalities we have that $\alpha'_i - \alpha_{\widetilde{J}(i,z)} \leqslant c_{i,\widetilde{J}(i,z)}(\alpha) < \alpha'_i + 1 - \alpha_{\widetilde{J}(i,z)}$. Hence, $c_{i,\widetilde{J}(i,z)}(\alpha) = \alpha'_i - \alpha_{\widetilde{J}(i,z)} = \alpha_i - \alpha_{\widetilde{J}(i,z)} - 1$ and, by item(i), α is $(i,1)$-removable. □

Summarizing, given a composition $\alpha \in \mathcal{C}_n$ and an index $i \in [n-1]$, we can determine z such that α is (i,z)-removable by following the steps below:

1. Compute the possible values of z by the range $[1, \alpha_i - \alpha_{k_\alpha(i)}]$ (cf. Equation 7).
2. If $z = 1$ and $\alpha_i > 0$, then α is $(i,1)$-removable.
3. For each $2 \leqslant z \leqslant \alpha_i - \alpha_{k_\alpha(i)}$, proceed as follows:

   (a) Start with $\widetilde{\alpha}$.
   (b) Compute $c_{i,j}(\widetilde{\alpha})$ and $\widetilde{J}(i,z)$ (cf. Equation 6).
   (c) If $c_{i,\widetilde{J}(i,z)}(\alpha) = \alpha_i - \alpha_{\widetilde{J}(i,z)} - z$, then α is (i,z)-removable. Define α' by:

   $$\alpha'_i = \alpha_i - z\,,\ \alpha'_{\widetilde{J}(i,z)} = \alpha_{\widetilde{J}(i,z)} + z - 1,\ \text{ and } \alpha'_m = \alpha_m,\ \text{ for } m \neq i \text{ or } \widetilde{J}(i,z),$$

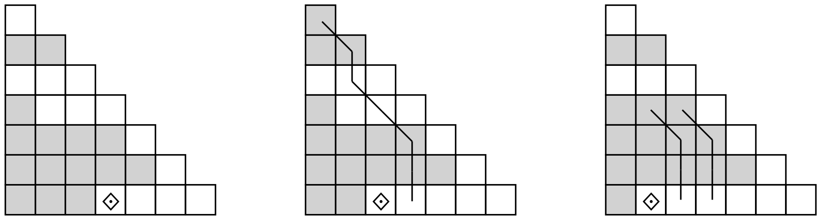
   as stated in Lemma 5.1.

Example 6. Let $\alpha = (4,5,4,1,0,2,0) \in \mathcal{C}_8$. Let us determine when α is $(1,z)$-removable, with $z \in [\alpha_1] = [4]$. Since $k_\alpha(1) = 4$, it follows that $z \leqslant \alpha_1 - \alpha_4 = 3$, i.e., α is not $(1,4)$-removable. Table 5.1 presents the $(1,z)$-removals α' of α, for $z = 1,2,3$. We remember that $c_{1,j}(\alpha) = (0,0,0,0,1,2,2,3,4)$. In each case, we have that $c_{i,\widetilde{J}(i,z)}(\alpha) = \alpha_i - \alpha_{\widetilde{J}(i,z)} - z$.

z	$c_{1,j}(\widetilde{\alpha})$	$\widetilde{J}(1,z)$	$\alpha'_{\widetilde{J}(1,z)}$	α'
1	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{2},\mathbf{3},3)$	8	–	$(\mathbf{3},5,4,1,0,2,0)$
2	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{2},2,2)$	7	1	$(\mathbf{2},5,4,1,0,2,\mathbf{1})$
3	$(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},0,1,1,1,1)$	4	3	$(\mathbf{1},5,2,\mathbf{3},0,2,0)$

Table 5.1: The $(1,z)$-removals of $\alpha = (4,5,4,1,0,2,0)$.

