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Permutation forcing $(0,1)$-matrices

Richard A. Brualdi

Department of Mathematics, University of Wisconsin
Madison, WI 53706, U.S.A.
brualdi@math.wisc.edu

Lei Cao

Department of Mathematics, Nova Southeastern University
Ft. Lauderdale, FL 33314, U.S.A.
lcao@nova.edu

John L. Goldwasser*

Department of Mathematics, West Virginia University
Morgantown, WV 26505, U.S.A.
jgoldwas@math.wvu.edu

Abstract

Let A be an $n \times n$ $(0,1)$-matrix with a nonzero permanent and let k be an integer with $2 \leq k \leq n$. Then A is $12\cdots k$-permutation forcing provided that every permutation matrix with $P \leq A$ (entrywise inequality) contains a $12\cdots k$-pattern. After relating and contrasting our investigations to previous work, we conjecture that the number of 0's in an $n \times n$ $12\cdots k$-permutation forcing $(0,1)$-matrix is at least $n(k-2)-k(k-3)/2$. If $k = 2, 3$ or n, this follows easily. We prove the conjecture for $k = n-1$.

1 Introduction

Let k and n be positive integers with $1 \leq k \leq n$. Let $\mathcal{S}_n$ be the set of permutations σ_n of $\{1, 2, \ldots, n\}$. Corresponding to the set $\mathcal{S}_n$ is the set $\mathcal{P}_n$ of $n \times n$ permutation matrices:

$$\sigma_n = i_1 i_2 \cdots i_n \in \mathcal{S}_n \leftrightarrow P_{\sigma_n} = [p_{kl}] \text{ where } p_{kl} = 1 \text{ if } i_k = l \text{ and } 0 \text{ otherwise.}$$

Now let π_k be a permutation of $\{1, 2, \ldots, k\}$. Corresponding to π_k, there are two natural sets of permutations which partition $\mathcal{S}_n$ (equivalently, permutation matrices which partition $\mathcal{P}_n$):

* Corresponding author.

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1. $\mathcal{S}_n(\pi_k) = \{\sigma_n : \sigma_n$ contains the pattern $\pi_k\}$; σ_n *forces* π_k, that is, σ_n has a subsequence in the same relative order as π_k.

2. $\mathcal{S}_n(\overline{\pi}_k) = \{\sigma_n : \sigma_n$ does not contain the pattern π_k $\}$; σ_n *avoids* π_k, that is no subsequence of σ_n is in the same relative order as π_k.

The corresponding subsets of $\mathcal{P}_n$ are $\mathcal{P}_n(\pi_k)$ and $\mathcal{P}_n(\overline{\pi}_k)$. If the permutation σ_n, and its corresponding permutation matrix P_{σ_n}, contain the pattern π_k, we then call σ_n a *π_k-permutation* and P_{σ_n} a *π_k-permutation matrix*.

Permutations avoiding a specified pattern is an important topic in combinatorics [1]. The basic idea can be extended from one permutation to the set $\mathcal{P}(A)$ of all permutations determined by an $n \times n$ $(0,1)$-matrix A, namely, the set $\mathcal{P}(A)$ of all $n \times n$ permutation matrices P such that $P \leq A$ (componentwise ordering). The following questions then arise [2, 4]:

(i) How many 1's can an $n \times n$ (0,1)-matrix A have if it *avoids* all $n \times n$ π_k-permutation matrices P, that is, no permutation matrix $P \leq A$ is a π_k-permutation matrix (A is *π_k-permutation avoiding*)?

(ii) How many 1's can an $n \times n$ (0,1)-matrix A have if it *forces* all $n \times n$ permutation matrices $P \leq A$ to be π_k-permutation matrices, that is, all permutation matrices $P \leq A$ are π_k-permutation matrices (A is *π_k- permutation forcing*)?

Here it is natural to assume that there is at least one permutation matrix $P \leq A$.

An even more general direction is contained in the Füredi-Hajnal conjecture concerning for a given $k \leq n$, the maximum number of 1's possible in an $n \times n$ $(0,1)$-matrix A that avoids a pattern prescribed by a permutation π_k of $\{1, 2, \ldots, k\}$ (that is, every $k \times k$ submatrix of A is π_k-avoiding). Marcus and Tardos [8] (see also [1]) proved the Füredi-Hajnal conjecture by showing that such an A has at most $2k^4\binom{k^2}{k}n$ 1's. The results of Marcus and Tardos have been improved and extended to higher dimensional matrices; see [6, 7] and the references contained therein.

