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Forts, (fractional) zero forcing, and Cartesian products of graphs*

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Abstract

The fort hypergraph, the fort number, and the fractional zero forcing number are introduced. The fort number and the fractional zero forcing number are determined for well-known graph families and Vizing-like lower bounds are established for these parameters. Results on hypergraph transversals and matchings and their fractional versions are applied to the

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zero forcing number, fort number, and fractional zero forcing number, establishing a Vizing-like lower bound for the zero forcing number of a Cartesian product of graphs for certain families of graphs. A family of graphs achieving this lower bound is exhibited.

1 Introduction

In this work we establish lower bounds for the zero forcing number of the Cartesian product of graphs under certain conditions. *Zero forcing* is a process on a graph, where vertices are either filled or unfilled. An initial set of filled vertices can force unfilled vertices to become filled by applying a color change rule. While there are many color change rules (see [13, Chapter 9]), we will use the *(standard) zero forcing color change rule* which states that a filled vertex u can change an unfilled vertex w to filled if w is the only unfilled neighbor of u; this is referred to as *u forcing w* and repeated application of this color change rule is referred to as a zero forcing process. Since the vertex set of a graph is finite, there comes a point in which no more forcings are possible. If at this point all vertices of the graph G are filled, then we say that the initial set of filled vertices is a *zero forcing set* of G. The *zero forcing number* of G, denoted $\mathrm{Z}(G)$, is the minimum cardinality of a zero forcing set of G.

One of the original applications for zero forcing is to bound the maximum nullity over all symmetric matrices associated with a graph. More precisely, given a graph G with $V(G) = \{1, 2, \dots, n\}$ and edge set $E(G)$, the *set of symmetric matrices associated with G* is defined by

$$\mathcal{S}(G) = \{A = [a_{ij}] \in S_n(\mathbb{R})\colon \text{for all } i \neq j, a_{ij} \neq 0 \Leftrightarrow \{i, j\} \in E(G)\}$$

where $S_n(\mathbb{R})$ denotes the set of $n \times n$ real symmetric matrices. Let $\operatorname{null} A$ denote the nullity of A. The *maximum nullity* of G is defined by $\mathrm{M}(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G)\}$. The original study [3] showed that the zero forcing number is an upper bound for the maximum nullity: $M(G) \leq \mathrm{Z}(G)$. The *Cartesian product* of two graphs G and G', denoted $G \Box G'$, has vertex set $V(G) \times V(G')$, where $A \times B = \{(a, b) : a \in A, b \in B\}$, and edge set $\{(g_1, g_1')(g_2, g_2') : g_1' = g_2' \text{ and } g_1 g_2 \in E(G)\} \cup \{(g_1, g_1')(g_2, g_2') : g_1 = g_2 \text{ and } g_1' g_2' \in E(G')\}$. That original study gave an upper bound for the zero forcing number of the Cartesian product of graphs [3]: $\mathrm{Z}(G \Box G') \leq \min\{\mathrm{Z}(G)|V(G')|, |V(G)|\,\mathrm{Z}(G')\}$. This upper bound is achieved simply by constructing a zero forcing set of $G \Box G'$ using the zero forcing sets of G or G'. However, a *lower bound* is more elusive as proving a lower bound requires showing that no smaller subset of vertices can possibly be a zero forcing set.

To approach a lower bound, we can appeal to the next theorem of Hogben, Lin and Shader.

Theorem 1.1. [13, Theorem 9.22, Corollary 9.23] *Let G and G' be graphs each of which has an edge. Then*

$$\mathrm{Z}(G \Box G') \geq M(G \Box G') \geq M(G)M(G') + 1.$$

If $\mathrm{M}(G) = \mathrm{Z}(G)$ *and* $\mathrm{M}(G') = \mathrm{Z}(G')$*, then*

$$\mathrm{Z}(G \,\Box\, G') \geq \mathrm{Z}(G)\,\mathrm{Z}(G') + 1.$$

All bounds are sharp.

We consider whether the second bound for $\mathrm{Z}(\cdot)$ in Theorem 1.1 holds for all graphs. This question was originally asked by Hogben, Lin and Shader in [13] and we present it here as a formal conjecture.

Conjecture 1.2. *If G and G' are graphs each containing an edge, then* $\mathrm{Z}(G \Box G') \geq \mathrm{Z}(G)\,\mathrm{Z}(G') + 1$.

If true, this bound would be sharp. We provide examples of large families of graphs that obtain equality in Section 5.

These types of bounds on graph products, known as Vizing-like bounds, are not uncommon in graph theory. Vizing conjectured that $\gamma(G \Box G') \geq \gamma(G)\gamma(G')$ where $\gamma(\cdot)$ is the domination number [24]. Similar bounds have been found for other graph parameters; see, e.g., [20].

As a tool to study Conjecture 1.2, we develop an alternative view of zero forcing using hypergraph transversals. A *hypergraph transversal* (or edge cover) is a set of vertices that intersects every edge. We will let $\tau(H)$ denote the *transversal number* of the hypergraph H, which is the cardinality of the minimum transversal. Brimkov, Fast and Hicks first gave this alternative characterization of zero forcing in the context of forts [8].

A *fort* is a nonempty subset of vertices whereby whenever its complement is filled, no more forces are possible. Specifically, a fort is a nonempty subset $F \subseteq V(G)$ such that for each $v \notin F$, $|N_G(v) \cap F| \neq 1$, where $N_G(v)$ or $N(v)$ denotes the open neighborhood of v.

Theorem 1.3 (See [8] and [10])**.** *A set of vertices $S \subseteq V(G)$ is a zero forcing set if and only if S intersects every fort.*

With this characterization of zero forcing, we focus on the *hypergraph of forts*, denoted $\mathcal{F}_G$. (We will more formally define $\mathcal{F}_G$ in Section 2). From this perspective, Theorem 1.3 can be reinterpreted as follows.

Theorem 1.4. *For any graph G,*

$$\mathrm{Z}(G) = \tau(\mathcal{F}_G).$$

This characterization allows the zero forcing number to be computed as the optimal value of the following binary integer program. A *minimal fort* is a fort that does not contain any fort except itself. As presented in Model 2 in [8], all forts are used, but the result is the same when only minimal forts are used, as in the next description:

$$\text{minimize} \quad \sum_{v\in V(G)} x_v \tag{1.1a}$$

$$\text{subject to} \quad \sum_{v\in F} x_v \geq 1, \qquad \text{for all minimal forts } F, \tag{1.1b}$$

$$x_v \in \{0,1\}, \qquad \text{for all } v \in V(G). \tag{1.1c}$$

One advantage of this view, not yet taken before, is that it provides a meaningful interpretation of a *fractional zero forcing number*. This is done by changing the constraint (1.1c) from $x_v \in \{0,1\}$ to $x_v \in [0,1]$. We will denote the resulting optimal value of this relaxation as $\mathrm{Z}^*(G)$. Note that this fractional variation of zero forcing is different from a previous one introduced in [14] via a three-color forcing game, which was shown to be equal to the skew forcing number $\mathrm{Z}^-(G)$. This distinction is illustrated in Example 3.3 where it is shown that $\mathrm{Z}^*(K_n) = \frac{n}{2}$, since $\mathrm{Z}^-(K_n) = n-2$.

Returning to our main question, Conjecture 1.2, we leverage known results about Cartesian products of hypergraph transversals. Namely, we show in Theorem 4.10 that $\mathrm{Z}(G\Box G') \geq \mathrm{Z}(G)\,\mathrm{Z}(G')+1$ whenever $\mathrm{Z}(G) = \mathrm{Z}^*(G)$. As a result, we can provide an affirmative answer to Conjecture 1.2 for many families of graphs (by showing that $\mathrm{Z}(G) = \mathrm{Z}^*(G)$ in those cases). It is interesting to compare Theorem 4.10 to Theorem 1.1: While more graphs G' are known for which $\mathrm{Z}(G') = \mathrm{M}(G')$ than for which $\mathrm{Z}(G') = \mathrm{Z}^*(G')$, once such a graph G' is found it has wider application to Cartesian products: By Theorem 4.10, the bound $\mathrm{Z}(G\Box G') \geq \mathrm{Z}(G)\,\mathrm{Z}(G') + 1$ applies without restriction on G, whereas to obtain $\mathrm{Z}(G\Box G') \geq \mathrm{Z}(G)\,\mathrm{Z}(G') + 1$ from Theorem 1.1 requires $\mathrm{Z}(G) = \mathrm{M}(G)$ in addition to $\mathrm{Z}(G') = \mathrm{M}(G')$. For $G = C_5 \circ K_1$ (called the pentasun), it is well known that $Z(G) = 3 > 2 = M(G)$. So if G' is any graph with $\mathrm{Z}(G') = \mathrm{Z}^*(G')$ (examples of such graphs are listed in Table 4.1), then Theorem 4.10 confirms the bound in Conjecture 1.2, while Theorem 1.1 does not.

The additional new parameter defined next also has a natural hypergraph interpretation (see Observation 2.2).

Definition 1.5. The *fort number* of a graph G, denoted by $\mathrm{ft}(G)$, is the maximum cardinality of a collection of disjoint forts in G.

While applying hypergraph results, we also prove the corresponding Vizing-like bounds for all graphs G and G' for the parameters Z^* and ft. Specifically, in Proposition 4.25 and Corollary 4.28 we show that for each pair of graphs G and G',

$$\mathrm{ft}(G\Box G') \geq \mathrm{ft}(G)\,\mathrm{ft}(G') \text{ and } \mathrm{Z}^*(G\Box G') \geq \mathrm{Z}^*(G)\,\mathrm{Z}^*(G').$$

Additional contributions of this study include the following:

- A determination of the exact values of these parameters, including Z^*, for many families of graphs including complete graphs, complete bipartite graphs, cycles, hypercubes, select coronas and more (see Section 3; these results are summarized in Table 6.1).
- A full characterization when $Z^*(G) = Z(G)$ for trees (see Theorem 4.17).
- A result that equality within Conjecture 1.2 is achieved whenever G is a star-clique path (see Section 5).
- Numerous interesting open questions covering many different aspects of this study (summarized in Section 6.3 and Table 6.2).

2 Forts, hypergraph transversals, and fractional zero forcing

In this section we more carefully define the fort hypergraph and use it to define a (new) fractional zero forcing number and fractional fort number, which are equal. These ideas are then used in Section 4 to leverage known results on hypergraph matchings and hypergraph transversals.

Let $H = (V(H), E(H))$ be a hypergraph, where $V(H)$ is a finite nonempty set (called the set of vertices) and the set of edges $E(H)$ is a set of nonempty subsets of $V(H)$. As in [6], we require that $V(H) = \cup_{e \in E(H)} e$. The *degree* of a vertex in a hypergraph is the number of edges that it is an element of. A set of vertices that intersects every edge of H is a *transversal* of H. The *transversal number* of H, denoted by $\tau(H)$, is the minimum cardinality of a transversal of H. A hypergraph is *simple* if no edge is a proper subset of another edge, *uniform* if every edge has the same number of vertices, and *regular* if every vertex is in the same number of edges; the term *k-uniform* refers to a uniform hypergraph in which every edge has k vertices, and analogously for *k-regular*.

Throughout, we will consider the hypergraph of minimal forts. One might think it would be natural to consider the hypergraph of all forts rather than the hypergraph of minimal forts. Such a fort hypergraph is also a reasonable object of study. However, a transversal of all minimal forts is also a transversal of all forts (because every fort contains a minimal fort) and it is convenient to consider only minimal forts. Furthermore, by restricting consideration to minimal forts, a simple hypergraph is obtained.

Definition 2.1. Let G be a graph. The *hypergraph of minimal forts* or *fort hypergraph* of G, denoted $\mathcal{F}_G$, is the hypergraph with

$$E(\mathcal{F}_G) = \{F : F \text{ is a minimal fort of } G\} \text{ and } V(\mathcal{F}_G) = \cup_{F \in E(\mathcal{F}_G)} F.$$

Note that a vertex of G that is not in any minimal fort of G is not a vertex of $\mathcal{F}_G$. However, if v is an isolated vertex of G, then $\{v\}$ is a minimal fort of G, so $v \in V(\mathcal{F}_G)$. A fort hypergraph is simple but need not be uniform nor regular.

For convenience, we may abuse notation and use $\mathcal{F}_G$ to refer to either the set of minimal forts of G or the hypergraph of minimal forts of G. In all cases, the context will be clear. As mentioned in Theorem 1.3, the strong relationship between zero forcing and hypergraph transversals is that $\mathrm{Z}(G) = \tau(\mathcal{F}_G)$.

Given a hypergraph H, a set of disjoint edges is a *matching* of H. A *maximum matching* is a matching of maximum cardinality. The *matching number* of a hypergraph H, denoted here by $\mu(H)$ is the number of edges in a maximum matching.[1]

Observation 2.2. *For any graph G,*

$$\mathrm{ft}(G) = \mu(\mathcal{F}_G).$$

For any hypergraph H, $\mu(H) \leq \tau(H)$ (as one will need at least one vertex from each edge in the matching to cover those edges). In the context of zero forcing, $\mu(\mathcal{F}_G) \leq \tau(\mathcal{F}_G)$ translates to $\mathrm{ft}(G) \leq \mathrm{Z}(G)$.

In analogy with equations (1.1a) – (1.1c), the matching problem for a hypergraph H can be formulated as follows:

$$\text{maximize} \quad \sum_{e \in E(H)} x_e \tag{2.1a}$$

$$\text{subject to} \quad \sum_{e \in E(H):\, v \in e} x_e \leq 1, \quad \text{for all } v \in V(H), \tag{2.1b}$$

$$x_e \in \{0,1\}, \quad \text{for all } e \in E(H). \tag{2.1c}$$

The fractional transversal number and fractional matching number are discussed in [6, Chapter 3] and [22, Chapter 1]. The fractional transversal problem for a hypergraph H can be formulated as a linear program obtained by relaxing constraint (1.1c) to $x_v \in [0,1]$, for all $v \in V(H)$. The fractional matching problem for a hypergraph H is the dual linear program obtained by relaxing constraint (2.1c) to $x_e \in [0,1]$, for all $e \in E(H)$. An assignment of values $x_v \in [0,1]$ to vertices is called a *weighting of vertices* and an assignment of values $x_F \in [0,1]$ to forts (edges of the fort hypergraph) is called a *weighting of forts*.