Figure 10 illustrates this process using diagrams. Eventually, after a sequence of ladder moves, illustrated by the polygonal paths, one obtains the covering diagrams resulting from removing a box in the first row, which is marked by a diamond-shaped box.



(3, 5, 4, 1, 0, 2, 0) (2, 5, 4, 1, 0, 2, 1) (1, 5, 4, 3, 0, 2, 0)

Figure 10: Diagrams of the $(1, z)$-removals of $\alpha = (4, 5, 4, 1, 0, 2, 0)$.

The product order of $\mathcal{C}_n$ is given by $\alpha' \leqslant \alpha$ if and only if $\alpha'_i \leqslant \alpha_i$ for all $i \in [n-1]$. The next corollary states that $\mathcal{PC}_n$ is a refinement of the product order of $\mathcal{C}_n$.

Corollary 5.3. *Let α and α' be compositions in $\mathcal{C}_n$ such that $\alpha' \leqslant \alpha$. Then $\alpha' \preceq \alpha$ in the poset $\mathcal{PC}_n$.*

Proof. The partial order $\preceq$ in $\mathcal{PC}_n$ is defined on covering pairs and then extended by transitivity via Equations (a1) – (a4). If α and α' form a covering pair in the product order, then $|\alpha| = |\alpha'| + 1$ and there exists $i \in [n]$ such that $\alpha'_i = \alpha_i - 1$ and $\alpha'_k = \alpha_k$ for every $k \neq i$. Equivalently, α is $(i, 1)$-removable, and hence, by Proposition 5.2(iii), α covers α' with respect to $\preceq$. □

Proposition 5.4. *Let α be (i, z_1)-removable and also (i, z_2)-removable. Then:*

(i) $z_1 = z_2$ if and only if $\widetilde{J}(i, z_1) = \widetilde{J}(i, z_2)$.
(ii) $z_1 > z_2$ if and only if $\widetilde{J}(i, z_1) < \widetilde{J}(i, z_2)$.

Proof. Let α' and α'' be the compositions covered by α such that $\alpha'_i = \alpha_i - z_1$ and $\alpha''_i = \alpha_i - z_2$, respectively.

(i): Clearly, if $z_1 = z_2$ then $\widetilde{J}(i, z_1) = \widetilde{J}(i, z_2)$. Conversely, if $\widetilde{J}(i, z_1) = \widetilde{J}(i, z_2) = j$ then, by condition (a4) of α' and α'', we have that $c_{i,j}(\alpha) = \alpha'_i - \alpha_j = \alpha''_i - \alpha_j$, i.e., $\alpha'_i = \alpha''_i$. Hence $z_1 = \alpha_i - \alpha'_i = \alpha_i - \alpha''_i = z_2$.

(ii): Assume that $z_1 > z_2$ and suppose that $\widetilde{J}(i, z_1) > \widetilde{J}(i, z_2)$. Applying condition (a4) to the covering α and α'', we obtain $\alpha_{\widetilde{J}(i,z_2)} + c_{i,\widetilde{J}(i,z_2)}(\alpha) = \alpha''_i$. Since $\alpha'_i < \alpha''_i < \alpha_i$, it follows that

$$\alpha'_i < \alpha_{\widetilde{J}(i,z_2)} + c_{i,\widetilde{J}(i,z_2)}(\alpha) < \alpha_i.$$

However, since $i < \widetilde{J}(i, z_2) < \widetilde{J}(i, z_1)$, the inequality above contradicts Proposition 3.3 (iv) for the covering α and α'.

The converse follows from assertion (i) and the analogous fact if $z_1 < z_2$ then $\widetilde{J}(i, z_1) > \widetilde{J}(i, z_2)$. □

Hence, Proposition 5.4 says that $\widetilde{J}(i,\cdot)$ is an injective (5.4(i)) and strictly decreasing (5.4(ii)) function of z. It means that as the position of the removed box moves leftward, the index of the corresponding row decreases.