In this paper we consider π_k-permutation forcing in an $n \times n$ (0,1)-matrix, particularly the natural case of $\pi_k = 12\cdots k$, that is, $12\cdots k$-forcing in an $n \times n$ $(0,1)$-matrix. Given the paper [3] in which $k = 3$ is considered (the case $k = 2$ is trivial), the next natural case to consider is $k = n - 1$.

Recall that the *permanent* of an $n \times n$ matrix A is defined as

$$\operatorname{per}(A) = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

Thus if A is a $(0,1)$-matrix, $\operatorname{per}(A)$ counts the number of $n \times n$ permutation matrices P with $P \leq A$. If A has a zero permanent, then A is vacuously π_k-permutation forcing for every π_k. Thus we shall always assume that A has a nonzero permanent. An $n \times n$ (0,1)-matrix $A \neq O$ has *total support* provided each 1 of A is a 1 of some permutation matrix $P \leq A$. The matrix A is *fully indecomposable* provided A does not have a zero submatrix $O_{r,s}$ with $r, s \geq 1$ and $r + s = n$; otherwise A is *partly decomposable*. Thus, in regards to permutation matrices contained in a matrix, a fully indecomposable matrix has no extraneous 1's. It follows from the Frobenius-König theorem that a fully indecomposable matrix A has total support and that a matrix with total support is, after row and column permutations, a direct sum of fully indecomposable matrices. The $(0, 1)$-matrix A is *nearly decomposable* provided that A is fully indecomposable and the replacement of a 1 of A with a 0 always results in a partly decomposable matrix. It is known that an $n \times n$ nearly decomposable $(0, 1)$-matrix has at most $3(n - 1)$ 1's (see [5], page 11).

Example 1.1 Let

$$B_{8,5} = \left[\begin{array}{c|c|c||c|c|c|c|c}
1 & & & & & & & \\ \hline
1 & 1 & & & & & & \\ \hline
1 & 1 & 1 & & & & & \\ \hline\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ \hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right].$$

(Here and elsewhere empty positions in a displayed matrix are to be interpreted as 0's.) Then $B_{8,5}$ is a partly decomposable matrix with 19 0's and is 12345-permutation forcing. ◇

The matrix in Example 1.1 is an instance of the construction of the $n \times n$ partly decomposable $(0, 1)$-matrix given by

$$B_{n,k} = \left[\begin{array}{cc} T_{k-2} & O_{k-2,n-(k-2)} \\ \hline \multicolumn{2}{c}{J^*_{n-k+2,n}} \end{array}\right]$$

where T_{k-2} is a $(k - 2) \times (k - 2)$ triangular matrix with 1's on and below the main diagonal and $J^*_{n-k+2,n}$ is an all 1's matrix except for a 0 in its upper right corner. The matrix $B_{n,k}$ is $123 \cdots k$-permutation forcing and the number of its $0'$s is

$$f(n, k) := n(k - 2) - \frac{k(k - 3)}{2}. \tag{1}$$

For future reference, note that $f(n, n - 1) = \binom{n}{2} - 2$.

Conjecture 1.2 Let k and n be integers with $2 \leq k \leq n$. Then the number of 0's in an $n \times n$ $12 \cdots k$-permutation forcing (0,1)-matrix with a nonzero permanent is at least $f(n, k)$ as given in (1). ◇

The only permutation in $\mathcal{S}_n$ that does not contain the pattern 12 is the permutation $n(n - 1) \cdots 21$. Thus an $n \times n$ 12-permutation forcing $(0, 1)$-matrix is a matrix A such that A has at least one 0 on its back (*Hankel*) diagonal and hence has at least $f(n,2) = 1$ zeros. In contrast, an $n \times n$ (0,1)-matrix is $12\cdots n$-permutation forcing provided that the only permutation matrix $P \leq A$ is the identity matrix I_n; thus if $n \geq 2$, such an A is partly decomposable and indeed after row and column permutations is a triangular matrix B with $I_n \leq B$ (see Example 1.6). The $n \times n$ 123-permutation forcing $(0,1)$-matrices with the minimum number n of 0's are characterized in [3]. We consider here the more general case of $n \times n$ $12\cdots k$-permutation forcing $(0,1)$-matrices where $k \leq n$, focussing primarily on $12\cdots(n-1)$-permutation forcing, a natural next case in view of these results just mentioned.

Note that if $k > 2$, then the matrix $B_{n,k}$ is partly decomposable. This raises the consideration of $12\cdots k$-forcing $(0,1)$-matrices which, in addition, are fully indecomposable.

