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On prism-hamiltonian bipartite graphs and toughness

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Abstract

Responding to an open question posed by Špacapan and Horak in 2024, we prove that there exists an infinite family of 2-connected bipartite graphs G_{nc} with 2-walks, thus having $\tau(G_{nc}) = \frac{1}{2}$, which are not prism-hamiltonian.

1 Introduction

A graph G is *hamiltonian* if it contains a *Hamiltonian cycle*, that is, a cycle which passes through every vertex of G exactly once. Determining if a graph is hamiltonian is a highly studied and long-standing problem in graph theory. It is well-known that this is an NP-complete problem for arbitrary graphs, so much work has also been devoted to studying hamiltonicity and related properties for restricted classes of graphs. For instance, graphs with k-walks are closely related to those with Hamiltonian cycles. For an integer k, denote by $k \times G$ the multigraph of G obtained by multiplying each edge in G by k. We say a graph G contains a *k-walk* if $(2k) \times G$ contains a spanning subgraph W such that each vertex in W has even degree at most $2k$. Equivalently, a k-walk is a spanning closed walk of a graph which visits every vertex at most k times. Thus, a 1-walk is a Hamiltonian cycle. Jackson and Wormald [7] provided a hierarchy which, when limited to $k = 1$, describes how close a graph is to being hamiltonian. In their hierarchy, they included, from most to least restrictive, the existence of a k-walk, the existence of a $(k+1)$-tree, and the existence of a $(k+1)$-walk. They defined a graph G to have a *k-tree* if G contains a spanning tree of maximum degree k, so a 2-tree in G is a Hamiltonian path.

In this paper, we will focus on 2-walks and prism-hamiltonicity. The *prism* over a graph G is the Cartesian product of G and K_2, denoted $G \square K_2$, where K_2 is the complete graph on two vertices. Equivalently, the prism is obtained by taking two copies of G and joining the corresponding vertices in each copy by a *vertical edge.* A graph G is *prism-hamiltonian* if $G \square K_2$ is hamiltonian. Kaiser, Král, Rosenfeld, Ryjáček, and Voss [6] observed that being prism-hamiltonian is between having a Hamiltonian path (a 2-tree) and having a 2-walk in Jackson and Wormald's hierarchy. It is easy to see that containing a Hamiltonian path is a sufficient condition

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for a graph G to be prism-hamiltonian. One can simply follow the path in one copy of G, take a vertical edge, follow the path back in the second copy of G, and take another vertical edge to complete the cycle. However, the converse is not necessarily true. For instance, the complete bipartite graph $K_{2,4}$ is prism-hamiltonian (see [6]), but does not have a Hamiltonian path. Due to the close relationship between hamiltonicity and prism-hamiltonicity, it is natural to extend problems and results from hamiltonian graphs to prism-hamiltonian graphs; Salehi Nowbandegani [9] proved a number of such extensions.

Another property which is strongly correlated to hamiltonicity is *toughness.* Chvátal [3] first introduced toughness in 1973. Let $\omega(G)$ denote the number of components in a graph G, and let a *cut-set* S of G be a set $S \subseteq V(G)$ such that $\omega(G - S) > 1$. Chvátal defined toughness as follows. Let G be a graph and t a positive real number. Then G is *t-tough* if $\omega(G - S) > 1$ implies $|S| \geq t \cdot \omega(G - S)$ for every set $S \subseteq V(G)$. Obviously, a t-tough graph is s-tough for all $s < t$. If G is not complete, then there is a largest t such that G is t-tough; this t will be called the toughness of G and denoted by $\tau(G)$. On the other hand, a complete graph contains no cut-set and so is t-tough for every t. Accordingly, we set $\tau(K_n) = +\infty$ for every n. Adopting the convention that a minimum taken over the empty set is infinite, we can write

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)}\right\}$$

where S ranges over all cut-sets of G. We will refer to any set which achieves this minimum as a *tough-set.*

It is not hard to see that every hamiltonian graph is 1-tough. In [1], Bauer, Broersma, and Schmeichel wrote a survey on toughness results related to Hamiltonian cycles. Chvátal [3] conjectured that there is a constant k such that every k-tough graph is hamiltonian. Chvátal originally believed this was true for $k = 2$, but Bauer, Broersma, and Veldman [2] proved this was not the case in the year 2000 by showing that for arbitrary $\epsilon > 0$ there exist non-hamiltonian graphs which are $(\frac{9}{4} - \epsilon)$-tough. Thus, we know k is at least $\frac{9}{4}$.