5.2 Insertable compositions

Given a composition α in $\mathcal{C}_n$, $i \in [n-1]$ and $z \in [n-i-\alpha_i]$, we say that $\alpha \in \mathcal{C}_n$ is (i,z)-insertable if there exists a composition $\alpha'' \in \mathcal{C}_n$ such that $\alpha''_i = \alpha_i + z$ and α is covered by α''.

For every composition $\alpha \in \mathcal{C}_n$, denote by α^* the composition in $\mathcal{C}_n$ such that $\alpha^*_i = n - i - \alpha_i$, for every $i \in [n-1]$.

Lemma 5.5. *We have the following properties:*

(i) For $i < j \leqslant n+1$, we have $c_{i,j}(\alpha^*) = j - i - 1 - c_{i,j}(\alpha)$.
(ii) $\alpha' \preceq \alpha$ if and only if $\alpha^* \preceq (\alpha')^*$.

Proof. We will prove (i) by induction on j. If $j = i+1$ then $c_{i,i+1}(\alpha^*) = 0 = j - i - 1 - c_{i,j}(\alpha)$.

Suppose that the equation holds for j. There are two cases to analyze: If $\alpha_j < \alpha_i - c_{i,j}(\alpha)$ then $\alpha^*_j = n - j - \alpha_j > n - j - \alpha_i + c_{i,j}(\alpha) = \alpha^*_i + i - j + c_{i,j}(\alpha) = \alpha^*_i - c_{i,j}(\alpha^*) - 1$, i.e., $\alpha^*_j \geqslant \alpha^*_i - c_{i,j}(\alpha^*)$. By definition of $c_{i,j}$, we have that

$$c_{i,j+1}(\alpha^*) = c_{i,j}(\alpha^*) = j - i - 1 - c_{i,j}(\alpha) = j - i - c_{i,j+1}(\alpha).$$

If $\alpha_j \geqslant \alpha_i - c_{i,j}(\alpha)$ then $\alpha^*_j = n - j - \alpha_j \leqslant n - j - \alpha_i + c_{i,j}(\alpha) = \alpha^*_i + i - j + c_{i,j}(\alpha) = \alpha^*_i - c_{i,j}(\alpha^*) - 1$, i.e., $\alpha^*_j < \alpha^*_i - c_{i,j}(\alpha^*)$. By definition of $c_{i,j}$, we have that

$$c_{i,j+1}(\alpha^*) = c_{i,j}(\alpha^*) + 1 = j - i - c_{i,j}(\alpha) = j - i - c_{i,j+1}(\alpha).$$

This proves assertion (i).

Assertion (ii) is easily obtained once we show conditions (a1) to (a4) using the definition of α^* and (i). $\square$

The next lemma states the relationship between removable and insertable compositions.

Lemma 5.6. *α is (i,z)-insertable if and only if α^* is (i,z)-removable.*

Proof. Suppose that α is (i,z)-insertable. Then, there exists α'' such that α is covered by α'' and $\alpha''_i = \alpha_i + z$. By Lemma 5.5, α^* covers $(\alpha'')^*$ and $(\alpha'')^*_i = n - i - \alpha''_i = n - i - \alpha_i - z = \alpha^*_i - z$. The reciprocal is analogous. $\square$

We now seek a criterion for a composition α to be (i,z)-insertable. A first approach is to apply Lemma 5.6 by which it is equivalent to have α^* be (i,z)-removable.

Instead, we will define a composition $\widehat{\alpha}$ to play an analogous role to $\widetilde{\alpha}$ in the removing process. Set the composition $\widehat{\alpha}$ as $\widehat{\alpha}_i = \alpha_i + z$ and $\widehat{\alpha}_m = \alpha_m$ for $m \neq i$. Denote by $\widehat{J}_\alpha(i,z)$ the greatest $j > i$ such that $c_{i,j}(\alpha) = c_{i,j}(\widehat{\alpha})$, i.e.,

$$\widehat{J}_\alpha(i,z) = \max\{j > i \colon c_{i,j}(\alpha) = c_{i,j}(\widehat{\alpha})\}. \tag{8}$$

If it is clear, we will denote $\widehat{J}(i,z) = \widehat{J}_\alpha(i,z)$.