Example 1.3 Consider the 5×5 matrix

$$A_{5,4} = \left[\begin{array}{c|c|c|c|c} 1 & 1 & & & \\ \hline & 1 & 1 & & \\ \hline 1 & 1 & 1 & 1 & \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & & 1 \end{array}\right].$$

Then A is fully indecomposable and is 1234-permutation forcing. A larger example is the 10×10 fully indecomposable $(0,1)$-matrix

$$A_{10,7} = \left[\begin{array}{c|c|c|c|c|c|c||c|c|c} 1 & 1 & 1 & & & & & & & \\ \hline & & 1 & 1 & & & & & & \\ \hline & & & 1 & 1 & & & & & \\ \hline & & & & 1 & 1 & 1 & & & \\ \hline\hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & & 1 & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & 1 \end{array}\right].$$

The matrix $A_{10,7}$ has 40 0's. It can be readily checked that $A_{10,7}$ is 1234567-permutation forcing and fully indecomposable. $\diamond$

The matrices in Example 1.3 are special cases of the following construction for $k < n$:

$$A_{n,k} = \left[\begin{array}{c||c} A_1 & O_{\lceil k/2\rceil,\lfloor k/2\rfloor} \\ \hline\hline J_{n-\lceil k/2\rceil,n-\lfloor k/2\rfloor} & A_2 \end{array}\right], \qquad (2)$$

where, as in Example 1.3, A_1 contains a zig-zag path of 1's from its upper left corner to its lower right corner with at least two 1's in each row, and A_2 contains a zig-zag path of 1's from its upper left corner to its lower right corner with at least two 1's in each column. Then A is $12\cdots k$-permutation forcing. Indeed, every permutation matrix $P \leq A$ corresponds to a permutation of the form $i_1 \cdots i_2 \cdots i_{\lceil k/2\rceil} \cdots j_1 \cdots j_2 \cdots j_{\lfloor k/2\rfloor} \cdots$ where $i_1 < i_2 < \cdots < i_{\lfloor k/2\rfloor} < j_1 < j_2 < \cdots < j_{\lceil k/2\rceil}$. The number of 0's in (2) is

$$g(n,k) := n(k-2) - \left\lfloor \frac{k^2-8}{4} \right\rfloor .$$

A computation shows that

$$g(n,k) - f(n,k) = \left\lceil \frac{(k-2)(k-4)}{4} \right\rceil .$$

Hence if $2 \leq k \leq 4$, then $g(n,k) = f(n,k)$ and, if Conjecture (1.2) is correct, both $A_{n,k}$ and $B_{n,k}$ have the minimum number of 0's in a $12\cdots k$-permutation forcing $(0,1)$-matrix.

If $k = 2$, then $f(n,2) = 1$ and a 0 on the Hankel diagonal gives a 12-permutation forcing $(0,1)$-matrix. It is not difficult to verify the conjecture for $k = 3$ and $k = n$.

Theorem 1.4 (see also [3]) *Let $n \geq 3$ be an integer and let A be an $n \times n$ 123-permutation forcing matrix (with nonzero permanent). Then the number of 0's in A is at least $f(n,3) = n$.*

Proof. Cyclically shifting the 1's of the Hankel diagonal gives n permutation matrices whose 1's are pairwise disjoint none of which contains a 123-pattern and so A must contain at least n 0's. □

Theorem 1.5 *Let $n \geq 2$ be an integer and let A be an $n \times n$ $12\cdots n$-permutation forcing matrix with nonzero permanent. Then the number of 0's in A is at least $f(n,n) = \binom{n}{2}$. Such a matrix has a row and a column with exactly one 1 and is not fully indecomposable.*

Proof. As already noted, A is $12\cdots n$-permutation forcing is equivalent to: I_n is the only permutation matrix $P \leq A$ and hence the permanent of A equals 1. Of each pair of symmetric entries about the main diagonal, one of the entries must then be 0 and hence A has at least $\binom{n}{2}$ 0's. If each row (or column) contained at least two 1's, then it is easy to see that the permanent of A would be at least 2. Hence, some row (and some column) contains only one 1 and A is partly decomposable. □

Example 1.6 There are many $12\cdots n$-permutation forcing matrices with precisely $f(n,n) = \binom{n}{2}$ 0's. The lower triangular matrix with all 1's on and below the main diagonal is one. Each of the others can be obtained from this one by a simultaneous permutation of the rows and columns. In the example (3) shown below, the simultaneous permutation is 13578642.