From a graph-theoretical viewpoint, the transversal problem seeks a set of vertices of minimum cardinality that intersect every edge. Whereas, the fractional-transversal problem seeks to assign a minimum total weight to the vertices such that each edge has weight at least 1. Similarly, the matching problem seeks a maximum collection of disjoint edges. Whereas, the fractional-matching problem seeks to assign a maximum total weight to the edges such that the edges containing a single vertex have a combined weight of at most 1.

Given a hypergraph H, we denote its fractional transversal number by $\tau^*(H)$ and its fractional matching number by $\mu^*(H)$. From here, we can define the fractional zero forcing number and fractional fort number of a graph.

[1]The matching number of H is often denoted by $\nu(H)$ in the literature.

Definition 2.3. For a graph G, the *fractional zero forcing number* is defined by

$$\mathrm{Z}^*(G) = \tau^*(\mathcal{F}_G)$$

or

$$\mathrm{Z}^*(G) = \min\left\{\sum_{v \in V(G)} x_v \colon \sum_{v \in F} x_v \geq 1 \text{ for all } F \in \mathcal{F}_G \text{ and } x_v \geq 0 \text{ for all } v \in V(G)\right\}$$

where $\mathcal{F}_G$ is the hypergraph or set of minimal forts, respectively.

Definition 2.4. For a graph G, the *fractional fort number* is defined by

$$\mathrm{ft}^*(G) = \mu^*(\mathcal{F}_G)$$

or

$$\mathrm{ft}^*(G) = \max\left\{\sum_{F \in \mathcal{F}_G} x_F \colon \sum_{F \in \mathcal{F}_G \colon v \in F} x_F \leq 1, \text{ for all } v \in V(G) \text{ and } x_F \geq 0 \text{ for all } F \in \mathcal{F}_G\right\}$$

where $\mathcal{F}_G$ is the hypergraph or set of minimal forts, respectively.

Remark 2.5. Since the relaxations of the integer programs in (1.1a)–(1.1c) and (2.1a)–(2.1c) are dual linear programs, the duality theorem of linear programming implies that $\tau^*(H) = \mu^*(H)$ [23, Corollary 7.1g]. Thus for all graphs G, we have the following:

$$\mathrm{ft}(G) \leq \mathrm{ft}^*(G) = \mathrm{Z}^*(G) \leq \mathrm{Z}(G). \tag{2.2}$$

It is sometimes convenient to denote a weighting of vertices $v \to x_v$ by a *weight function* $\omega : V(G) \to [0, 1]$ with $\omega(v) = x_v$. We say a weight function ω is *valid* for G if it satisfies the constraint $\sum_{v \in F} x_v \geq 1$ for all $F \in \mathcal{F}_G$. An *optimal* weight function is a valid weight function ω such that $\sum_{v \in V(G)} \omega(v) = \mathrm{Z}^*(G)$.

Most variations of zero forcing are introduced using an alternative color-change rule. However, a natural color-change rule for $\mathrm{Z}^*(G)$ appears elusive. Hence, we ask the following open question.

Question 2.6. *Is there a color-change rule that can be used to compute* Z^**?*

It is known that determining whether $\mathrm{Z}(G) \leq k$ is NP-complete [1, 25], and it is unclear whether computing the parameters $\mathrm{Z}^*(G) = \mathrm{ft}^*(G)$ can be done in polynomial time. Many fractional graph theoretic parameters (e.g., fractional chromatic number, fractional matching, etc.) can be computed in polynomial time as a linear program with (at most) a quadratic number of constraints. In contrast, the fractional fort number and the fractional zero forcing number potentially require understanding all of the minimal forts, for which there may be exponentially many even for trees (see [5]). Hence, the linear program approach does not directly give a polynomial time approach for $\mathrm{Z}^*(G) = \mathrm{ft}^*(G)$. Even so, the theoretical considerations of $\mathrm{Z}^*(G)$ and $\mathrm{ft}^*(G)$ are powerful tools in studying $\mathrm{Z}(G)$ and $\mathrm{ft}(G)$; we leave this question for future research.

Question 2.7. *Can whether* $\mathrm{ft}^*(G) = \mathrm{Z}^*(G) \leq k$ *be determined in polynomial time? Is there an approach toward computing* $\mathrm{ft}^*(G) = \mathrm{Z}^*(G)$ *that does not require potentially enumerating all minimal forts?*

3 Fort number and fractional zero forcing number of families of graphs

We begin this section by determining the fort number and fractional zero forcing number of several families of graphs, including (but not limited to) paths, cycles, complete graphs, and complete bipartite graphs; these results are summarized in Table 6.1. We also introduce some tools for computing fractional zero forcing number. Finally, we examine graphs with the smallest and largest possible fractional zero forcing numbers (among connected graphs of fixed order).

Example 3.1. Let P_n denote a path graph of order n. Since $\mathrm{Z}(P_n) = 1$ (and $\mathrm{ft}(G) \geq 1$ for every graph G), it follows that

$$\mathrm{ft}(P_n) = \mathrm{Z}^*(P_n) = \mathrm{Z}(P_n) = 1.$$

For $n \geq 4$, $\mathcal{F}_{P_n}$ is never regular (since each of the two leaves is in every fort but other vertices are not).

The next lemma is useful for graphs that have forts with cardinality two, including complete graphs. In a graph G, we say that vertices u and w are *twins* if $v \in N_G(u)$ if and only if $v \in N_G(w)$ for $v \neq u, w$.

Lemma 3.2. *Let G be a graph. Then the following hold:*

- $F_0 = \{u, w\}$ *is a fort of G if and only if u and w are twins.*
- *If u and w are twins and F is a fort of G such that $u \in F$ and $w \notin F$, then* $(F \setminus \{u\}) \cup \{w\}$ *is also a fort of G.*
- *Whenever u and w are twins and there is a fort F with $u \in F$ but $w \notin F$, every optimal weight function has* $\omega(u) = \omega(w) = \frac{1}{2}$.

Proof. Observe that $F_0 = \{u, w\}$ is a fort of G if and only if $v \in N(u) \Leftrightarrow v \in N(w)$ for every $v \neq u, w$, which is exactly the statement that u and w are twins.

Now assume u and w are twins, F is a fort of G, $u \in F$, and $w \notin F$. Let $F' = (F \setminus \{u\}) \cup \{w\}$ and let $v \in V(G) \setminus F'$. If $v \neq u$, then $|N(v) \cap F'| = |N(v) \cap F| \neq 1$. If $v = u$, then $|N(v) \cap F'| = |N(w) \cap F| \neq 1$. Therefore, F' is a fort of G. Since F_0 is a fort, $\omega(u) + \omega(w) \geq 1$. Since F and F' are both forts of G it follows that $\omega(u) = \omega(w) = 1/2$ for any optimal weight function ω. □

If every minimal fort has exactly 2 vertices, then $\mathcal{F}_G$ is a 2-uniform hypergraph and thus is a graph, as is the case in the next example.

Example 3.3. Let K_n denote the complete graph of order n, where $n \geq 2$. Recall that $\mathrm{Z}(K_n) = n - 1$. Also, observe that each minimal fort of K_n is pair of two vertices, and any such pair is a minimal fort. Therefore, $\mathrm{ft}(K_n) = \lfloor \frac{n}{2} \rfloor$. Furthermore, Lemma 3.2 implies that an optimal weight function has $\omega(u) = 1/2$ for all vertices $u \in V(K_n)$. Thus, $\mathrm{Z}^*(K_n) = \frac{n}{2}$. Observe that the fort hypergraph of the complete graph is a graph and $\mathcal{F}_{K_n} \cong K_n$.

Example 3.4. Consider the complete bipartite graph $K_{p,q}$ where we have a partition of the vertices V_1, V_2 with $|V_1| = p$, $|V_2| = q$, and $uv \in E$ if and only if $u \in V_1$ and $v \in V_2$.

Assume first that $p, q \geq 2$. Every pair of vertices in V_1 (or pair in V_2) constitutes a fort of $K_{p,q}$. No set of one vertex from each partite set is a fort. Therefore, $\mathrm{ft}(K_{p,q}) = \lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$. Furthermore, since any set of 2 vertices with both in V_1 or both in V_2 is a fort of $K_{p,q}$, Lemma 3.2 implies that the optimal weight function has $\omega(u) = 1/2$ for all vertices $u \in V(K_{p,q})$. Thus $\mathrm{Z}^*(K_{p,q}) = \frac{p+q}{2}$.

For $p = 1$ and $q \geq 2$, the formula for fort number remains valid: $\mathrm{ft}(K_{1,q}) = \lfloor \frac{q}{2} \rfloor$. However, $\mathrm{Z}^*(K_{1,q}) = \frac{q}{2}$, because the only minimal forts are pairs of leaves. In both cases, the fort hypergraph is 2-uniform, and hence is a graph (that is disconnected when $p, q \geq 2$). When $p = q$ or $p = 1$, then the fort hypergraph is regular.

Observation 3.5. *If every pair of distinct minimal forts of G is disjoint, then* $\mathrm{ft}(G) = \mathrm{Z}^*(G) = \mathrm{Z}(G)$.

Example 3.6. Let $\overline{K_n}$ denote the empty graph of order n. Each set consisting of a single vertex is a minimal fort and $\mathcal{F}_{\overline{K_n}}$ is 1-uniform and 1-regular. From Observation 3.5, $\mathrm{ft}(\overline{K_n}) = \mathrm{Z}^*(\overline{K_n}) = \mathrm{Z}(\overline{K_n}) = n$.

The next remarks provide upper and lower bounds on the fractional zero forcing number that are used in determining its value for various families.

Remark 3.7. Suppose that G is a connected graph of order $n \geq 2$. Then each fort of G must contain at least two vertices. Hence, we can weight each vertex $\frac{1}{2}$ to cover each fort with weight at least 1, and it follows that $\mathrm{Z}^*(G) \leq \frac{n}{2}$. Furthermore, by Example (3.3), this bound is sharp.

This bound can be improved when all forts are larger or some vertices are not in any fort: Let φ be the minimum cardinality of a fort and let $|V(\mathcal{F}_G)| = n'$. We see that $\mathrm{Z}^*(G) \leq \frac{n'}{\varphi}$ by considering the valid weight function $\omega(v) = \frac{1}{\varphi}$ for each vertex v in $V(\mathcal{F}_G)$ and $\omega(v) = 0$ for $v \notin V(\mathcal{F}_G)$.

We can obtain a lower bound on the fractional fort number even when only some minimal forts of a graph are known.

Remark 3.8. Let S be a set of s minimal forts of G, and let d denote the largest number of forts in S that contain any one vertex of G (so d is the maximum degree of a vertex in the subhypergraph with edge set S). We see that $\frac{s}{d} \leq \mathrm{ft}^*(G) = \mathrm{Z}^*(G)$ by weighting each fort in S as $\frac{1}{d}$ and every other minimal fort weighted zero. In particular, if m is the number of edges in the fort hypergraph $\mathcal{F}_G$ and Δ is the maximum vertex degree in $\mathcal{F}_G$, then $\frac{m}{\Delta} \leq \mathrm{ft}^*(G)$.

Remark 3.9. By the two previous remarks (and with the notation used there),

$$\frac{m}{\Delta} \leq \mathrm{ft}^*(G) = \mathrm{Z}^*(G) \leq \frac{n'}{\varphi}.$$

If $\frac{m}{\Delta} = \frac{n'}{\varphi}$ for a graph G, then $\mathrm{ft}^*(G) = \mathrm{Z}^*(G) = \frac{n'}{\varphi}$. In particular, if the fort hypergraph of G is both φ-uniform and Δ-regular, then there are $n'\Delta = \varphi m$ edge-to-vertex incidences and thus $\mathrm{ft}^*(G) = \mathrm{Z}^*(G) = \frac{n'}{\varphi} = \frac{m}{\Delta}$.

Example 3.10. Let C_n denote a cycle of order n, where $n \geq 3$. Recall that $\mathrm{Z}(C_n) = 2$. Number the vertices of C_n as $0, 1, \dots, n-1$ in cycle order and perform arithmetic modulo n.

First consider the case $n = 2k$. Notice that the disjoint sets $F_e = \{0, 2, \dots, 2k-2\}$ and $F_o = \{1, 3, \dots, 2k-1\}$ are each forts. Hence, $\mathrm{ft}(C_{2k}) = \mathrm{Z}^*(C_{2k}) = \mathrm{Z}(C_{2k}) = 2$.

Now let $n = 2k+1$. Each fort contains at least $k+1$ vertices, so $\mathrm{Z}^*(C_{2k+1}) \leq \frac{2k+1}{k+1}$ by Remark 3.7. The sets of the form $F_\ell = \{\ell, \ell+2, \dots, \ell+2k = \ell-1\}$ are all minimal forts. There are $2k+1$ such forts and each vertex appears in $k+1$ forts. Thus, $\frac{2k+1}{k+1} \leq \mathrm{Z}^*(C_{2k+1})$ by Remark 3.8. So $\mathrm{Z}^*(C_{2k+1}) = \frac{2k+1}{k+1} < 2 = \mathrm{Z}(C_{2k+1})$. Note also that $\mathrm{ft}(C_{2k+1}) = 1$ because every fort contains at least $k+1$ vertices.

The next two examples make use of coronas of graphs. Let G_1 and G_2 be graphs. The *corona* of G_1 with G_2, denoted $G_1 \circ G_2$, is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and joining the ith vertex of G_1 to every vertex in the ith copy of G_2.