The next lemma relates $\widetilde{\alpha}$ and $\widehat{\alpha}$. In particular, we will see that the corresponding J's for the process of either removing from α^* or insertion in α are equivalent.

Lemma 5.7. *Let $\alpha \in \mathcal{C}_n$, $i \in [n-1]$ and $z \in [n-i-\alpha_i]$.*

(i) $\widetilde{\alpha^*} = (\widehat{\alpha})^*$.

(ii) $\widehat{J}_\alpha(i,z) = \widetilde{J}_{\alpha^*}(i,z)$.

Proof. (i): It follows directly by the definitions: $(\widetilde{\alpha^*})_i = \alpha^*_i - z = n - i - \alpha_i - z$ and $(\widehat{\alpha})^*_i = n - i - \widehat{\alpha}_i = n - i - \alpha_i - z$. For $m \neq i$, $(\widetilde{\alpha^*})_m = \alpha^*_m = n - i - \alpha_m$ and $(\widehat{\alpha})^*_m = n - i - \widehat{\alpha}_m = n - i - \alpha_m$.

(ii): By item (i), notice that $c_{i,j}(\alpha^*) = c_{i,j}(\widetilde{\alpha^*}) = c_{i,j}((\widehat{\alpha})^*)$. Hence, by Lemma 5.5(i), $c_{i,j}(\alpha^*) = c_{i,j}(\widetilde{\alpha^*})$ if and only if $c_{i,j}(\alpha) = c_{i,j}(\widehat{\alpha})$. □

Lemma 5.8. *If α is (i,z)-insertable then there exists a unique composition α'' such that α is covered by α'' and $\alpha''_i = \alpha_i + z$. In this case, α is covered by the composition α'' defined by $\alpha''_i = \alpha_i + z$, $\alpha''_{\widehat{J}(i,z)} = \alpha_{\widehat{J}(i,z)} - z + 1$, and $\alpha''_m = \alpha_m$ for $m \neq i$ or $\widehat{J}(i,z)$.*

Proof. By Lemma 5.6, α^* is (i,z)-removable. Then, it follows from Lemma 5.1 that the unique α' such that α^* covers α' is defined by $\alpha'_i = \alpha^*_i - z$, $\alpha'_{\widetilde{J}_{\alpha^*}(i,z)} = \alpha_{\widetilde{J}_{\alpha^*}(i,z)} + z - 1$ and $\alpha'_m = \alpha^*$ for $m \neq i$ or $\widetilde{J}_{\alpha^*}(i,z)$. Put $\alpha'' = (\alpha')^*$ and notice that, by Lemma 5.7, $\widetilde{J}_{\alpha^*}(i,z) = \widehat{J}_\alpha(i,z)$. □

Since α'' in Lemma 5.8 is unique, we will call α'' the (i,z)-insertion of α.

The next proposition shows that we can also lower the upper bound of z in the insertion process.

Proposition 5.9. *Let $\alpha \in \mathcal{C}_n$, $i \in [n-1]$, and $z \in [n-i-\alpha_i]$. We have the following:*

(i) α is (i,z)-insertable if and only if $c_{i,\widehat{J}(i,z)}(\alpha) = \alpha_i - \alpha_{\widehat{J}(i,z)} + z - 1$.

(ii) If α is (i,z)-insertable then $z \leqslant \alpha_{k_{\alpha^*}(i)} - \alpha_i + k_{\alpha^*}(i) - i$.

(iii) If $\alpha_i + i < n$ then α is always $(i,1)$-insertable.