$$\left[\begin{array}{c|c|c|c|c|c|c|c} 1 & & & & & & & \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & & 1 & & & & & \\ \hline 1 & & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & & 1 & & 1 & & & \\ \hline 1 & & 1 & & 1 & 1 & 1 & 1 \\ \hline 1 & & 1 & & 1 & & 1 & \\ \hline 1 & & 1 & & 1 & & 1 & 1 \end{array}\right]. \tag{3}$$

◇

The main goal of this paper is to prove the correctness of Conjecture 1.2 for $k = n-1$. To do this we show that if $n > 5$, then every fully indecomposable $12\cdots(n-1)$-permutation forcing (0,1) matrix has more than $f(n, n-1)$ 0's, and it is then straightforward to prove the result for partly decomposable matrices. As already applied, we use the symbol ◇ to denote the end of an example, remark, and conjecture.

2 Proof of the Conjecture for $k = n - 1$

In this section we show that Conjecture 1.2 holds for $12\cdots(n-1)$-permutation forcing $n \times n$ $(0,1)$-matrices.

Remark 2.1 We describe the set of $12\cdots(n-1)$-permutations of $\{1, 2, \ldots, n\}$ and calculate their number α_n. They can be partitioned according to their first element a as given below:

1. The number of this type equals α_{n-1} (corresponding to permutations of $\{2, 3, \ldots, n\}$.

2. There are $(n-1)$ possibilities, $2134\cdots n$, $2314\cdots n$, $23415\cdots n$, $23451\cdots n$, $2345\cdots n1$.

k. $(3 \le k \le n-1)$ There are $(n-3)$ possibilities of the form $k12\cdots(k-1)(k+1)\cdots n$.

n. There is only one possibility, namely $n12\cdots(n-1)$.

Thus we have

$$\alpha_n = \alpha_{n-1} + (n-1) + (n-3) + 1 = \alpha_{n-1} + 2n - 3,$$

where $\alpha_n = (n-1)^2 + 1$ satisfies this recursion for $n \ge 2$. Notice that of these $(n-1)^2 + 1$ permutations, there is only one that ends with a 1, namely $2345\cdots n1$ and only one that begins with an n, namely, $n12\cdots(n-1)$.

In proving Conjecture 1.2 for $12\cdots(n-1)$-permutation forcing $n\times n$ $(0,1)$-matrices A, the next theorem implies that we need only consider partly decomposable matrices.

Theorem 2.2 *Let $n \geq 6$ be an integer, and let A be an $n \times n$ $12\cdots(n-1)$-permutation forcing $(0,1)$-matrix with a nonzero permanent. If A contains at most $f(n,n-1) = \left(\binom{n}{2}-2\right)$ $0's$, then A is partly decomposable.*

Our proof of Theorem 2.2 is based on a number of forthcoming lemmas. That Conjecture 1.2 is true for $k = n-1$ is a consequence of Theorem 2.2 as shown in the next theorem.

Theorem 2.3 *Let $n \geq 3$ be an integer and let A be a $12\cdots(n-1)$-permutation forcing $n \times n$ $(0,1)$-matrix with a nonzero permanent. Then A has at least $f(n,n-1) = \left(\binom{n}{2}-2\right)$ 0's.*

Proof. If $n = 3$, then clearly A has at least one 0. If $n = 4$, then by Theorem 1.4 A has at least $f(4,3) = 4$ 0's. If $n = 5$, the separate proof is in the Appendix. We now assume that $n \geq 6$ and use induction.

If A is $12\cdots(n-1)$-permutation forcing and has at most $f(n,n-1)$ 0's, then by Theorem 2.2, A is partly decomposable and so contains an $r\times(n-r)$ zero submatrix for some r with $1 \leq r \leq n-1$. Hence there are permutation matrices P and Q such that

$$PAQ = \begin{bmatrix} B & O_{r,n-r} \\ * & C \end{bmatrix},$$

where B is $r \times r$ and C is $(n-r)\times(n-r)$. By taking the rows and columns within B and C in the same relative order as in A in the decomposition shown for PAQ, we can make it so that B is $r \times r$ and $12\cdots r$-permutation forcing and C is $(n-r)\times(n-r)$ and $12\cdots(n-r-1)$-permutation forcing. Hence by induction and Theorem 1.5, the number of 0's in A is at least