Example 3.11. Let G_1 be a graph of order r, let $\widehat{G} = G_1 \circ 2K_1$, and let F_i be the set of two leaves adjacent to the ith vertex of G_1 for $i = 1, \dots, r$. Then F_i is a minimal fort and these forts are disjoint, so $\mathrm{ft}(\widehat{G}) \geq r$. Since choosing one leaf from each F_i gives a zero forcing set, $\mathrm{Z}(\widehat{G}) \leq r$. Because $\mathrm{ft}(G) \leq \mathrm{Z}^*(G) \leq \mathrm{Z}(G)$ for every graph G, $\mathrm{ft}(\widehat{G}) = \mathrm{Z}^*(\widehat{G}) = \mathrm{Z}(\widehat{G}) = r$.

Example 3.12. Let G_1 be a graph of order $r \geq 2$, $G = G_1 \circ K_2$, and $W = V(G) \setminus V(G_1)$. Then $r \leq \mathrm{ft}(G)$ since each pair of adjacent vertices in W is a fort. Since every fort of G contains at least two vertices of W, we weight each vertex of W as $\frac{1}{2}$ and weight each vertex of G_1 as zero, which shows that $\mathrm{Z}^*(G) \leq r$. Thus $\mathrm{ft}(G) = \mathrm{Z}^*(G) = r$.

The next result is stated in [16] for connected graphs of any order, so does not exclude $G_2 = K_1$, but it is not correct when $G_2 = K_1$ (e.g., $\mathrm{Z}(P_{4r} \circ K_1) = 2r$ is a counterexample). We present a proof that covers the graphs in Example 3.12. The notation $v \to w$ is widely used to indicate v forces w and we use that notation here. In addition, given $U \subseteq V(G)$ we use the notation $G[U]$ to refer to the subgraph with vertex set $V(G[U]) = U$ and edge set $E(G[U]) = \{uv \in E(G) : u, v \in U\}$.

Proposition 3.13. *Let G_1 and G_2 be graphs of orders n_1 and $n_2 \geq 2$, respectively, such that G_2 has no isolated vertices. Then $\mathrm{Z}(G_1 \circ G_2) = n_1\,\mathrm{Z}(G_2) + \mathrm{Z}(G_1)$.*

Proof. Let $G = G_1 \circ G_2$, let $V(G_1) = \{u_1, \ldots, u_{n_1}\}$, let $G_2^{(i)}$ denote the copy of G_2 joined to u_i, let $V(G_2^{(i)}) = \{x_1^{(i)}, \ldots, x_{n_2}^{(i)}\}$, and let $\hat{G}^{(i)} = G\Big[V\left(G_2^{(i)}\right) \cup \{u_i\}\Big]$. First, choose zero forcing sets S_1 for G_1 and S_i for $G_2^{(i)}$. Then $S_1 \cup \bigcup_{i=1}^{n_1} S_2^{(i)}$ is a zero forcing set of G, alternating forces in $G_2^{(i)}$ that have u_i filled with forces in G_1 by u_i such that all vertices of $G_2^{(i)}$ are filled. Thus $\mathrm{Z}(G) \leq n_1 \mathrm{Z}(G_2) + \mathrm{Z}(G_1)$.

Suppose S is a minimum zero forcing set of G. Define $S_1 = S \cap V(G_1)$ and $\hat{S}^{(i)} = S \cap V\left(\hat{G}^{(i)}\right)$. By [13, Proposition 9.16], $\mathrm{Z}\left(\hat{G}^{(i)}\right) = \mathrm{Z}\left(G_2^{(i)}\right) + 1$. Since at most one vertex of $\hat{G}^{(i)}$ can be forced by a vertex outside $\hat{G}^{(i)}$, $|\hat{S}^{(i)}| \geq \mathrm{Z}(G_2)$. Since S is a minimum zero forcing set of G, $|\hat{S}^{(i)}| \leq \mathrm{Z}(G_2) + 1$. Let $I = \{i : |\hat{S}^{(i)}| = \mathrm{Z}(G_2) + 1\}$ and $B = \{u_i : i \in I\}$. We show that B is a zero forcing set of G_1, which implies that $\mathrm{Z}(G) = |S| \geq n_1 \mathrm{Z}(G_2) + \mathrm{Z}(G_1)$.

There are four possible types of forces: 1) $u_i \to u_j$, which requires all vertices of $G_2^{(i)}$ to be filled; 2) $x_k^{(i)} \to x_j^{(i)}$, which requires u_i to be filled; 3) $u_i \to x_j^{(i)}$, which requires every vertex of $G_2^{(i)}$ except $x_j^{(i)}$ to be filled; 4) $x_j^{(i)} \to u_i$, which requires every neighbor of $x_j^{(i)}$ in $G_2^{(i)}$ to be filled. Since G_2 has no isolated vertices, a force $u_i \to x_j^{(i)}$ can be replaced by $x_k^{(i)} \to x_j^{(i)}$ where $x_k^{(i)} \in N_{G_2^{(i)}}(x_j^{(i)})$ without affecting any other forces. If a force $x_j^{(i)} \to u_i$ takes place, it is the first force involving a vertex in $G_2^{(i)}$. Since G_2 has no isolated vertices, we may replace S by $\left(S \setminus \{x_k^{(i)}\}\right) \cup \{u_i\}$ where $x_k^{(i)} \in N_{G_2^{(i)}}(x_j^{(i)})$, and replace $x_j^{(i)} \to u_i$ by $x_j^{(i)} \to x_k^{(i)}$ without affecting any other forces. Thus we may assume every force is one of the first two types. Given that only the first two types of forces are performed, B must be a zero forcing set of G_1. □

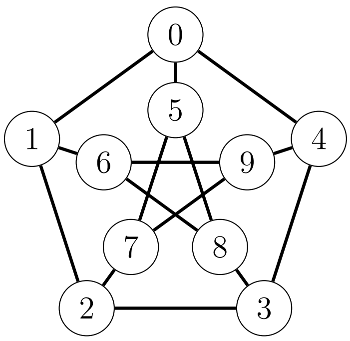

Figure 3.1: Petersen Graph

Example 3.14. Let P be the Petersen graph given in Fig. 3.1. The minimal forts of P are $\{0,1,3,8\}, \{0,1,9,7\}, \{0,2,3,5\}, \{0,2,4,7\}, \{0,8,2,9\}, \{0,3,6,7\}, \{0,8,4,6\}$, $\{0,9,5,6\}, \{1,2,4,9\}, \{8,1,2,5\}, \{1,3,4,6\}, \{1,3,5,9\}, \{8,1,4,7\}, \{1,5,6,7\}$, $\{9,2,3,6\}, \{2,4,5,6\}, \{8,2,6,7\}, \{3,4,5,7\}, \{8,9,3,7\}, \{8,9,4,5\}$ [19].

Since $\mathcal{F}_P$ is 4-uniform and 8-regular, we have $\mathrm{Z}^*(P) = \mathrm{ft}^*(P) = \frac{20}{8} = 2.5 = \frac{10}{4}$ and since we can find two disjoint forts in the list, we also have $\mathrm{ft}(P) = 2$. The zero forcing number of the Petersen graph P is $\mathrm{Z}(P) = 5$ [3].

Let $Q_d = \underbrace{K_2 \square K_2 \square \cdots \square K_2}_{d \text{ times}}$ denote the *hypercube* of dimension d. The *failed zero forcing number* of a graph G is the largest cardinality of a set of vertices of G that is not a zero forcing set. Note that the failed zero forcing number of G is equal to the order minus the minimum cardinality of a fort of G, because $V(G) \setminus F$ is a failed zero forcing set for every fort F, and $V(G) \setminus W$ is a fort for every maximal failed zero forcing set W.

Proposition 3.15. *For $d \geq 2$, $\mathrm{Z}^*(Q_d) = \frac{2^d}{d}$.*

Proof. First we show that the neighborhood $N(v)$ is a fort for any vertex v. Observe that the argument is the same for every vertex. We use the d-tuple representation with $v = 00\ldots0$, so $u \in N(v)$ if and only if there is exactly one 1 in u. Then $w \in V(Q_d) \setminus N(v)$ has a neighbor in $N(v)$ if and only if w has exactly two 1s or $w = v$. In either case, w has at least two neighbors in $N(v)$. Then $\frac{2^d}{d} \leq \mathrm{Z}^*(Q_d)$ by Remark 3.8 using the 2^d forts of the form $N(x)$ for $x \in V(Q_d)$ because each $y \in V(Q_d)$ is in exactly d such forts since y has d neighbors.

By [2, Theorem 4.2], the failed zero forcing number of Q_d is $2^d - d$. Therefore, the minimum forts of Q_d have cardinality d. Thus $\mathrm{Z}^*(Q_d) \leq \frac{2^d}{d}$ by Remark 3.7. □

The next result shows that the difference between $\mathrm{ft}(G)$ and $\mathrm{Z}^*(G)$ can be (at least) $\frac{n}{6}$ asymptotically, just as K_n shows the difference between $\mathrm{Z}(G)$ and $\mathrm{Z}^*(G)$ can be (at least) $\frac{n}{2}$ asymptotically (where n is the order of G). The join of disjoint graphs G and G' is denoted by $G \vee G'$.

Proposition 3.16. *For $s \geq 2$, let $G = sK_3 \vee K_1$ and let $n = 3s + 1$. Then $\mathrm{ft}(G) = \frac{n-1}{3} + 1$, $\mathrm{Z}^*(G) = \frac{n-1}{2}$, and $\mathrm{Z}(G) = \frac{2(n-1)}{3} + 1$.*

Proof. Let c be the vertex of the K_1, so $\deg_G c = 3s$. Label the vertices of the jth copy of K_3 by x_j, y_j, z_j. Then each of $\{x_j, y_j\}$, $\{x_j, z_j\}$, and $\{y_j, z_j\}$ is a fort for $j = 1, \ldots, s$; we call such a fort a *standard* 2*-fort*. Any set of the form $\{c, w_1, \ldots, w_s\}$ where $w_j \in \{x_j, y_j, z_j\}$ is a fort; we call such a fort a *standard* $(s+1)$*-fort*. By choosing forts $\{x_j, y_j\}, j = 1, \ldots, s$ and $\{c, z_1, \ldots, z_s\}$, we see that $\mathrm{ft}(G) \geq s + 1$.

Now consider a minimal fort F that is not a standard 2-fort. Since F must have at least two vertices, it has some vertex other than c; without loss of generality, $z_1 \in F$. Note that F cannot contain two elements of $\{x_j, y_j, z_j\}$ for any $j = 1, \ldots, s$ or it would contain a standard 2-fort. Thus $x_1, y_1 \notin F$. Since $z_1 \in N_G(x_1)$, x_1 must have another neighbor in F, i.e., $c \in F$. Now for $j = 2, \ldots, s$, $c \in N(x_j)$, so at least one of x_j, y_j, z_j must be in F. Since F is minimal, F is a standard $(s+1)$-fort. Thus $\mathrm{ft}(G) = s + 1 = \frac{n-1}{3} + 1$.

For each $j = 1, \ldots, s$ and each vertex $w_j \in \{x_j, y_j, z_j\}$, w_j is contained in a standard 2-fort, so by Lemma 3.2, $\omega(w_j) = \frac{1}{2}$ for any optimal weight function ω.

Starting with $\omega(w_j) = \frac{1}{2}$ for each vertex $w_j \in \{x_j, y_j, z_j\}$ and assigning $\omega(c) = 0$ results in a valid weight function, which is optimal. Thus $\mathrm{Z}^*(G) = \frac{n-1}{2}$.

A set that contains c and two vertices from each copy of K_3 is a zero forcing set so $\mathrm{Z}(G) \leq 2s + 1$. Any zero forcing set B must contain an element of each fort. Thus $2s$ elements are needed by the standard 2-forts. With exactly these $2s$ vertices in B, there is a standard $(s + 1)$-fort that does not contain an element of B. Thus, $\mathrm{Z}(G) \geq 2s + 1$. □

Remark 3.17. Observe that fort number and fractional zero forcing number sum over the connected components. Furthermore, isolated vertices and connected components with order at least two behave very differently, because $\mathrm{ft}(K_1) = \mathrm{Z}^*(K_1) = 1$, whereas $\mathrm{ft}(G) \leq \mathrm{Z}^*(G) \leq \frac{n}{2}$ for a connected graph of order $n \geq 2$.

As is done in the study of zero forcing number, we can focus on connected graphs of order at least two and then infer results for all graphs by Remark 3.17.

Finally, we characterize graphs having the lowest possible value of fractional zero forcing number and highest possible fort number and fractional zero forcing number. Note that $\mathrm{Z}^*(G) \geq 1$ for every graph G and this bound is achieved by P_n (cf. Example 3.1). The next result shows that paths are the only graph achieving this lowest value.

Proposition 3.18. *For a graph G, $\mathrm{Z}^*(G) = 1$ if and only if G is a path graph.*

Proof. Suppose that $\mathrm{Z}^*(G) = 1$. In what follows, we show that there exists a vertex $v \in V(G)$ such that $v \in F$ for all $F \in \mathcal{F}_G$. Then $\{v\}$ intersects every fort and so is a zero forcing set. Thus $\mathrm{Z}(G) = 1$, which implies G is a path graph [13, Theorem 9.12(1)]. To this end, suppose that no such $v \in V(G)$ exists. Let ω be an optimal weight function and let $F \in \mathcal{F}_G$. Then, $\sum_{v \in F} \omega(v) \geq 1$ and there exists a vertex $u \in F$ with $\omega(u) > 0$. Since u is not in every fort, there exists a fort $\hat{F} \in \mathcal{F}_G$ such that $u \notin \hat{F}$. Then $\sum_{v \in V(G)} \omega(v) \geq \sum_{v \in \hat{F}} \omega(v) + \omega(u) \geq 1 + \omega(u) > 1$, which contradicts $\mathrm{Z}^*(G) = 1$. □

Note that $\mathrm{ft}(G) \geq 1$ for every graph G. This bound is achieved by P_n, C_{2k+1}, and numerous other graphs.