Proof. (i): By Lemma 5.6 and Proposition 5.2, α^* is (i,z)-removable if and only if $c_{i,\widetilde{J}_{\alpha^*}(i,z)}(\alpha^*) = \alpha^*_i - \alpha^*_{\widetilde{J}_{\alpha^*}(i,z)} - z$. By Lemmas 5.5 and 5.7, the latter equation is equivalent to $c_{i,\widehat{J}(i,z)}(\alpha) = \alpha_i - \alpha_{\widehat{J}(i,z)} + z - 1$.

(ii): By Lemma 5.6, α^* is (i,z)-removable. Since $k_{\alpha^*}(i) = \min\{k > i\colon \alpha^*_k < \alpha^*_i\}$ then $z \leqslant \alpha^*_i - \alpha^*_{k_{\alpha^*}(i)} = n - i - \alpha_i - (n - k_{\alpha^*}(i) - \alpha_{k_{\alpha^*}(i)}) = \alpha_{k_{\alpha^*}(i)} - \alpha_i + k_{\alpha^*}(i) - i$.

(iii): Under this condition we have that $\alpha^*_i > 0$. Then, α^* is $(i,1)$-removable, i.e., α is $(i,1)$-insertable. □

Proposition 5.10. *Let α be (i,z_1)-insertable and also (i,z_2)-insertable. Then:*

(i) $z_1 = z_2$ if and only if $\widehat{J}(i,z_1) = \widehat{J}(i,z_2)$.

(ii) $z_1 > z_2$ if and only if $\widehat{J}(i,z_1) < \widehat{J}(i,z_2)$.

Proof. It follows directly by Lemmas 5.6 and 5.7 and by Proposition 5.4. □

Summarizing, given a composition $\alpha \in \mathcal{C}_n$ and an index $i \in [n-1]$, we can determine if α is (i, z)-insertable by following the steps below:

1. Start with α^*.
2. Compute the possible values of z by the range $[1, \alpha_{k_{\alpha^*}(i)} - \alpha_i + k_{\alpha^*}(i) - i]$ (cf. Equation 7).
3. If $\alpha_i + i < n$ then α is $(i, 1)$-insertable.
4. For each $2 \leqslant z \leqslant \alpha_{k_{\alpha^*}(i)} - \alpha_i + k_{\alpha^*}(i) - i$, proceed as follows:

   (a) Start with $\widehat{\alpha}$.

   (b) Compute $c_{i,j}(\widehat{\alpha})$ and $\widehat{J}(i, z)$ (cf. Equation 8).

   (c) If $c_{i,\widehat{J}(i,z)}(\alpha) = \alpha_i - \alpha_{\widehat{J}(i,z)} + z - 1$, then α is (i, z)-insertable. Define α'' by:

   $$\alpha''_i = \alpha_i + z\,,\ \alpha''_{\widehat{J}(i,z)} = \alpha_{\widehat{J}(i,z)} - z + 1,\ \text{and}\ \alpha''_m = \alpha_m,\ \text{for}\ m \neq i\ \text{or}\ \widehat{J}(i, z),$$

   as stated in Lemma 5.8.

Example 7. Let $\alpha = (2, 5, 4, 1, 0, 2, 1) \in \mathcal{C}_8$. Let us determine when α is $(1, z)$-insertable, with $z \in [5]$. Notice that $\alpha^* = (5, 1, 1, 3, 3, 0, 0)$ and $k_{\alpha^*}(1) = \min\{k > 1 \colon \alpha^*_k < 5\} = 2$. Hence, $z \leqslant \alpha_2 - \alpha_1 + k_{\alpha^*}(1) - 1 = 4$, i.e., α is not $(1, 5)$-insertable. Table 5.2 presents the $(1, z)$-insertions α'' of α, for $z \in [4]$. We remember that $c_{1,j}(\alpha) = (0, 0, 0, 0, 1, 2, 2, 2, 2)$. In each case, we have that $c_{i,\widehat{J}(i,z)}(\alpha) = \alpha_i - \alpha_{\widehat{J}(i,z)} + z - 1$.