$$r(n-r) + \binom{r}{2} + \binom{n-r}{2} - 2 = \binom{n}{2} - 2.$$

□

If A is an $n \times n$ (0,1)-matrix and P is a permutation matrix such that $P \leq A$, then P induces a *pairing* $S(A,P)$ of the $n(n-1)$ entries of A where P has a 0 as follows: If P has 1's in positions (i,j) and (s,t), then the entries of A in positions (i,t) and (s,j) are *P-paired*. This pairing then determines a *graph* $G(A,P)$ with vertices $\{1,2,\ldots,n\}$ and edges $\{t,j\}$, called *P-edges*, provided that $a_{it} = a_{sj} = 1$ (so $G(A,P)$ is a complete graph if and only if $A = J_n$). By removing the 1's in positions (i,j) and (s,t) of P and inserting 1's in positions (i,t) and (s,j) corresponding to this edge, we get a new permutation matrix $P' \leq A$. This *interchanges* one of the two submatrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with the other.

Example 2.4 With $n = 5$ and P as shown, the pairing is indicated by using the same labels for entries:

$$P = \left[\begin{array}{c|c|c|c|c} & & 1 & & \\ \hline 1 & & & & \\ \hline & 1 & & & \\ \hline & & & 1 & \\ \hline & & & & 1 \end{array}\right] \to \left[\begin{array}{c|c|c|c|c} a & b & 1 & f & j \\ \hline 1 & e & a & h & g \\ \hline e & 1 & b & c & i \\ \hline h & c & f & 1 & d \\ \hline g & i & j & d & 1 \end{array}\right] \quad (4)$$

Notice for example that in the matrix in (4), the 2×2 submatrices determined by the f's and g's have no overlapping rows or overlapping columns and thus determine a 4×4 submatrix with two possible independent interchanges:

$$\left[\begin{array}{c|c|c|c|c} & & 1 & f & \\ \hline 1 & & & & g \\ \hline & 1 & & & \\ \hline & & f & 1 & \\ \hline g & & & & 1 \end{array}\right].$$

◇

We now obtain some of the properties of $n \times n$ $12 \cdots (n-1)$-permutation forcing matrices

Proposition 2.5 *If k and l are integers with $1 \leq k, l \leq n$ such that $|k - l| \geq 2$, then there is a unique $n \times n$ $12 \cdots (n-1)$-permutation matrix $P = [p_{ij}]$ such that $p_{kl} = 1$.*

Proof. If $|k - l| \geq 2$, the only $12 \cdots (n-1)$-permutation matrix P containing $p_{kl} = 1$ is the permutation matrix P such that the submatrix of P obtained by deleting row k and column l is the identity matrix I_{n-1}. □

For example, if $n = 7$, then the unique 123456-permutation matrix $P = [p_{ij}]$ with $p_{52} = 1$ corresponds to the permutation 1345267.

Lemma 2.6 *Let A be an $n \times n$ $12 \cdots (n-1)$-permutation forcing $(0, 1)$-matrix, and let $P \leq A$ be a permutation matrix. Then $G(A, P)$ does not contain two disjoint P-edges.*

Proof. Two disjoint P-edges determine a 4×4 submatrix of A that contains two row and column disjoint submatrices equal to J_2 and results in a permutation matrix $Q \leq A$ without a $12 \cdots (n-1)$-pattern. □

Lemma 2.7 *Let $A = [a_{ij}]$ be an $n \times n$ $12 \cdots (n-1)$-permutation forcing $(0, 1)$-matrix with precisely $\left(\binom{n}{2} - 2\right)$ zeros. If $P \leq A$ is a permutation matrix, then $G(A, P)$ has at least two and at most four P-edges, and these P-edges have a common vertex w forming a subgraph equal to a star centered at w.*

Proof. By Lemma 2.6, $G(A, P)$ cannot have two disjoint P-edges. If $G(A, P)$ had a triangle on vertices r, s, t, then the 3×3 submatrix of A containing the three corresponding 1's in P is J_3, so A cannot be $12 \cdots (n-1)$-permutation forcing. Hence all P-edges share a common vertex resulting in a star $K_{1,t}$. Moreover, a 1 of P in a position (r, s) can be in a switch with a 1 of P in a position (i, j) only if $i = r - 1$ or r+1 or $j = s - 1$ or $s + 1$; otherwise we obtain a permutation matrix $Q \leq A$ without a $12 \cdots (n-1)$-pattern. Hence there are at most four vertices that can be switched in P with this common vertex implying that there are at most four P-edges in $G(A, P)$. Since there are $\left(\binom{n}{2} - 2\right)$ 0's, there must be at least two P-edges. □