Similarly, one can obtain lower bounds on the toughness of prism-hamiltonian graphs. Observe that, as pointed out by Špacapan and Horak [12], if G is not $\frac{1}{2}$-tough, then $G \square K_2$ is not 1-tough, and hence every prism-hamiltonian graph must be $\frac{1}{2}$-tough. Kaiser, Král, Rosenfeld, Ryjáček, and Voss [6] conjectured that there is a constant k such that every k-tough graph is prism-hamiltonian, and they showed that for arbitrary $\epsilon > 0$ there exist non-prism-hamiltonian graphs which are $(\frac{9}{8} - \epsilon)$-tough. Thus, we know k is at least $\frac{9}{8}$ for prism-hamiltonian graphs.

Restricting ourselves to only bipartite graphs, we notice that every bipartite graph has toughness at most 1. Let G be a bipartite graph with the bipartition $V(G) = A \cup B$ such that $|A| \leq |B|$. Then we have $\tau(G) \leq \frac{|A|}{\omega(G-|A|)} = \frac{|A|}{|B|} \leq 1$. Thus, Špacapan and Horak propose the following conjecture:

Conjecture 1.1. *[12] Every 1-tough bipartite graph is prism-hamiltonian.*

We can gain insight on this conjecture by asking for which graphs G with $\frac{1}{2} \leq \tau(G) \leq 1$ is G prism-hamiltonian and for which is it not? For instance, in their paper,

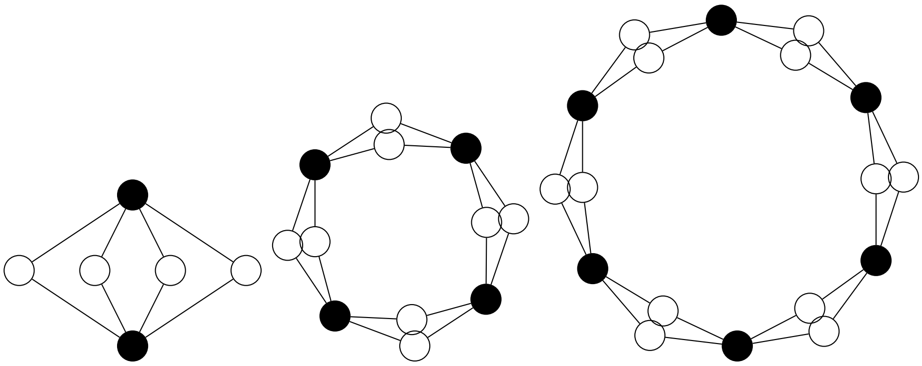

Figure 1: The initial graphs G_1, G_2, and G_3 for constructing crown graphs G_{1c}, G_{2c}, and G_{3c}.

Špacapan and Horak presented a $\frac{1}{2}$-tough graph that is not prism-hamiltonian. Yet some $\frac{1}{2}$-tough graphs are prism-hamiltonian, such as $K_{2,4}$. Therefore, Špacapan and Horak concluded their paper with the following open question:

Question 1.2. *[12] Is there a* 2*-connected bipartite graph* G *with* $\tau(G) \geq \frac{1}{2}$ *which is not prism-hamiltonian?*

An even stronger property than a graph having $\tau(G) = \frac{1}{2}$ is a graph containing a 2-walk. In Lemma 2.1 of their paper, Jackson and Wormald [7] point out that if a graph G has a k-walk then G is $\frac{1}{k}$-tough. Meanwhile, not every graph which is $\frac{1}{k}$-tough contains a k-walk. Ellingham and Zha [5] provided instances of graphs that are $(\frac{17}{24} - \epsilon)$-tough, and hence $\frac{1}{2}$-tough, with no 2-walk.

The examples H_k in Section 6 of the paper by Kaiser et al. [6] are $2k$-connected, have a 2-walk, and are not prism-hamiltonian, but these graphs are not bipartite. Špacapan and Horak wondered what the impact of imposing a bipartite requirement would be on sufficiently tough 2-connected graphs and if these constraints would be enough to guarantee a graph is prism-hamiltonian. Thus, in response to Question 1.2, we will prove the main theorem of this paper.