z	$c_{1,j}(\widehat{\alpha})$	$\widehat{J}(1, z)$	$\alpha''_{\widehat{J}(1,z)}$	α''
1	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{2}, 3)$	8	–	$(\mathbf{3}, 5, 4, 1, 0, 2, 1)$
2	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{2}, 3, 4)$	7	0	$(\mathbf{4}, 5, 4, 1, 0, 2, \mathbf{0})$
3	$(\mathbf{0}, \mathbf{0}, \mathbf{0}, 1, 2, 3, 3, 4, 5)$	3	2	$(\mathbf{5}, 5, \mathbf{2}, 1, 0, 2, 1)$
4	$(\mathbf{0}, \mathbf{0}, 1, 2, 3, 4, 4, 5, 6)$	2	2	$(\mathbf{6}, \mathbf{2}, 4, 1, 0, 2, 1)$

Table 5.2: The $(1, z)$-insertions of $\alpha = (2, 5, 4, 1, 0, 2, 1)$.

Figure 11 shows this process using diagrams. Eventually, after a sequence of inverse of ladder moves, illustrated by the polygonal paths, one produces the covering diagrams formed by inserting a box in the first row, marked by a diamond-shaped box.

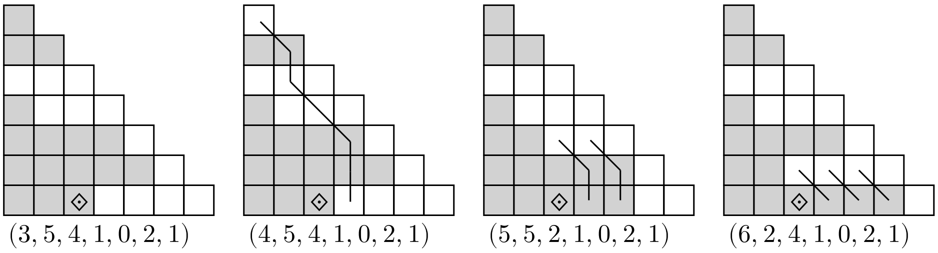


Figure 11: Diagrams of the $(1, z)$-insertions of $\alpha = (2, 5, 4, 1, 0, 2, 1)$.

5.3 Monk's rule

Coşkun-Taşkın (see [4], Theorem 4.1) obtained a result analogous to Theorem 4.3 (or Proposition 4.5) using the language of Tower diagrams constructed from the inverse permutation Lehmer code. As an application, their approach yields an algorithm that describes Monk's rule, which in turn can be used to derive an algorithm for Pieri's rule.

Similarly, Theorem 4.3 provides an insertion process on compositions that leads to an algorithm for computing Monk's Rule for permutations (see Lascoux-Schützenberger [7] for the Schubert polynomials theory). Specifically, for a permutation $w \in S_\infty$, let $\mathfrak{S}_w$ denote the corresponding Schubert polynomial. Then:

$$\mathfrak{S}_{s_r}\mathfrak{S}_w = \sum_{\substack{i \leqslant r < j \\ \ell(w\cdot(i,j)) = \ell(w)+1}} \mathfrak{S}_{w\cdot(i,j)}.$$

Define $\mathcal{C}_\infty = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$. In the context of compositions, given $\alpha \in \mathcal{C}_\infty$ and $r > 0$, we need to find all (i, z) such that α is (i, z)-insertable and $i \leqslant r < \widehat{J}(i, z)$. Note that Proposition 5.9(i) provides an explicit criterion to determine whether α is insertable, while Propositions 5.9(ii) and 5.10 further restrict the possible values of z.

Example 8. Let $w = 37621854 \in S_8$ for which we have $L(w) = \alpha = (2, 5, 4, 1, 0, 2, 1)$. Let us compute the Monk's Rule for $r = 4$. Table 5.3 displays the possible insertions (i, z) with $i \leqslant 4$, along with the corresponding values of $\widehat{J}$. Only those satisfying $\widehat{J} \geqslant 5$ contribute to the result.