Let A be a $12 \cdots (n-1)$-permutation forcing $(0, 1)$ matrix with precisely $\binom{n}{2} - 2$ zeros. Suppose $P \leq A$ and that $G(A, P)$ has t edges (so $t = 2, 3,$ or 4). Suppose j is the apex of the star, with (i, j) the corresponding position of A in P (so $a_{ij} = 1$). That means there are t positions (r, s) in P such that (i, j) and (r, s) are P-paired, and a_{is} and a_{rj} are both 1. Consider the $(t + 1) \times (t + 1)$ submatrix $M(A, P)$ of A containing these t 1's and the 1 in position (i, j). All entries of $M(A, P)$ in the row i or column j of A are 1's. Along with the $(t+1)$ 1's of $M(A, P)$ in P, this accounts for $(3t + 1)$ 1's, with the other entries of $M(A, P)$ P-paired.

If $t = 2$, then the two other entries (besides the $3t+1 = 7$ 1's) are P-paired. They cannot both be 1's (since $G(A, P)$ is a star) and they cannot both be 0's (because then there would have to be another pair of P-paired entries both of which are 1). So A has $\binom{n}{2} - 2$ P-pairs which have a 0 and a 1, and two P-pairs which have two 1's. All P-pairs outside of $M(A, P)$ have one 0 and one 1.

If $t = 3$, then $M(A, P)$ has 1's in these $3t + 1 = 10$ positions, with its 6 other entries in 3 P-pairs. None of these P-pairs can have two 1's. Since $M(A, P)$ is 123-permutation forcing, by Theorem 1.4, it has at least four 0's. If two of these three P-pairs had two 0's, then some other P-pair would have to have two 1's, which cannot happen. Hence one of them has two 0's and the other two of them have one 0 and one 1. All P-pairs outside of $M(A, P)$ have one 0 and one 1.

While we do not need it for our proofs, if $t = 4$ then using the fact that $f(5, 4) = 8$ (see the Appendix) it is not hard to show that $M(A, P)$ has 4 P-pairs with two 1's, 2 pairs with two 0's, and 4 pairs with one 0 and one 1, while all P-pairs outside of $M(A, P)$ have one 0 and one 1. Thus we have proved the following lemma.

Lemma 2.8 *Let A be an $n \times n$ $12 \cdots (n-1)$-permutation forcing matrix with precisely $\binom{n}{2} - 2$ 0's and let $P \leq A$ be a permutation matrix. Then every P-pair of entries (i, j) and (r, s) in A such that j and s are not both vertices in the star $G(A, P)$ (so (i, j) and (r, s) are not contained in $M(A, P)$) has one 0 and one 1.*

If the n-permutation σ_n contains a $12 \cdots (n-1)$-pattern, where $P \leq A$ is the corresponding permutation matrix, and where A is an unknown $n \times n$ $12 \cdots (n-1)$-permutation forcing matrix with precisely $\binom{n}{2} - 2$ 0's, we know that $G(A, P)$ is a star. Any element of σ_n which can be transposed with at least two other of its elements resulting in a $12 \cdots (n-1)$-permutation could be the apex.

Example 2.9 Refer to Lemma 2.8. Let $\sigma_n = 1345267$. Then 2 must be the apex and there could be edges to vertices 1,3,5, and 6 (and there must be at least two such edges). If $\sigma_n = 2345167$, then 1 must be the apex, and there could be edges to vertices 2,5, and 6 (so t cannot equal 4). If $\sigma_n = 2134567$, then vertex 1, 2, or 3 could be the apex and t would have to equal 2 in all three cases; if 3 were the apex, then 13 and 23 must be the edges of the star. Note that each of those two transpositions results in a 12...6-permutation.

Lemma 2.10 *Let $n \geq 6$ be an integer, and let A be an $n \times n$ $12\cdots(n-1)$-permutation forcing $(0,1)$-matrix with total support whose number of 0's is $\left(\binom{n}{2} - 2\right)$. Then A has 0's in each of the positions $(1,3),(3,1),(n-2,n)$ and $(n,n-2)$.*