By Remark 3.7, $\mathrm{Z}^*(G) \leq \frac{n}{2}$ for every connected graph of order $n \geq 2$. This bound is achieved by K_n, $K_{p,q}$ with $p, q \geq 2$, and some other graphs. The next result uses forts of order two to characterize graphs with $\mathrm{Z}^*(G) = \frac{n}{2}$.

Proposition 3.19. *Let G be a connected graph of order $n \geq 2$. Then every vertex of G is in a fort of cardinality two if and only if $\mathrm{Z}^*(G) = \frac{n}{2}$.*

Proof. If every vertex of G is in a fort of cardinality two, then $\omega(v) = \frac{1}{2}$ for every vertex v is an optimal weight function by Lemma 3.2. If z is not in any fort of cardinality two, then $\omega(v) = \frac{1}{2}$ for every vertex $v \neq z$ and $\omega(z) = 0$ is a valid weight function, so $\mathrm{Z}^*(G) < \frac{n}{2}$. □

Proposition 3.20. *Let G be a graph of order n with no isolated vertices. Then $\mathrm{ft}(G) = \frac{n}{2}$ if and only if all of the following conditions are true:*

(1) *$n = 2k$ is even,*

(2) *there exists a partition of the vertices of G into $F_i = \{x_i, y_i\}, i = 1, \ldots, k$, and*

(3) *for each $i \neq j$, $S_{i,j} \cap E(G) = S_{i,j}$ or $S_{i,j} \cap E(G) = \emptyset$, where $S_{i,j} = \{x_i x_j, x_i y_j, y_i x_j, y_i y_j\}$.*

Proof. Let G be a graph satisfying conditions (1)–(3). We show that each F_i is a fort: Let $u \notin F_i$. Then there exists $j \neq i$ such that $u \in F_j$. Moreover, $|N(u) \cap F_i| = 2$ if $S_{i,j} \cap E(G) = S_{i,j}$ and $|N(u) \cap F_i| = 0$ if $S_{i,j} \cap E(G) = \emptyset$. So F_i is a fort, and therefore $\mathrm{ft}(G) = \frac{n}{2}$.

Now assume G is a graph such that $\mathrm{ft}(G) = \frac{n}{2}$. Then $n = 2k$ is even and there are disjoint forts $F_i = \{x_i, y_i\}, i = 1, \ldots, k$ that partition the vertices of G. For $i \neq j$, define $S_{i,j} = \{x_i x_j, x_i y_j, y_i x_j, y_i y_j\}$. If $S_{i,j} \cap E(G) = \emptyset$, then there is nothing to prove. So assume $S_{i,j} \cap E(G) \neq \emptyset$, and without loss of generality, $x_i x_j \in E(G)$. Then $|N(x_i) \cap F_j| \neq 1$ implies $x_i y_j \in E(G)$ and $|N(x_j) \cap F_i| \neq 1$ implies $x_j y_i \in E(G)$. Then $|N(y_i) \cap F_j| \neq 1$ implies $y_i y_j \in E(G)$. Thus $S_{i,j} \cap E(G) = S_{i,j}$. □

The graphs K_{2k} and $K_{2p,2q}$ are easy examples of such graphs, as is a graph constructed from the sth *friendship graph* $\mathrm{Fr}_s = sK_2 \vee K_1$, as seen in the next example.

Example 3.21. The graph $G = \mathrm{Fr}_s \vee K_1 = sK_2 \vee K_2$ is an example of a graph satisfying the conditions in Proposition 3.20: Let F_0 denote the set of two vertices each of which has degree $2s + 1$ in G. Then $G - F_0$ consists of s copies of K_2. For $i = 1, \ldots, s$, let $F_i = V(K_2)$ for the ith copy of K_2 in $G - F_0$. Each F_i is a fort for $i = 0, \ldots, s$. Using the notation of Proposition 3.20, $S_{i,j} \cap E(G) = \emptyset$ when $1 \leq i \neq j \leq s$ and $S_{i,j} \cap E(G) = S_{i,j}$ when either $i = 0$ or $j = 0$. Note that while $\mathrm{ft}(G) = s + 1 = \mathrm{Z}^*(G)$, $\mathrm{Z}(G) = s + 2$ (since it is well known that $\mathrm{Z}(\mathrm{Fr}_s) = s + 1$ and adding a vertex adjacent to every other vertex (a universal vertex) raises the zero forcing number by one [13, Theorem 9.5, Proposition 9.16]).

Proposition 3.20 does not cover all graphs G of order n such that $\mathrm{Z}^*(G) = \frac{n}{2}$. There are odd order graphs with this property, including K_{2k+1} and the graphs in the next example.

Example 3.22. Let G be a graph constructed from K_n with $n \geq 4$ by deleting a disjoint set of edges E_0 such that either every vertex is an endpoint of an edge in E_0 or at least two vertices are not endpoints of edges in E_0. Then every vertex is in a 2-element fort because the endpoints of an edge deleted are a fort and any set of two vertices neither of which is an endpoint of a deleted edge is a fort. Thus $\mathrm{Z}^*(G) = \frac{n}{2}$.

4 Bounds on the zero forcing number of the Cartesian product

In this section, we apply results on fractional zero forcing number to establish lower bounds on the zero forcing number of the Cartesian product of two graphs.

For two hypergraphs, H_1, H_2, we define $H_1 \times H_2$ to be the hypergraph with

$$V(H_1 \times H_2) = V(H_1) \times V(H_2) \text{ and } E(H_1 \times H_2) = \{e_1 \times e_2 \mid e_1 \in E(H_1), e_2 \in E(H_2)\}.$$

Proposition 4.1. *Let G and G' be graphs, let F_G be a fort of G, and let $F_{G'}$ be a fort of G'. Then $F_G \times F_{G'}$ is a fort of $G \Box G'$.*

Proof. Note that $x = (v, v') \in V(G \Box G') \setminus (F_G \times F_{G'})$ implies $v \notin F_G$ or $v' \notin F_{G'}$. Observe first that if both $v \notin F_G$ and $v' \notin F_{G'}$, then $|N_{G \Box G'}(x) \cap (F_G \times F_{G'})| = 0$. So without loss of generality suppose $v \notin F_G$ and $v' \in F_{G'}$. Let $|N_G(v) \cap F_G| = k$, so $k \neq 1$. For each $w \in F_G$, $(w, v') \in F_G \times F_{G'}$ is a neighbor of $x = (v, v')$ if and only if w is a neighbor of v in G. So, it follows that $|N_{G \Box G'}(x) \cap (F_G \times F_{G'})| = k \neq 1$. Thus, $F_G \times F_{G'}$ is a fort of $G \Box G'$. □

The next result is immediate from the previous proposition.

Corollary 4.2. *For graphs G and G',*

$$\tau(\mathcal{F}_G \times \mathcal{F}_{G'}) \leq \mathrm{Z}(G \Box G'), \tau^*(\mathcal{F}_G \times \mathcal{F}_{G'}) \leq \mathrm{Z}^*(G \Box G'), \text{ and } \mu(\mathcal{F}_G \times \mathcal{F}_{G'}) \leq \mathrm{ft}(G \Box G').$$

Theorem 4.3. [6, Theorem 15 (Chapter 3)] *For two hypergraphs, H and H',*

$$\mu(H)\mu(H') \leq \mu(H \times H') \leq \tau^*(H)\mu(H') \leq \tau^*(H)\tau^*(H')$$
$$= \tau^*(H \times H') \leq \tau^*(H)\tau(H') \leq \tau(H \times H') \leq \tau(H)\tau(H')$$

Using the hypergraphs $H = \mathcal{F}_G$ and $H' = \mathcal{F}_{G'}$ and taking into account Observation 2.2 and Theorem 1.4, the following string of inequalities is given by Theorem 4.3.

Theorem 4.4. *For two graphs, G and G',*

$$\mathrm{ft}(G)\,\mathrm{ft}(G') \leq \mu(\mathcal{F}_G \times \mathcal{F}_{G'}) \leq \mathrm{Z}^*(G)\,\mathrm{ft}(G') \leq \mathrm{Z}^*(G)\,\mathrm{Z}^*(G')$$
$$= \tau^*(\mathcal{F}_G \times \mathcal{F}_{G'}) \leq \mathrm{Z}^*(G)\,\mathrm{Z}(G') \leq \tau(\mathcal{F}_G \times \mathcal{F}_{G'}) \leq \mathrm{Z}(G)\,\mathrm{Z}(G'),$$

where $\mathcal{F}_G$ and $\mathcal{F}_{G'}$ are the hypergraphs of minimal forts of G and G' respectively.

The following corollary is an immediate consequence of Theorem 4.4.

Corollary 4.5. *Let G and G' be graphs.*

1. *If $\mu(\mathcal{F}_G \times \mathcal{F}_{G'}) = \tau^*(\mathcal{F}_G \times \mathcal{F}_{G'})$, then $\mathrm{ft}(G') = \mathrm{Z}^*(G')$.*
2. *If $\mu(\mathcal{F}_G \times \mathcal{F}_{G'}) = \tau(\mathcal{F}_G \times \mathcal{F}_{G'})$, then $\mathrm{ft}(G') = \mathrm{Z}^*(G') = \mathrm{Z}(G')$.*

Since $H_1 \times H_2 \cong H_2 \times H_1$, the roles of G and G' can be reversed in Theorems 4.3 and 4.4. Thus Corollary 4.5 remains true with G' replaced by G in the conclusions. The next result is immediate from Theorem 4.4 and Corollary 4.2.

Corollary 4.6. *For two graphs G and G',*

$$\mathrm{Z}^*(G)\,\mathrm{Z}(G') \leq \mathrm{Z}(G\Box G').$$

Note that Corollary 4.6 implies that if $\mathrm{Z}^*(G) = \mathrm{Z}(G)$, then

$$\mathrm{Z}(G)\,\mathrm{Z}(G') \leq \mathrm{Z}(G\Box G').$$

Families of graphs where $\mathrm{Z}^*(G) = \mathrm{Z}(G)$ are discussed in Section 4.1. For now, we note that we can strengthen the bound given above by using the next lemma.

Lemma 4.7. *Let G and G' be graphs each containing an edge. Then $\mathrm{Z}(G\Box G') \geq \tau(\mathcal{F}_G \times \mathcal{F}_{G'}) + 1$.*

Proof. Suppose B is a minimum transversal of $\mathcal{F}_G \times \mathcal{F}_{G'}$, i.e., B is a subset of $V(G\Box G')$ of minimum size such that $B \cap (F \times F') \neq \emptyset$ for each $F \in \mathcal{F}_G$ and $F' \in \mathcal{F}_{G'}$. Let C_G be a nontrivial component of G, $C_{G'}$ be a nontrivial component of G', and let $C = C_G \Box C_{G'}$, which is a component of $G \Box G'$. Now let $(g, g') \in B \cap V(C)$. We will show that there exists $x \in N_G(g)$ such that $(x, g') \notin B$. Once this is done, an identical argument will show that there exists $x' \in N_{G'}(g')$ such that $(g, x') \notin B$, and thus that every vertex in $B \cap V(C)$ has two neighbors in $V(C) \setminus B$. Since B is an arbitrary minimum transversal of $\mathcal{F}_G \times \mathcal{F}_{G'}$ and $V(C) \setminus B$ is a fort of $G\Box G'$, this completes the proof.

Let $\{g_i\}_{i=1}^m$ be an enumeration of $N_G(g)$ and let $g_0 = g$; note that $m \geq 1$ since C_G is nontrivial. If $(g_1, g') \notin B$, then we are done. Otherwise, since B is of minimum size there exists a fort F_1 of G and a fort F' of G' such that $(g_1, g') \in F_1 \times F'$ and $(g_0, g') \notin F_1 \times F'$, i.e., $g_1 \in F_1$ and $g_0 \notin F_1$. Let k be such that $1 \leq k \leq m-2$ and suppose that $\{(g_i, g')\}_{i=1}^k \subseteq B$ and there exists a set of forts $\{F_i\}_{i=1}^k$ of G such that for each F_i we have $g_i \in F_i$ and $g_j \notin F_i$ for $j = 0, \ldots, i-1$. If $(g_{k+1}, g') \notin B$, then we are done. Otherwise, since B is of minimum size there exists a fort F_{k+1} of G such that $g_{k+1} \in F_{k+1}$ and $g_i \notin F_{k+1}$ for $i = 0, \ldots, k$. So we may suppose that $\{(g_i, g')\}_{i=1}^{m-1} \subseteq B$ and there exists a set of forts $\{F_i\}_{i=1}^{m-1}$ of G such that for each F_i we have $g_i \in F_i$ and $\{g_j\}_{j=0}^{i-1} \cap F_i = \emptyset$. If $(g_m, g') \notin B$, then we are done. So suppose, by way of contradiction, that $(g_m, g') \in B$. Then since B is of minimum size it follows that there exists a fort F_m of G such that $g_m \in F_m$ and $g_i \notin F_m$ for $i = 0, \ldots, m-1$. This is a contradiction because it implies that $|N_G(g) \cap F_m| = 1$. Thus it follows that at least one member of $\{(g_i, g')\}_{i=1}^m$ is a member of $V(C) \setminus B$. □

The bound in Lemma 4.7 is sharp as seen by the next example.

Example 4.8. Consider the infinite class of graphs $G = K_r\Box P_s$, for $r, s \geq 2$. Since each pair of vertices in K_r is a fort, if $B \cap F \neq \emptyset$ for each fort $F \in \mathcal{F}_{K_r} \times \mathcal{F}_{P_s}$, then $|B| \geq r - 1$ and so

$$r \leq |B| + 1 \leq \mathrm{Z}(K_r\Box P_s) = r,$$

where the last equality was established in [3].

The bound in Lemma 4.7 does not always have equality, and, as is witnessed by the next example, the gap grows arbitrarily large.