Hence,

$$\begin{aligned}\mathfrak{S}_{s_4}\mathfrak{S}_{(2,5,4,1,0,2,1)} &= \mathfrak{S}_{(3,5,4,1,0,2,1)} + \mathfrak{S}_{(4,5,4,1,0,2,0)} + \mathfrak{S}_{(2,6,4,1,0,2,1)} \\ &\quad + \mathfrak{S}_{(2,5,5,1,0,2,1)} + \mathfrak{S}_{(2,5,4,2,0,2,1)} + \mathfrak{S}_{(2,5,4,3,0,2,0)} + \mathfrak{S}_{(2,5,4,4,0,0,1)}.\end{aligned}$$

Acknowledgements

We thank the anonymous referees for their comments on the manuscript.

i	$c_{i,j}(\alpha)$	z	$c_{i,j}(\widehat{\alpha})$	$\widehat{J}$	α''
1	$(0,0,0,0,1,2,2,2,2)$	1	$(0,0,0,0,1,2,2,2,3)$	8	$(3,5,4,1,0,2,1)$
		2	$(0,0,0,0,1,2,2,3,4)$	7	$(4,5,4,1,0,2,0)$
		3	$(0,0,0,1,2,3,3,4,5)$	3	–
		4	$(0,0,1,2,3,4,4,5,6)$	2	–
2	$(0,0,0,1,2,3,3,4,5)$	1	$(0,0,0,1,2,3,4,5,6)$	6	$(2,6,4,1,0,2,1)$
3	$(0,0,0,0,1,2,2,3,4)$	1	$(0,0,0,0,1,2,3,4,5)$	6	$(2,5,5,1,0,2,1)$
4	$(0,0,0,0,0,1,1,1,1)$	1	$(0,0,0,0,0,1,1,1,2)$	8	$(2,5,4,2,0,2,1)$
		2	$(0,0,0,0,0,1,1,2,3)$	7	$(2,5,4,3,0,2,0)$
		3	$(0,0,0,0,0,1,2,3,4)$	6	$(2,5,4,4,0,0,1)$

Table 5.3: The Monk's Rule for $\alpha = (2,5,4,1,0,2,1)$ and $r = 4$.

References

[1] N. Bergeron and S. Billey, RC-Graphs and Schubert polynomials, *Exp. Math.* 2(1993), 257–269.

[2] A. Björner and F. Brenti, *Combinatorics of Coxeter Groups*, Springer-Verlag, 2005.

[3] D. Bolognini and P. Sentinelli, The Lehmer complex of a Bruhat interval, *ArXiv preprint*, 2025; `arXiv:2501.03037`.

[4] O. Coşkun and M. Taşkın, Tower diagrams and Pieri's rule, *Discrete Math.* 341(11) (2018), 3089–3105. 2018.

[5] H. Denoncourt, A refinement of weak order intervals into distributive lattices, *Ann. Comb.* 17 (2013), 655–670.

[6] T. Ikeda and H. Naruse, Excited Young diagrams and equivariant Schubert calculus, *Trans. Amer. Math. Soc.* 361(10) (2008), 5193–5221. 2008.

[7] A. Lascoux and M.-P. Schützenberger, Polynômes de Schubert, *C.R. Acad. Sci. Paris Sér. I Math*, 29 (1982), 447–450.

[8] L. Manivel, *Symmetric Functions, Schubert Polynomials, and Degeneracy Loci*, Amer. Math. Soc., 2001.

[9] A. Matszangosz, On the cohomology ring of real flag manifolds: Schubert cycles, *Math. Ann.* 381 (2021), 1537–1588.

[10] L. Rabelo and L. A. B. San Martin, Cellular homology of real flag manifolds, *Indag. Math.* 30(5) (2019), 745–772.

[11] R. Stanley, *Enumerative Combinatorics: Volume 1*, Cambridge University Press, USA, 2nd ed., 2011.

(Received 17 June 2025; revised 18 Mar 2026)