Proof. Suppose, for instance, that position $(1,3)$ contains a 1. Since A has total support, the unique $12\cdots(n-1)$-permutation matrix $Q \leq A$ with a 1 in position (1,3) corresponds to the permutation $31245\cdots n$. By Lemma 2.8, the graph $G(A,Q)$ must be a star in which vertex 3 is the apex and can be joined only to vertices $1, 2,$ and 4. Since $(n-2,n)$ and $(n,n-2)$ are Q-paired, by Lemma 2.8, one of them must contain a 1, say $(n-2,n)$ contains a 1. The unique permutation $R \leq A$ containing this 1 corresponds to the permutation $12\cdots(n-3)n(n-2)(n-1)$. Putting parts of these two permutations together, we obtain the permutation matrix corresponding to the permutation $31245\cdots n(n-2)(n-1)$, a contradiction to A being $12\cdots(n-1)$-permutation forcing. Proofs for the other three cases are similar. □

Lemma 2.11 *Let $n \geq 6$ be an integer and let A be an $n \times n$ $12\cdots(n-1)$-permutation forcing $(0,1)$-matrix with total support whose number of 0's is $\left(\binom{n}{2} - 2\right)$. Then the following hold:*

(1) *A has $0's$ in positions $(3,3)$ and $(n-2,n-2)$.*

(2) *For all i with $4 \leq i \leq n-3$, positions $(i,1),(1,i),(n,i)$, and (i,n) contain 0's.*

(3) *Each of the positions $(n-2,1),(n,3),(1,n-2)$ and $(3,n)$ contains a 0, and thus there are at least $(n-4)$ 0's in each of rows and columns 1 and n.*

Proof.

(1) If A has a 1 in position $(3,3)$, then a permutation matrix $P \leq A$ with a 1 in that position corresponds to a permutation beginning with $123\cdots$ or $213\cdots$. If the latter, the permutation must be $21345\cdots n$, which is impossible since then $(n-2,n)$ is paired with $(n,n-2)$, contradicting the fact that there is a 0 in both these positions. Since positions $(1,3)$ and $(3,1)$ both contain 0's, the permutation cannot be of the form $123\cdots$. The argument is similar if A has a 1 in position $(n-2,n-2)$.

(2) Consider i with $4 \leq i \leq n-3$ such that position $(i,1)$ contains a 1. The unique $12\cdots(n-1)$-permutation forcing matrix $P \leq A$ with a 1 in position $(i,1)$ corresponds to the permutation $23\cdots i1(i+1)...n$ which has $(n-2,n)$ paired with $(n,n-2)$, a contradiction since both of these positions contain 0's. Similar arguments work for the other three cases.

(3) Suppose that position $(n-2,1)$ contains a 1. The unique permutation matrix P with a $123\cdots(n-1)$-pattern and with a 1 in position $(n-2,1)$ corresponds to the permutation $234\cdots(n-2)1(n-1)n$, so vertex 1 is the apex of the star in $G(A,P)$ with vertices 2, $(n-2)$ and $(n-1)$ possibly in the star and $(n-3,1)$ P-paired with $(n-2,n-2)$. If vertex $(n-2)$ is in the star then both these entries are 1, while if vertex $(n-2)$ is not in the star, then by Lemma 2.8 one is 1 and the other is 0. Either way this is a contradiction since by (1) and (2) both are 0. The other three cases follow similarly. Now using (2) we conclude that each of rows and columns 1 and n have at least $(n-4)$ zeros.

□

Lemma 2.12 *Let A be an $n \times n$ fully indecomposable $12\cdots(n-1)$-permutation forcing $(0,1)$-matrix with less than $f(n,n-1) = \binom{n}{2} - 2$ 0's. Then it is possible to change some 1's of A to 0's to get a fully indecomposable $12\cdots(n-1)$-permutation forcing $(0,1)$-matrix A' with exactly $f(n,n-1)$ 0's.*

Proof. By replacing certain 1's of A with 0's we can obtain a nearly decomposable matrix A', clearly still $12\cdots(n-1)$-permutation forcing. As we remarked in the introduction, A' has at most $3(n-1)$ 1's, and thus at least $n^2 - 3(n-1)$ 0's. Since $n^2 - 3(n-1) \geq f(n,n-1)$ for $n \geq 3$, the result follows. □

We can now give the proof of Theorem 2.2.

Proof. (Theorem 2.2.) By Lemma 2.12 we may assume that A has exactly $f(n,n-1)$ 0's. First suppose that position $(n,1)$ contains a 1. Since row n and column 1 each have at least $(n-4)$ zeros by (3) of Lemma 2.11, the $(n-1)\times(n-1)$ matrix $A(n|1)$ obtained from A by deleting row n and column 1 has at most

$$f(n,n-1) - 2(n-4) = \binom{n}{2} - (n-1) - (n-5) = \binom{n-1}{2} - (n-5)$$

0's. But $A(n|1)$ is a $12\cdots(n-1)$-forcing $(n-1)\times(n-1)$ $(0,1)$-matrix with nonzero permanent and so by Theorem 1.5 has at least $\binom{n-1}{2}$ 0's, a contradiction since $n \geq 6$. A similar argument works if position $(1,n)$ contains a 1.