Example 4.9. Consider $P_r \Box P_r$, for $r \geq 3$. Since every fort of P_r contains the endpoints of P_r, for $V(P_r) = \{1, 2, \ldots, r\}$ enumerated in path order, choosing $B = \{(1, 1)\}$ provides a transversal of $\mathcal{F}_{P_r} \times \mathcal{F}_{P_r}$. Thus, $\tau(\mathcal{F}_{P_r} \times \mathcal{F}_{P_r}) + 1 = 2$, but $3 \leq r = \mathrm{Z}(P_r \Box P_r)$, where the last equality was established in [3].

Theorem 4.10. *Let G and G' be graphs each containing an edge, and suppose $\mathrm{Z}(G') = \mathrm{Z}^*(G')$. Then*

$$\mathrm{Z}(G \Box G') \geq \mathrm{Z}(G)\,\mathrm{Z}(G') + 1.$$

In particular, Conjecture 1.2 holds whenever one of the two graphs has its zero forcing number equal to its fractional zero forcing number.

Proof. Let $\mathcal{F}_G$ and $\mathcal{F}_{G'}$ be the hypergraphs of forts for G and G' respectively. Since G and G' each contain nontrivial components, from Lemma 4.7 and Theorem 4.4, it follows that

$$\mathrm{Z}(G \Box G') \geq \tau(\mathcal{F}_G \times \mathcal{F}_{G'}) + 1 \geq \tau(\mathcal{F}_G)\tau^*(\mathcal{F}_{G'}) + 1 = \mathrm{Z}(G)\,\mathrm{Z}^*(G') + 1 = \mathrm{Z}(G)\,\mathrm{Z}(G') + 1.$$

□

Table 4.1 in the next section lists families of graphs G' for which $\mathrm{Z}(G') = \mathrm{Z}^*(G')$, to which Theorem 4.10 applies. The following corollary is an immediate consequence of Theorem 4.10 and Corollary 4.5.

Corollary 4.11. *If there exist graphs G and G', with G' containing an edge, such that $\mu(\mathcal{F}_G \times \mathcal{F}_{G'}) = \tau(\mathcal{F}_G \times \mathcal{F}_{G'})$, then for any graph G'' containing an edge, $\mathrm{Z}(G' \Box G'') \geq \mathrm{Z}(G')\,\mathrm{Z}(G'') + 1$.*

4.1 Graphs for which $\mathrm{Z}^*(G) = \mathrm{Z}(G)$

We begin with a characterization for when trees have $\mathrm{Z}(G) = \mathrm{Z}^*(G)$. Recall a *path cover* of a graph G is a set of vertex-disjoint paths of G such that every vertex is in one of the paths and each path is an induced path in G; the *path cover number* $\mathrm{P}(G)$ is the minimum size of a path cover of G. It is well-known that $\mathrm{Z}(G) \geq \mathrm{P}(G)$ and $\mathrm{P}(G) \geq \frac{\ell(G)}{2}$ where $\ell(G)$ is the number of leaves in G. For trees, however, $\mathrm{Z}(T) = \mathrm{P}(T)$. Trees having $\mathrm{P}(T) = \frac{\ell(G)}{2}$ play a key role, so we begin by examining such trees. Observe that if $\mathfrak{P}$ is a path cover of T such that $|\mathfrak{P}| = \frac{\ell(G)}{2}$, then every path in $\mathfrak{P}$ must contain two leaves of T.

A *high-degree vertex* of a graph is a vertex of degree at least three. A *generalized star* is a tree that has at most one high-degree vertex (a path is a generalized star). A path P in a tree T is a *pendent path* of vertex $u \in V(T)$ if P is a component of $T - u$ and (in T) P is connected to u by one of its end-points. A *pendent generalized star* of a tree T is a connected induced subgraph R of T such that: (a) there is exactly one high-degree vertex u of T in R (u is called the *center* of R); (b) $\deg_T(u) = k + 1$

and exactly k of the components of $T-u$ are pendent paths of u; (c) R is induced by the vertices of these k pendent paths and u. Pendent generalized stars are illustrated in Example 4.13.

It is known that any tree contains a pendent generalized star or is a generalized star [13, Lemma 2.25]. The method of computation of $\mathrm{P}(T)$ in Algorithm 2.26 in [13] removes pendent generalized stars one at a time until what remains is a generalized star.

Proposition 4.12. *Let T be a tree such that $\mathrm{P}(T) = \frac{\ell(T)}{2}$. Then T has a unique minimum path cover.*

Proof. The proof is by induction on $\mathrm{P}(T)$ and the base case $P(T) = 1$ is immediate. So assume $\mathrm{P}(T) \geq 2$. A generalized star R has $\mathrm{P}(R) = \frac{\ell(R)}{2}$ if and only if R is a path. Thus T is not a generalized star, so T has a pendent generalized star R with center u. From the definition of pendent generalized star, u has at least two pendent paths. Since every path in a minimum path cover of T must contain two leaves, u has exactly two pendent paths and R is a path. Any path cover that does not include R as one of its paths has a path that contains at most one leaf of T. So $R \in \mathfrak{P}$ for any minimum path cover $\mathfrak{P}$ of T. Removing the vertices of R results in a smaller tree T' that retains the property that $\mathrm{P}(T') = \frac{\ell(T')}{2}$. Thus T' has a unique minimum path cover by the induction hypothesis, so T also has a unique minimum path cover. □

We define $\mathcal{T}$ to be the family of trees T such that $P(T) = \frac{\ell(T)}{2}$ and no path P in the minimum path cover $\mathfrak{P}$ contains two adjacent high-degree vertices.

Example 4.13. For $i = 1, 2$, the tree T_i shown in Figure 4.1 has $P(T_i) = 3 = \frac{\ell(T_i)}{2}$. The unique minimum path cover $\mathfrak{P}$ for each T_i consists of the horizontal paths. Tree T_1 has no adjacent high-degree vertices on any of these paths and thus $T_1 \in \mathcal{T}$, whereas T_2 has a pair of adjacent high-degree vertices in $P_{(1)}$ and thus $T_2 \notin \mathcal{T}$. Observe that $P_{(2)}$ and $P_{(3)}$ are pendent generalized stars of $T_i, i = 1, 2$ but $P_{(1)}$ is not.

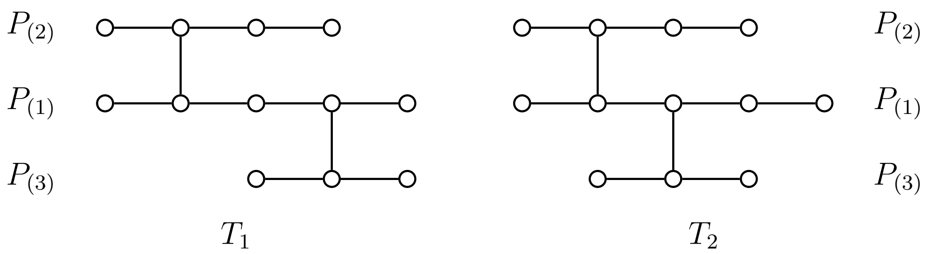

Figure 4.1: Trees $T_1 \in \mathcal{T}$ and $T_2 \notin \mathcal{T}$

Lemma 4.14. *For a tree T, every fort contains two leaves.*

Proof. It suffices to prove the following statement: Let T be a tree. Then, every set of $\ell(T) - 1$ leaves of T is a zero forcing set of T. We proceed via strong induction on the order n of the tree. The base case, $n = 2$, is clear since the only fort of P_2

is $V(P_2)$. For the induction step, let $n \geq 2$, and assume that the result holds for all trees of order between 2 and n. Let T be a tree of order $n+1$ and let S be a set of $\ell(T)-1$ leaves of T. Suppose that each vertex in S is filled and each vertex in $V(T) \setminus S$ is unfilled. Since each vertex in S is a leaf, they each force a vertex in $V(T) \setminus S$. After these forcings are applied, $T - S$ is a tree of order between 2 and n such that all but at most one leaf is filled. Hence, the induction hypothesis implies that all vertices in $T - S$ can be forced and it follows that S is a zero forcing set of T. □

Lemma 4.15. *For any tree T, $\mathrm{Z}^*(T) \leq \frac{\ell(T)}{2} \leq \mathrm{Z}(T)$.*

Proof. By Lemma 4.14 every fort of T must contain at least two leaves, so $\mathrm{Z}^*(T) \leq \frac{\ell(T)}{2}$. Since each path in a path cover covers at most two leaves and $\mathrm{Z}(T) = \mathrm{P}(T)$, it follows that $\frac{\ell(T)}{2} \leq \mathrm{Z}(T)$. □

Lemma 4.16. *Let T be a tree such that $\mathrm{P}(T) = \frac{\ell(T)}{2}$ and let $\mathfrak{P}$ be its minimum path cover. Then every fort of T contains two leaves belonging to the same path of $\mathfrak{P}$.*

Proof. Recall that for any tree, a minimum zero forcing set can be constructed by arbitrarily choosing either end of each path in a minimum path cover (see [3]). By the contrapositive, if a set of vertices is not a zero forcing set, at least one path in $\mathfrak{P}$ must have both its leaves unfilled. Hence, every fort of T contains two leaves from the same path in $\mathfrak{P}$. □

Theorem 4.17. *For any tree T, $\mathrm{Z}^*(T) = \mathrm{Z}(T)$ if and only if $T \in \mathcal{T}$, in which case,* $\mathrm{ft}(T) = \mathrm{Z}^*(T) = \mathrm{Z}(T) = \frac{\ell(T)}{2}$.

Proof. Suppose first that $T \in \mathcal{T}$, so $\mathrm{Z}(T) = \mathrm{P}(T) = \frac{\ell(T)}{2}$. Let $\mathfrak{P} = \{P_{(1)}, \ldots, P_{(\frac{\ell(T)}{2})}\}$ be the minimum path cover. Let $F_{(k)}$ be the vertices of $P_{(k)}$ of degree at most two. Then $F_{(k)}$ is a fort because there are no adjacent high-degree vertices in $P_{(k)}$ so $\mathrm{ft}(T) \geq \frac{\ell(T)}{2}$. Thus, $\mathrm{Z}(T) = \mathrm{Z}^*(T) = \mathrm{ft}(T) = \frac{\ell(T)}{2}$.

It remains to show that $T \notin \mathcal{T}$ implies $\mathrm{Z}^*(T) < \mathrm{Z}(T)$. If $\mathrm{P}(T) > \frac{\ell(T)}{2}$, then $\mathrm{Z}(T) > \frac{\ell(T)}{2} \geq \mathrm{Z}^*(T)$ and we are done. So we assume $\mathrm{P}(T) = \frac{\ell(T)}{2}$. Let $\mathfrak{P} = \{P_{(1)}, \ldots, P_{(\frac{\ell(T)}{2})}\}$ be the minimum path cover. Without loss of generality, let $P_{(1)} \in \mathfrak{P}$ be a path that has two adjacent high-degree vertices u and w.

We show that any fort containing the two leaves of $P_{(1)}$ contains another leaf. Suppose for a contradiction that all other leaves are filled. Then, all those vertices will force until they reach $P_{(1)}$ by forcing u and w. Those two adjacent vertices will then force the rest of $P_{(1)}$ and hence T. Therefore, there can be no fort of T that excludes all leaves not contained in $P_{(1)}$.

Together with Lemma 4.16, we have that every fort of T either contains two leaves of a path that is not $P_{(1)}$ or it contains both leaves of $P_{(1)}$ and at least one additional leaf. Consider the following weighting on the vertices: $x_v = 0$ if v is not a leaf, $x_v = \frac{1}{4}$ if v is a leaf in $P_{(1)}$, and $x_v = \frac{1}{2}$ if v is a leaf not in $P_{(1)}$. For either case for forts, this weighting provides that each fort has a weight sum of at least 1 with a total weight over all vertices of $\frac{\ell(T)}{2} - \frac{1}{2} < \frac{\ell(T)}{2}$. □

Proposition 4.18. *Let T be a tree of order n. Whether or not $T \in \mathcal{T}$ can be determined in $O(n^3)$ operations.*

Proof. It is well-known that the degrees of the vertices can be determined in $O(n)$ operations and this determines the leaves of T as well as $\ell(T)$. Geneson, Haas, and Hogben showed that a minimum path cover of T can be found in $O(n^3)$ operations [11]. If $\mathrm{P}(T) = \frac{\ell(T)}{2}$, then the adjacent high-degree vertex condition can be checked in linear time. □

As shown in Examples 3.10 and 3.11, even cycles and double foliations $G_1 \circ 2K_1$ satisfy $\mathrm{Z}^*(G) = \mathrm{Z}(G)$. We can generalize these examples. A graph G is a *graph of two parallel paths* if $\mathrm{P}(G) = 2$ and the graph can be drawn in the plane in such a way that the paths are parallel line segments, the edges between the two paths do not cross, and each edge is drawn as a straight line segment. A graph that consists of two connected components, each of which is a path, is a graph of two parallel paths, but a single path is not. A *polygonal path* is a graph that can be constructed by starting with a cycle and adding one cycle at a time by identifying an edge of a new cycle with an edge of the most recently added cycle that has a vertex of degree 2. Every polygonal path is a graph on two parallel paths but not conversely.

The *circumference* $c(G)$ of a graph G that is not a forest is the length of the longest cycle in G. A graph is *outerplanar* if it can be drawn on the plane such that its edges intersect only at their endpoints and all vertices lie on the outer (infinite) face. Let G be an outerplanar graph with $\mathrm{P}(G) = 2$ that has a cycle and let $\widehat{G}$ be the subgraph induced by vertices that are part of a cycle; note that $\mathrm{P}(\widehat{G}) = 2$. If $\widehat{G}$ had a cut vertex, then $2 < \mathrm{P}(\widehat{G})$. So the order of $\widehat{G}$ is $c(G)$ and $\widehat{G}$ has a *Hamilton cycle* (a cycle that includes all the vertices). Denote the vertex set of $\widehat{G}$ by $C(G)$.

The next example describes a family of graphs G on two parallel paths that have $\mathrm{Z}(G) = \mathrm{Z}^*(G)$, including polygonal paths of even order.

Example 4.19. It is well-known that a graph G has $\mathrm{Z}(G) = 2$ if and only if G is a graph of two parallel paths [21], [13, Theorem 9.12]. Let $\mathcal{P}$ be the set of graphs G such that G is a graph of two parallel paths and $c(G)$ is even. An example of a graph $G \in \mathcal{P}$ is shown in Figure 4.2. For this graph G, $C(G) = \{u_1, u_2, \dots, u_8\}$.

We can see that every graph $G \in \mathcal{P}$ has $\mathrm{ft}(G) = 2$, which implies that $\mathrm{Z}(G) = \mathrm{Z}^*(G)$: Denote the vertices of $C(G)$ by $u_1, \dots, u_{2k}$. Observe that G can be obtained from $G[C(G)]$ by adding at most four pendent paths with at most one path attached to each endpoint of a minimum path cover (except that if one of the paths is a single vertex of $C(G)$, then two pendent paths may be attached to that vertex).

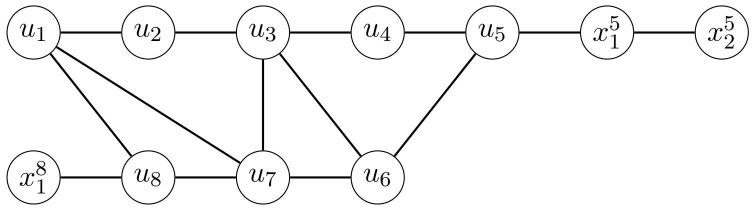

Figure 4.2: A graph $G \in \mathcal{P}$

Define F_e to be the set consisting of $W_e = \{u_2, u_4, \ldots, u_{2k}\}$ and the vertices of any path pendent from a vertex in W_e, and define F_o similarly. Then F_e and F_o are disjoint forts. For the graph G in Figure 4.2, we have $F_e = \{u_2, u_4, u_6, u_8, x_1^8\}$ and $F_o = \{u_1, u_3, u_5, u_7, x_1^5, x_2^5\}$.

Remark 4.20. The definition of a graph of two parallel paths given above is the original one [17], which is standard in the literature. We can see immediately that if G is a graph of two parallel paths, then $\mathrm{P}(G) = 2$ and G is outerplanar, since the drawing of G required by the definition of a graph of two parallel paths is an outerplanar drawing. The converse is also true: Assume G is outerplanar and $\mathrm{P}(G) = 2$. If G is a forest, then G is a graph of two parallel paths. So assume G has a cycle. So $\widehat{G}$ has a Hamilton cycle with vertex set $C(G)$, and we can draw $\widehat{G}$ by placing the vertices in $C(G)$ on a circle and drawing any other edges without crossing and ensuring $\mathrm{P}(\widehat{G}) = 2$. Furthermore, a drawing of G can be obtained from this drawing of $\widehat{G}$ by adding zero, one, two, three, or four pendant paths as described in Example 4.19. The drawing just produced can then be deformed into the form required to show G is a graph of two parallel paths.

Remark 4.21. It follows from known results that whether a graph G is a member of $\mathcal{P}$ can be determined in polynomial time: Recall that G is a graph of two parallel paths if and only if $\mathrm{Z}(G) = 2$, and whether $\mathrm{Z}(G) = 2$ can be determined in polynomial time [25]. So assume G is a graph for which we have determined that $\mathrm{Z}(G) = 2$, which implies $\mathrm{P}(G) = 2$. We can determine $\widehat{G}$, the subgraph induced by vertices on a cycle, by removing up to four pendent paths (which we do by finding a leaf and successively deleting vertices until we reach a vertex of degree at most three). Then G is in $\mathcal{P}$ if and only if the order of $\widehat{G}$ is even.

Example 4.22. Let G_1 be a graph of order $r \geq 2$ with $V(G_1) = \{u_1, \ldots, u_r\}$. Construct $\widetilde{G}$ from G_1 by adding two pendent paths to each vertex u_k of G_1; label the vertices of these paths by $\{x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(i_k)}\}$ and $\{y_k^{(1)}, y_k^{(2)}, \ldots, y_k^{(j_k)}\}$ in path order starting with the leaves. An example of such a graph $\widetilde{G}$ with its disjoint forts is shown in Figure 4.3.

Since $\{x_1^{(1)}, \ldots, x_r^{(1)}\}$ is a zero forcing set of G, $\mathrm{Z}(\widetilde{G}) \leq r$. Since we have disjoint forts $F_k = \{x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(i_k)}, y_k^{(1)}, y_k^{(2)}, \ldots, y_k^{(j_k)}\}$ for $k = 1, \ldots, r$, $\mathrm{ft}(\widetilde{G}) \geq r$. Because $\mathrm{ft}(G) \leq \mathrm{Z}^*(G) \leq \mathrm{Z}(G)$ for every graph G, we have $\mathrm{Z}(\widetilde{G}) = \mathrm{Z}^*(\widetilde{G}) = \mathrm{ft}(\widetilde{G})$. Let $\mathcal{L}$ denote the family of graphs just defined.

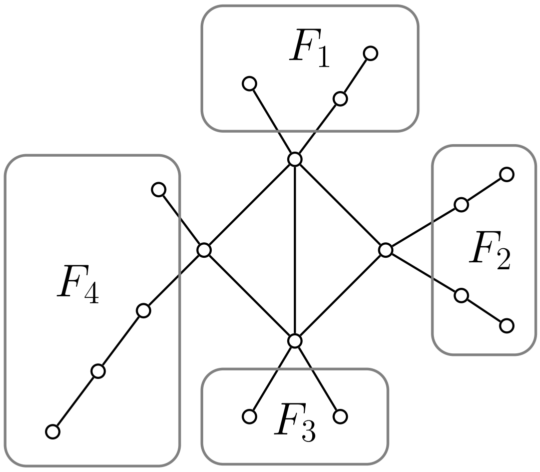

Figure 4.3: A graph $\widetilde{G} \in \mathcal{L}$ with its disjoint forts F_1, F_2, F_3, F_4

Remark 4.23. It can be determined whether a graph G is a member of $\mathcal{L}$ in polynomial time: Observe that $G \in \mathcal{L}$ if and only if every connected component of G is in $\mathcal{L}$, but we do not know what the components are initially, so we find them in the process as needed. It is well-known that the degrees of the vertices can be determined in linear time operations and this identifies any isolated vertices, the leaves of G, the vertices of degree two, and the high-degree vertices of G. If G has an isolated vertex, then $G \notin \mathcal{L}$ and we end the process. Create one list $\mathcal{W}$ of degree-two vertices and another list $\mathcal{H}$ of pairs, where the first entry of each pair is the label of a high-degree vertex of G and the second is initially zero; order the second list by first entries of the pairs.

For each leaf of the original graph G: Follow edges through a path until (a) a high-degree vertex u is reached or (b) another leaf is reached. Remove each degree-two vertex encountered this process from $\mathcal{W}$. In case (a), update $(u, k) \in \mathcal{H}$ to $(u, k+1)$. In case (b), if the leaves are adjacent, then $G \notin \mathcal{L}$ and we end; otherwise, delete this component from G and continue.

Once the previous process ends: If $\mathcal{W} \neq \emptyset$, then $G \notin \mathcal{L}$. Provided $\mathcal{W} = \emptyset$, then examine each pair $(u, k) \in \mathcal{H}$. If there is a pair (u, k) with $k \neq 2$, then $G \notin \mathcal{L}$; otherwise $G \in \mathcal{L}$.

Table 4.1 lists families of graphs G having $\mathrm{Z}(G) = \mathrm{Z}^*(G)$. Note there is some duplication (e.g. an even cycle is an even polygonal path, which is also in $\mathcal{P}$), but the results are displayed this way for ease of use.

It is worth noting that in each example above where $\mathrm{Z}^*(G) = \mathrm{Z}(G)$ it is also the case that $\mathrm{ft}(G) = \mathrm{Z}(G)$. In particular, $\mathrm{Z}^*(T) = \mathrm{Z}(T)$ implies $\mathrm{ft}(T) = \mathrm{Z}(T)$ whenever T is a tree. This motivates the following question.

Question 4.24. *Is it the case that, for all graphs G, $\mathrm{Z}^*(G) = \mathrm{Z}(G)$ if and only if $\mathrm{ft}(G) = \mathrm{Z}(G)$?*

By Theorem 4.17, Question 4.24 is answered in the affirmative for trees. For

result #	G	order	$\mathrm{Z}^*(G) = \mathrm{Z}(G)$
3.1	P_n	n	1
3.10	C_{2k}	$2k$	2
4.17	$T \in \mathcal{T}$		$\frac{\ell(T)}{2}$
4.19	polygonal path, even order	$2k$	2
4.19	$G \in \mathcal{P}$		2
3.11	$G_1 \circ 2K_1$	$3\|V(G_1)\|$	$\|V(G_1)\|$
4.22	$G \in \mathcal{L}$		$\frac{\ell(G)}{2}$

Table 4.1: Graphs G for which $\mathrm{Z}(G) = \mathrm{Z}^*(G)$.

arbitrary graphs G, one direction of Question 4.24 follows from the inequality in (2.2), that is, if $\mathrm{ft}(G) = \mathrm{Z}(G)$ then $\mathrm{Z}^*(G) = \mathrm{Z}(G)$. For the other direction, one might try to show that $\mathrm{Z}^*(G) \leq \frac{1}{2}(\mathrm{ft}(G) + \mathrm{Z}(G))$. If all minimal forts of G have size 2 (see Examples 3.11 and 3.12), then it is true that $\mathrm{Z}^*(G) \leq \frac{1}{2}(\mathrm{ft}(G) + \mathrm{Z}(G))$ because in this case the hypergraph of minimal forts $\mathcal{F}_G$ is a simple graph and Theorem 3 in Chapter 3 of [6] implies that

$$\tau^*(\mathcal{F}_G) \leq \frac{1}{2}(\mu(\mathcal{F}_G) + \tau(\mathcal{F}_G))$$

and the result follows from Theorem 1.4 and Observation 2.2. Unfortunately, $\mathrm{Z}^*(G) \leq \frac{1}{2}(\mathrm{ft}(G) + \mathrm{Z}(G))$ is not true for all G. Indeed, Example 3.10 demonstrates that $\mathrm{Z}^*(C_{2k+1}) = \frac{2k+1}{k+1}$ and $\mathrm{ft}(C_{2k+1}) = 1$. Thus,

$$\mathrm{Z}^*(C_{2k+1}) = \frac{2k+1}{k+1} > \frac{3}{2} = \frac{1}{2}(\mathrm{ft}(C_{2k+1}) + \mathrm{Z}(C_{2k+1})),$$

for $k \geq 2$.

4.2 Bounds on $\mathrm{ft}(G \square G')$ and $\mathrm{ft}^*(G \square G') = \mathrm{Z}^*(G \square G')$

In this section we establish Vizing-like bounds for the fort number and fractional zero forcing number of Cartesian products.

Proposition 4.25. *For all graphs G and G',*

$$\mathrm{Z}^*(G)\ \mathrm{Z}^*(G') \leq \mathrm{Z}^*(G \square G').$$

Proof. By Corollary 4.2 we have $\tau^*(\mathcal{F}_G \times \mathcal{F}_{G'}) \leq \mathrm{Z}^*(G \square G')$. On the other hand, from Theorem 4.3, $\tau^*(\mathcal{F}_G)\tau^*(\mathcal{F}_{G'}) = \tau^*(\mathcal{F}_G \times \mathcal{F}_{G'})$, which completes the proof of the inequality because $\tau^*(\mathcal{F}_G) = \mathrm{Z}^*(G)$. □

The next proposition shows the bound is sharp. Define $G_m = K_{2m} \square K_{2m}$ for $m \geq 2$.

Proposition 4.26. *For $m \geq 2$, any fort of G_m contains at least 4 vertices, and*

$$\mathrm{Z}^*(G_m) = \mathrm{Z}^*(K_{2m})\,\mathrm{Z}^*(K_{2m}) = m^2.$$

Proof. First note that $\mathrm{ft}(K_{2m}) = m$ (see Example 3.3). We show that any fort of G_m contains at least 4 vertices, which implies

$$m^2 = \mathrm{Z}^*(K_{2m})\,\mathrm{Z}^*(K_{2m}) \leq \mathrm{Z}^*(G_m) \leq \frac{|V(\mathcal{F}_{G_m})|}{4} \leq \frac{|V(G_m)|}{4} = m^2,$$

establishing the equality.

Let $V(G_m) = \{(i,j) : 1 \leq i,j \leq 2m\}$ and let F be a fort of G_m. Suppose first that for some $i_0 \in \{1,2,\ldots,2m\}$, $F \cap \{(i_0,j)\}_{j=1}^{2m} = \{(i_0,j_0)\}$. Then for each vertex (i_0,j_1) with $j_1 \neq j_0$, $(i_0,j_1) \notin F$. Since $N_{G_m}((i_0,j_1)) = \{(i_0,j)\}_{j\neq j_1} \cup \{(i,j_1)\}_{i\neq i_0}$ and $|\{(i_0,j)\}_{j\neq j_1} \cap F| = 1$, it follows that $\{(i,j_1)\}_{i\neq i_0} \cap F \neq \emptyset$. Since j_1 was chosen arbitrarily, for each $j \neq j_0$, $\{(i,j)\}_{i\neq i_0} \cap F \neq \emptyset$, and thus $|F| \geq 2m \geq 4$. By symmetry, $|F \cap \{(i,j_0)\}_{i=1}^{2m}| = 1$ implies $|F| \geq 4$.

Now suppose $\left|F \cap \{(i_0,j)\}_{j=1}^{2m}\right| \neq 1$ for each $i_0 \in \{1,2,\ldots,2m\}$ and $|F \cap \{(i,j_0)\}_{i=1}^{2m}| \neq 1$ for each $j_0 \in \{1,2,\ldots,2m\}$. Since $F \neq \emptyset$, for some $i_1,i_2,i_3,j_1,j_2 \in \{1,2,\ldots,2m\}$, with $i_2 \neq i_1$, $i_3 \neq i_1$, and $j_1 \neq j_2$, $\{(i_1,j_1),(i_1,j_2),(i_2,j_1),(i_3,j_2)\} \subseteq F$, and thus $|F| \geq 4$. So for any fort F of G_m, we have that $|F| \geq 4$. □

The gap in the bound in Proposition 4.25 can grow arbitrarily large, as seen in the next example.