Hence positions $(n,1)$ and $(1,n)$ both contain 0's. Now suppose that position $(n-1,1)$ contains a 1. Then the unique permutation matrix with a $12\cdots(n-1)$-pattern which contains this 1 corresponds to the permutation $23\cdots(n-1)1n$. The apex of the corresponding star is vertex 1, possibly joined to vertices $n-1, n$, and 2. But it cannot be joined to vertex $(n-1)$ (since position $(n-2,1)$ contains a 0) and it cannot be joined to vertex n (since position $(n, 1)$ contains a 0). Hence vertex 1 cannot be the apex, a contradiction, so position $(n-1, 1)$ contains a 0. Similarly, position $(1, n-1)$ contains a 0.

Since A is fully indecomposable, it must contain at least two 1's in each row and column. Hence positions $(1, 1), (1, 2)$, and $(2, 1)$ contain 1's and these are the only 1's in the first row and column of A. This implies that A contains the permutation matrix corresponding to the permutation $2134 \cdots n$, a contradiction since position $(3, 3)$ contains a 0. □

3 Coda

We believe that the matrices $A_{n,k}$ defined in Example 1.3, and more generally in (2), that have $g(n, k)$ 0's, have the fewest number of 0's among all fully indecomposable $12 \cdots k$-permutation forcing $n \times n$ $(0, 1)$-matrices.

Conjecture 3.1 If $4 < k \leq n$ and A is an $n \times n$ $12 \cdots k$-permutation forcing (0,1)-matrix with fewer than $g(n, k)$ 0's, then A is partly decomposable.

If Conjecture 3.1 is true, then it is straightforward to prove Conjecture 1.2 by induction, since if $m + p = n$ and $0 \leq i \leq m$ and $0 \leq j \leq p$, then

$$f(m, m - i) + f(p, p - j) = f(n, n - i - j) + ij.$$

Appendix

We give a proof of Theorem 2.2 in the special case of $n = 5$ that was not included in its proof.

Lemma $f(5, 4) = 8$. (So if A is 5×5 1234-permutation forcing then it has at least eight 0's.)

Proof. Easy to do if A is partly decomposable. (For example, if the component matrices are B which is 3×3 and 12-permutation forcing, and C which is 2×2 and 12-permutation forcing, then B has at least $f(3, 2) = 1$ zero and C has at least one zero, so A has at least $1 + 1 + 6 = 8$ 0's.)

So assume A is a fully indecomposable 5×5 1234-permutation forcing with only seven 0's.

Observation A: If position (i, j) contains a 1, then there are at most three 0's in row i and column j together. Otherwise, we get a contradiction by considering the 4×4 matrix $A(i|j)$, which is 123-permutation forcing (and has at least four 0's).

Observation B: If position (i, j) contains a 1 and $|i - j| > 1$, then there is at most one 0 in the union of row i and column j. Otherwise, the 4×4 matrix $A(i|j)$ gives a contradiction, since it is 1234-permutation forcing (and has at least six 0's)

First suppose position $(5,1)$ contains a 1. By Observation B, column 1 and row 5 together have at most one 0, so at least one of them has no 0's. Without loss of generality, we assume column 1 contains all 1's. Then $(5,1) = 1$ implies 23451. $(3,1) = 1$ implies 23145. If $(5,4) = 1$ we have 23154 impossible. If $(5,3) = 1$ we have 21453 impossible. Hence positions $(5,4)$ and $(5,3)$ contain 0's, which puts two 0's in row 5, contradiction. So we can assume positions $(5,1)$ and $(1,5)$ contain 0's.

If there is another 0 in column 1, then position $(3,1)$ contains a 0, and so does position $(4,1)$ because if either is 1 we contradict Observation B. However, that means position $(1,1)$ contains a 1 (column 1 has at least two 1's), contradicting Observation A, since row 1 and column 1 together would have at least four 0's.

Hence we can assume that the 0 in position $(5,1)$ is the only 0 in the union of column 1 and row 5. Now position $(5,3)$ contains a 1 implying we have 12453, and position $(4,1)$=1 implies we have 23451. Hence we have 21453 which is impossible. □

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