Example 4.27. Consider $G = C_m \Box C_m$ with $m \geq 5$. Enumerate the vertices of each cycle by $0,1,\ldots,m-1$ in cyclic order (so $V(G) = \{(i,j) : 0 \leq i,j \leq m-1\}$), and perform arithmetic modulo m. For each k with $0 \leq k \leq m-1$, define the *kth diagonal* $D_k = \{(i,i+k) : i = 0,\ldots,m-1\}$. We see that each D_k is a fort of G: Fix k and i and consider the vertex (i,j). For $j = i+k$, $(i,i+k) \in D_k$. For $j = i+k+1$, $(i,i+k+1) \sim (i,i+k)$ and $(i,i+k+1) \sim (i+1,i+k+1)$, and $j = i+k-1$ is similar. For $j \neq i+k, i+k+1, i+k-1$, $N[(i,j)] \cap D_k = \emptyset$, so D_k is a fort. For $k_1 \neq k_2$, $D_{k_1} \cap D_{k_2} = \emptyset$, so $\mathrm{Z}^*(G) \geq \mathrm{ft}(G) \geq m > 4 \geq \mathrm{Z}^*(C_m)\,\mathrm{Z}^*(C_m)$, with the last inequality following from $\mathrm{Z}^*(C_m) \leq \mathrm{Z}(C_m) = 2$.

Next we present a Vizing-like bound for fort number, which also follows from work of Anderson et al. in [4] as discussed below.

Corollary 4.28. *Let G and G' be graphs. Then $\mathrm{ft}(G) \cdot \mathrm{ft}(G') \leq \mathrm{ft}(G \Box G')$ and the bound is sharp.*

Proof. We have $\mathrm{ft}(G)\,\mathrm{ft}(G') = \mu(\mathcal{F}_G)\mu(\mathcal{F}_{G'}) \leq \mu(\mathcal{F}_G \times \mathcal{F}_{G'}) \leq \mathrm{ft}(G \Box G')$ where the equality follows from Observation 2.2, the first inequality results from Theorem 4.3 and the final inequality results from Corollary 4.2. Proposition 4.26 shows that the bound in Corollary 4.28 is sharp because $m^2 = \mathrm{ft}(K_{2m})\,\mathrm{ft}(K_{2m}) \leq \mathrm{ft}(G_m) \leq \mathrm{Z}^*(G_m) = m^2$. □

Example 4.27 shows the gap can be arbitrarily large since $\mathrm{ft}(C_m) \leq \mathrm{Z}^*(C_m)$.

In [4], Anderson et al. proved results concerning failed zero forcing partitions and their values for Cartesian products of graphs. These results can be interpreted in terms of the fort number, using definitions introduced in [4]. Let G be a graph and let $\Pi = \{\Pi_i\}_{i=1}^{k}$ be a partition of the vertex set of G such that for each $i \in [k]$, $V(G) - \Pi_i$ cannot force G. Then Π is a *failed zero forcing partition of* G. Define z_G to be the maximum number of sets in a failed zero forcing partition of G, i.e.,

$$z_G = \max\{j : |\Pi| = j \text{ and } \Pi \text{ is a failed zero forcing partition of } G\}.$$

The next result is a corollary of Theorem 1.3.

Corollary 4.29. *Let G be a graph. Then $z_G = \mathrm{ft}(G)$.*

Proof. Let $\mathcal{F} = \{F_i\}_{i=1}^{\mathrm{ft}(G)}$ be a collection of disjoint forts of G and let $\Pi = \{\Pi_i\}_{i=1}^{\mathrm{ft}(G)}$ be a partition of $V(G)$ such that for each $i \in [\mathrm{ft}(G)]$, $F_i \subseteq \Pi_i$. Since for each $\Pi_i \in \Pi$, Π_i contains a fort, by Theorem 1.3, $V - \Pi_i$ cannot force G. Thus Π is a failed zero forcing partition of G, and in particular $z_G \geq \mathrm{ft}(G)$.

Next let $\Pi = \{\Pi_i\}_{i=1}^{z_G}$ be a failed zero forcing partition of G. Since for each $i \in [z_G]$, $V(G) - \Pi_i$ does not zero force G, by Theorem 1.3, for each $i \in [z_G]$, Π_i contains a fort F_i. Since Π is a partition of $V(G)$, it follows that for each distinct $i, j \in [z_G]$, $F_i \cap F_j = \emptyset$. Thus, $\{F_i\}_{i=1}^{z_G}$ is a collection of disjoint forts of G, and so $\mathrm{ft}(G) \geq z_G$. □

Thus the next result is equivalent to Corollary 4.28.

Theorem 4.30. [4] *Let G and G' be graphs. Then $z_G \cdot z_{G'} \leq z_{G\Box G'}$.*

4.3 More Lower Bounds on $\mathrm{Z}(G\Box G')$

In this section we present a variety of additional lower bounds on the zero forcing number of a Cartesian product of graphs.

Theorem 4.31. [7] *Let H_1 and H_2 be two hypergraphs. Then $\tau(H_1 \times H_2) \geq \tau(H_1) + \tau(H_2) - 1$.*

Proposition 4.32. *Let G and G' be graphs each with an edge. Then,*

$$\mathrm{Z}(G\Box G') \geq \mathrm{Z}(G) + \mathrm{Z}(G')$$

and this bound is sharp.

Proof. By Lemma 4.7 and Theorem 4.31, we have

$$\mathrm{Z}(G\Box G') \geq \tau(\mathcal{F}_G \times \mathcal{F}_{G'}) + 1 \geq \tau(\mathcal{F}_G) + \tau(\mathcal{F}_{G'}) = \mathrm{Z}(G) + \mathrm{Z}(G').$$

To see that the bound is sharp, note that

$$\mathrm{Z}(K_r\Box P_n) = r = (r-1) + 1 = \mathrm{Z}(K_r) + \mathrm{Z}(P_n)$$

where the equalities involving Z were established in [3]. □

The next example shows that the lower bound in Proposition 4.32 is sometimes better than that in Theorem 1.1.

Example 4.33. Let $G = G' = C_5 \circ K_1$, which is called the pentasun. Then the lower bound for $\mathrm{Z}(G\Box G')$ in Proposition 4.32 is equal to $2Z(G) = 6$, while using Theorem 1.1, the lower bound is equal to $M^2(G) + 1 = 5$.

As is common with sharp bounds on graph parameters (e.g., $\mathrm{M}(G) \leq \mathrm{Z}(G)$), there exist families of graphs for which the gap grows arbitrarily large.

Example 4.34. For $s \geq 2$, we have that $\mathrm{Z}(P_s \Box P_s) = s \geq 2 = \mathrm{Z}(P_s) + \mathrm{Z}(P_s)$ where the equalities were established in [3].

Theorem 4.35 (Lovasz 1975 [18])**.** *Let H be a hypergraph with maximum degree Δ (i.e., each vertex is in at most Δ edges). Then,*

$$\frac{\tau(H)}{\tau^*(H)} \leq 1 + \ln \Delta.$$

Corollary 4.36. *Let G' be a graph containing an edge where every vertex in G' is in at most Δ minimal forts of G'. Then for every graph G that contains an edge,*

$$1 + \frac{\mathrm{Z}(G)\,\mathrm{Z}(G')}{1 + \ln \Delta} \leq \mathrm{Z}(G\Box G')$$

Proof. By Lemma 4.7 and Theorems 4.3 and 4.35 we have

$$\mathrm{Z}(G\Box G') \geq \tau(\mathcal{F}_G \times \mathcal{F}_{G'}) + 1 \geq \tau(\mathcal{F}_G)\tau^*(\mathcal{F}_{G'}) + 1 \geq \frac{\tau(\mathcal{F}_G)\tau(\mathcal{F}_{G'})}{1 + \ln \Delta} + 1. \qquad \Box$$

As with many bounds, there exist families of graphs for which the gap grows arbitrarily large. For instance, by [5], the path graph has an exponential number of minimal forts (in the order of the graph) and every fort of the path graph contains the two pendent vertices.

5 Graphs satisfying $\mathrm{Z}(G\Box G') = \mathrm{Z}(G)\,\mathrm{Z}(G') + 1$

In this section we exhibit a family of graphs attaining the bound $\mathrm{Z}(G\Box G') = \mathrm{Z}(G)\,\mathrm{Z}(G') + 1$. We begin with several definitions. A vertex v of a graph G is called a *cut-vertex* if deleting v and all edges incident to it increases the number of connected components of G. A *block* of a graph G is a maximal connected induced subgraph of G that has no cut-vertices. A graph is *block-clique* (also called 1-chordal) if every block is a clique. Note that a block-clique graph can be constructed iteratively as follows: G_1 is a clique. Given that G_{k-1} is block-clique, then define $G_k = G_{k-1} \cup G'_k$ where G'_k is a clique and $V(G_{k-1}) \cap V(G'_k) = \{v_k\}$ for some vertex v_k. The next result is well-known and useful.

Theorem 5.1. [15] *If G is a block-clique graph, then $\mathrm{Z}(G) = \mathrm{M}(G)$.*

Definition 5.2. A *star-clique path* is a graph G that can be constructed as $G = G_k$ where $G_j = \cup_{i=1}^{j} G'_i$ for $j = 1, \ldots, k$, each G'_i is a star or a clique (each with at least two vertices), and the following conditions are satisfied:

1. $V(G'_{i-1}) \cap V(G'_i) = \{v_i\}$ for some vertex v_i for $i \in \{2, 3, \ldots, k\}$ and $V(G'_j) \cap V(G'_i) = \emptyset$ for $j = 1, \ldots, i-2$.
2. $v_i \notin \{v_2, \ldots, v_{i-1}\}$ for $i \in \{3, \ldots, k\}$.
3. If G'_i is a star for some $i \in \{2, 3, \ldots, k-1\}$, then $\deg_{G'_i}(v_i) = 1$ and $\deg_{G'_i}(v_{i+1}) = 1$. If G'_1 is a star then $\deg_{G'_1}(v_2) = 1$, and if G'_k is a star then $\deg_{G'_k}(v_k) = 1$.

Figure 5.1 shows examples of star-clique paths. Since a star-clique path is a block-clique graph, the next result follows from Theorem 5.1.

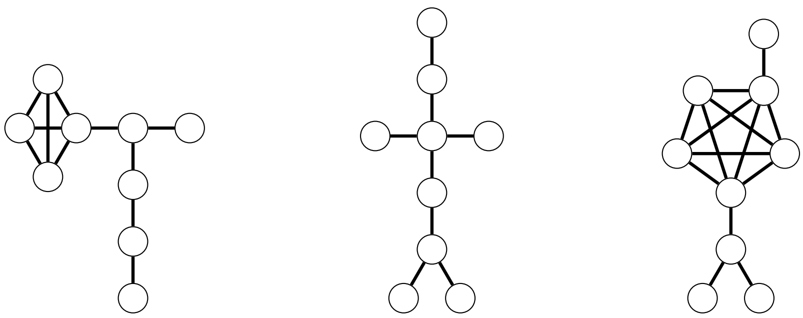

Figure 5.1: Examples of star-clique paths

Corollary 5.3. *If G is a star-clique path, then* $\mathrm{Z}(G) = \mathrm{M}(G)$.

Next, we state an additional result that we use in the proof of Proposition 5.5.

Lemma 5.4. [21] *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \ldots, W_h$ be the vertex sets for the connected components of $G - v$ and for $1 \leq i \leq h$, let $G_i = G[W_i \cup \{v\}]$. Then*

$$\mathrm{Z}(G) \geq \sum_{i=1}^{h} \mathrm{Z}(G_i) - h + 1.$$

Note that for $r \geq 2$, we have the following options for choosing a minimum zero forcing set B' for K_r and $K_{1,r}$: For K_r and any vertices u and w, we may choose B' so that $u \in B'$ and $w \notin B'$ (which implies w does not perform a force when forcing K_r with B'). For $K_{1,r}$ and any leaves u and w, we may choose B' so that $u \in B'$ and $w \notin B'$ (which implies w does not perform a force when forcing $K_{1,r}$ with B'). Thus for a star-clique path G constructed from $G'_1, \ldots, G'_k$ according to Definition 5.2. we can choose zero forcing sets B'_i for G'_i for $i = 1, \ldots, k$ with the following properties: For $i = 2, \ldots, k$, $v_i \in B'_i$ and $v_i \notin B'_{i-1}$ and v_i does not perform a force when forcing G'_{i-1} with B'_{i-1}. Such a zero forcing set of G is called a *canonical star-clique path set*.

Proposition 5.5. *Let G be a star-clique path constructed from $G'_1, \ldots, G'_k$ according to Definition 5.2. Then a canonical star-clique path set is a minimum zero forcing set of G and*

$$\mathrm{Z}(G) = \sum_{i=1}^{k} \mathrm{Z}(G'_i) - k + 1.$$

Proof. First we show by induction on k that $\mathrm{Z}(G) \geq \sum_{i=1}^{k} \mathrm{Z}(G'_i) - k + 1$. This is immediate when $k = 1$. Assume the result is true for $G_{k-1} = \cup_{i=1}^{k-1} G'_i$. Applying Lemma 5.4 to the cut-vertex v_k of G_k together with the induction assumption gives

$$\begin{aligned} \mathrm{Z}(G_k) &\geq \mathrm{Z}(G_{k-1}) + \mathrm{Z}(G'_k) - 2 + 1 \\ &\geq \left(\sum_{i=1}^{k-1} \mathrm{Z}(G'_i) - (k-1) + 1 \right) + \mathrm{Z}(G'_k) - 1 = \sum_{i=1}^{k} \mathrm{Z}(G'_i) - k + 1. \end{aligned}$$

By definition, $B_k = B'_1 \cup \bigcup_{i=2}^{k} (B'_i \setminus \{v_i\})$ is a zero forcing set of G_k. Therefore, $\mathrm{Z}(G) \leq |B_k| = \sum_{i=1}^{k} \mathrm{Z}(G'_i) - k + 1$. □

The next theorem is the main result of this section and gives extremal graphs for Conjecture 1.2.

Theorem 5.6. *Let G be a star-clique path and let $r \geq 2$. Then,*

$$\mathrm{Z}(K_r \Box G) = (r-1)\,\mathrm{Z}(G) + 1 = \mathrm{Z}(K_r)\,\mathrm{Z}(G) + 1.$$

Proof. Let G be a star-clique path constructed from $G'_1, \ldots, G'_k$ according to Definition 5.2. For purposes of this proof, we view $K_{1,1}$ as a clique and $K_{1,2}$ as $G'_i = K_2$ and $G'_{i+1} = K_2$, so we assume a star is $K_{1,s}$ with $s \geq 3$. Note first that $\mathrm{Z}(K_r \Box G) \geq (r-1)\,\mathrm{Z}(G) + 1$ by Theorem 1.1 and Corollary 5.3. Choose a canonical star-clique path set $\hat{B}$ for G. We describe vertices by the vertex label in G and the copy number $j = 1, \ldots, r$. In $K_r \Box G$, we define a set B consisting of the same canonical star-clique path set $\hat{B}$ in the jth copy of G for $1 \leq j \leq r-1$, together with one vertex u in the rth copy of G'_1 such that $u \neq v_2$ and u is not the center of G'_1 if G'_1 is a star. We show that the vertex set B defined is a zero forcing set:

If G'_1 is a clique, then v_2 can be forced by u in the jth of the copy of G'_1 for $j = 1, \ldots, r-1$. Then each of the unfilled vertices $w \neq v_2$ in the rth copy of G'_1 can be forced by its neighbor w in another (fully filled) copy of G'_1. Then v_2 in copy r can be forced by u in copy r. If G'_1 is a star, then the center vertex c can be forced by u in the jth copy of G'_1 for $j = 1, \ldots, r$. Then each unfilled vertex $w \neq v_2$ in the rth copy of G'_1 can be forced by its neighbor w in another copy of G'_1. Finally, v_2 can be forced by c in the jth copy of G'_1 for $j = 1, \ldots, r$.

Now the jth copy of G'_2 contains a zero forcing set of G'_2 for $j = 1, \ldots, r-1$ and v_2 is filled in the rth copy, so we can repeat the process just described for G'_1 for G'_2, and then additional G'_i as needed to fill all vertices of G. This together with Proposition 5.5 proves that $\mathrm{Z}(K_r \Box G) \leq (r-1)\,\mathrm{Z}(G) + 1$. □

Question 5.7. *Does the converse of Theorem 5.6 hold? That is, does*

$$Z(G\square G') = Z(G)\,Z(G') + 1,$$

imply that G is a complete graph of order at least two and G' is a star-clique path or vice versa?

Computations in *Sage* [19] have established that for $n \leq 9$, every graph G' of order n satisfying $Z(K_2\square G') = Z(K_2)\,Z(G') + 1$ is a star-clique path.

6 Concluding remarks

In this section we summarize values of the fort number, fractional zero forcing number, and zero forcing number for various families of graphs in Table 6.1. We also discuss a possible relationship between maximum nullity M and ft, Z^* in Section 6.2 and other open questions in Section 6.3.

6.1 Summary of parameter values for various graph families

In Table 6.1, the result number listed (from within this paper) establishes the fort number and fractional zero forcing number. In each case, the zero forcing number can be found in at least one of the numbered result, [13, Theorems 9.5, 9.12], or the listed reference.

6.2 A possible relationship between maximum nullity M and ft and/or Z^*

Recall that the zero forcing number $Z(G)$ is a well-known upper bound on the maximum nullity of a graph $M(G)$. It is natural to explore the relationship between $M(G)$ and the new parameters $\text{ft}(G)$ and $Z^*(G)$. We are unaware of any graphs for which $\text{ft}(G) > M(G)$, but we point out that $M(G) = Z(G) \geq Z^*(G) \geq \text{ft}(G)$ for all graphs of order 7 or less [9] and for all but one of the graph families listed in Table 6.1. For all but four of these graph families, the values of $M(G)$ and $Z(G)$ appear in [13, Theorems 9.5, 9.12] and/or [3]. The exceptions are $G_1 \circ 2K_1$, $K_1 \vee sK_3$, $K_2 \vee sK_2$, and $G_1 \circ K_2$.

There is a standard technique that can be used to show $M(G) = Z(G)$ for $G' \circ 2K_1$, $K_1 \vee sK_3$ and $K_2 \vee sK_2$ and to show that $Z^*(G) < M(G)$ for $G' \circ K_2$. Recall that the *minimum rank* of G is $\text{mr}(G) = \min\{\text{rank}\, A : A \in \mathcal{S}(G)\}$, and $\text{mr}(G) + M(G) = |V(G)|$. Viewing a graph G as the (possibly overlapping) union of $G_i, i = 1, \ldots, t$, it is known [13] that

$$\text{mr}\left(\bigcup_{i=1}^{t} G_i\right) \leq \sum_{i=1}^{t} \text{mr}(G_i).$$

For $G = G' \circ 2K_1$, we see that G is the union of stars, one for each vertex v of G' centered on v and having as leaves all neighbors of v in G. Since $\text{mr}(K_{1,q}) = 2$ for $q \geq 2$, we have $\text{mr}(G) \leq 2|V(G')|$, which implies $M(G) \geq |V(G')| = Z(G)$.

result #	G	order	$\mathrm{ft}(G)$	$\mathrm{Z}^*(G)$	$\mathrm{Z}(G)$
3.1	P_n	n	1	1	1
3.3	K_n	n	$\lfloor \frac{n}{2} \rfloor$	$\frac{n}{2}$	$n-1$
3.4	$K_{p,q}$, $2 \le p,q$ $K_{1,q}$, $2 \le q$	$p+q$ $1+q$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$ $\lfloor \frac{q}{2} \rfloor$	$\frac{p+q}{2}$ $\frac{q}{2}$	$p+q-2$ $q-1$
3.6	$\overline{K_n}$	n	n	n	n
3.10	C_{2k} C_{2k+1}	$2k$ $2k+1$	2 1	2 $\frac{2k+1}{k+1}$	2 2
3.11	$G_1 \circ 2K_1$	$3\|V(G_1)\|$	$\|V(G_1)\|$	$\|V(G_1)\|$	$\|V(G_1)\|$
3.12, 3.13	$G_1 \circ K_2$	$3\|V(G_1)\|$	$\|V(G_1)\|$	$\|V(G_1)\|$	$\|V(G_1)\|+\mathrm{Z}(G_1)$
3.14	Petersen graph	10	2	$\frac{5}{2}$	5
3.15, [3]	Q_d	2^d		$\frac{2^d}{d}$	2^{d-1}
3.16	$sK_3 \vee K_1$, $s \ge 2$	$3s+1$	$s+1$	$\frac{3s}{2}$	$2s+1$
3.21	$sK_2 \vee K_2$, $s \ge 2$	$2s+2$	$s+1$	$s+1$	$s+2$
4.17	$T \in \mathcal{T}$		$\frac{\ell(T)}{2}$	$\frac{\ell(T)}{2}$	$\frac{\ell(T)}{2}$
4.19	polygonal path, even order	$2k$	2	2	2
4.26, 4.28	$K_{2m} \square K_{2m}, m \ge 2$	$4m^2$	m^2	m^2	$4m^2-4m+2$

Table 6.1: Summary of values of fort number and fractional zero forcing number for families of graphs.

For $G = K_1 \vee sK_3$, we see that G is the union of s copies of K_4, one for each K_3 together with the vertex of the K_1. Since $\mathrm{mr}(K_4) = 1$ and $s \ge 2$, we have $\mathrm{mr}(G) \le s$, which implies $\mathrm{M}(G) \ge 2s + 1 = \mathrm{Z}(G)$. The case $G = K_2 \vee sK_2$ is similar.

Finally consider $G = G' \circ K_2$, where G is the union of G' and $|V(G')|$ copies of K_3. Thus $\mathrm{mr}(G) \le \mathrm{mr}(G') + |V(G')|$. This implies $\mathrm{M}(G) \ge \mathrm{M}(G') + |V(G')|$, which may be less than $|V(G')| + \mathrm{Z}(G')$. However, $\mathrm{ft}(G) = \mathrm{Z}^*(G) = |V(G')| < \mathrm{M}(G)$.

Question 6.1. *Is* $\mathrm{M}(G) \ge \mathrm{ft}(G)$ *for every graph* G*? Is* $\mathrm{M}(G) \ge \mathrm{Z}^*(G)$ *for every graph* G*?*

While we cannot prove even the weaker of the two inequalities, $\mathrm{M}(G) \ge \mathrm{ft}(G)$, we can prove that the fort number is at most the maximum nullity among combinatorially symmetric matrices described by the graph.

A matrix A is *combinatorially symmetric* if $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$. The *graph* $\mathcal{G}(A)$ of a combinatorially symmetric matrix A is the graph with vertices $\{1, \ldots, n\}$ and edges $\{ij : a_{ij} \neq 0, 1 \le i < j \le n\}$; whenever we write $\mathcal{G}(A)$, it is assumed that A is combinatorially symmetric. For a graph G with $V(G) = \{1, 2, \ldots, n\}$, the maximum nullity of combinatorially symmetric matrices described by G, $\mathrm{N}(G) = \max\{\operatorname{null} A : A \in \mathbb{R}^{n\times n}, \mathcal{G}(A) = G\}$, has not been as widely studied as $\mathrm{M}(G)$, but it is known that $\mathrm{N}(G) \leq \mathrm{Z}(G)$ [3].

The next result connects forts and null vectors. For a real vector $\mathbf{x} = [x_i]$, the *support* of $\mathbf{x}$ is the set of indices i for which $x_i \neq 0$.

Theorem 6.2. [12] *Let G be a graph of order n and let $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$. There exists a matrix $A \in S(G)$ such that $A\mathbf{x} = \mathbf{0}$ if and only if the support of $\mathbf{x}$ is a fort of G.*

For a graph G and fort F of G, the *incidence vector* $\mathbf{v} = [v_j]$ of F is the vector such that $v_j = 1$ if $j \in F$ and 0 if $j \notin F$.

Proposition 6.3. *Let G be a graph with disjoint forts $F_1, \ldots, F_k$, where $1 \leq k \leq n$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be the corresponding incidence vectors. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are null vectors for some $A \in \mathbb{R}^{n\times n}$ such that $\mathcal{G}(A) = G$. Thus,* $\mathrm{N}(G) \geq \mathrm{ft}(G)$.

Proof. Without loss of generality, assume the vertices within each fort are numbered consecutively, so that $F_i = \{v_{\ell_{i-1}+1}, \ldots, v_{\ell_i}\}$ where $\ell_0 = 0$. For $i = 1, \ldots, k$, apply Theorem 6.2 to choose matrices $A_i \in \mathbb{R}^{n\times n}$ such that $\mathcal{G}(A_i) = G$ and $A_i\mathbf{v}_i = \mathbf{0}$. Construct matrix A by choosing columns $\ell_{i-1}+1, \ldots, \ell_i$ from A_i (and columns $\ell_k + 1, \ldots, n$ from A_k if necessary). Then $\mathcal{G}(A) = G$ and $A\mathbf{v}_i = \mathbf{0}$ for $i = 1, \ldots, k$ because $A\mathbf{v}_i = A_i\mathbf{v}_i$ is the sum of the columns associated with the indices in F_i and the forts are disjoint. The last statement is immediate. □

6.3 Summary of open questions

This work provides a basis for many ripe and interesting questions. The conjectured lower bound on zero forcing number of a Cartesian product (Conjecture 1.2) remains open in general, and Question 5.7 asks whether Theorem 5.6 characterizes all graphs attaining equality in the bound in Conjecture 1.2.

Beyond the main conjecture, there are plenty of directions of study for these new parameters. Question 4.24 asks whether $\mathrm{Z}^*(G) = \mathrm{Z}(G)$ if and only if $\mathrm{ft}(G) = \mathrm{Z}(G)$ for all graphs G. Question 2.6 asks what color-change rule might be used to compute $\mathrm{Z}^*(G)$. Such a rule could potentially make it easier to compute or bound $\mathrm{Z}^*(G)$ without enumerating many forts. Question 2.7 asks about the complexity of computing $\mathrm{ft}^*(G)$ and $\mathrm{Z}^*(G)$; it is not immediately clear whether or not these fractional graph-theoretical parameters should be computable in polynomial time. Finally, Question 6.1 asks whether $\mathrm{ft}(G)$ and/or $\mathrm{Z}^*(G)$ provide a lower bound for $\mathrm{M}(G)$; this is interesting as the original motivation for the classical zero forcing number $\mathrm{Z}(G)$ was to provide an upper bound on $\mathrm{M}(G)$.

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Reference	Short description
Conjecture 1.2	A Vizing-like lower bound on zero forcing number of a Cartesian product.
Question 2.6	Color-change rule for computing $\mathrm{Z}^*(G)$.
Question 2.7	Complexity of computing $\mathrm{Z}^*(G)$ $(= \mathrm{ft}^*(G))$.
Question 4.24	Relationship between $\mathrm{Z}^*(G) = \mathrm{Z}(G)$ and $\mathrm{ft}(G) = \mathrm{Z}(G)$.
Question 5.7	Characterization of graphs attaining equality in the bound in Conjecture 1.2.
Question 6.1	Whether $\mathrm{ft}(G)$ and/or $\mathrm{Z}^*(G)$ bound $M(G)$ from below.

Table 6.2: A summary of open questions presented throughout this article.

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