Theorem 1.3. *There exists an infinite family of* 2*-connected bipartite graphs which have a* 2*-walk, and hence are* $\frac{1}{2}$*-tough, but are not prism-hamiltonian.*

Specifically, the crown graphs G_{nc} defined in Section 2 satisfy Theorem 1.3. In the next section, we prove Theorem 1.3, outsourcing more technical derivations to be proved later as supporting lemmas in Section 3. Our goal will be to first prove that crown graphs are 2-connected, bipartite, and have a 2-walk. We will then prove that these graphs are not prism-hamiltonian. The paper concludes with an outlook on future directions.

2 Proof of Theorem 1.3

Proof. We begin by defining a crown graph G_{nc} where n is any non-zero natural number. Let G_n be a graph with the bipartition (X, Y) such that $|X| = 2n$ and

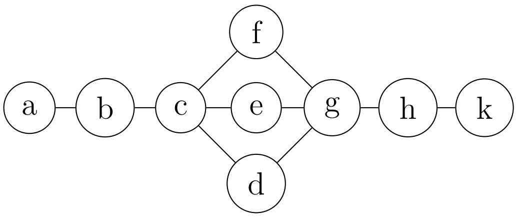

Figure 2: The subgraph H which will be used in the construction of the crown graphs.

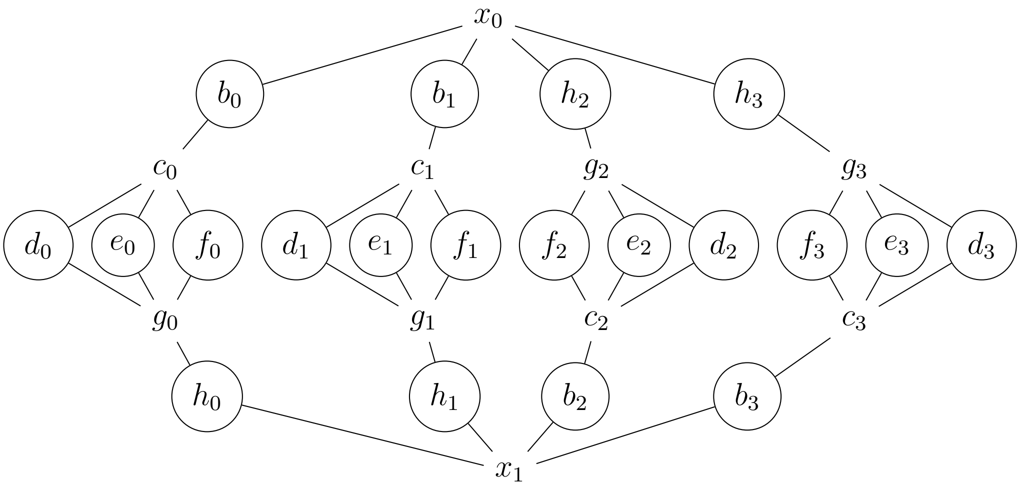

Figure 3: The crown graph G_{1c} as constructed in the proof of Theorem 1.3

$|Y| = 4n$ following the construction pattern of G_1, G_2, and G_3 as shown in Figure 1 where the black vertices belong to X and the white vertices belong to Y. Label the vertices in X in cyclic order as $x_0, x_1, \ldots, x_{2n-1}$ with subscripts interpreted modulo $2n$. For each path in G_n containing one vertex in Y and two vertices x_i, x_{i+1} in X, replace this with a copy H_j of the graph H from Figure 2, and let v_j be the vertex of H_j corresponding to v in H. Identify a and k in H with the black vertices such that for each $i = 0, 1, \ldots, 2n-1$ we have $k_{2i-2} = k_{2i-1} = a_{2i} = a_{2i+1} = x_i$. Thus, for each $i = 0, 1, \ldots, 2n-1$ both H_{2i} and H_{2i+1} have x_i and x_{i+1} as their terminal vertices. Now the crown graph G_{nc} is defined, and the labeled G_{1c} is depicted in Figure 3.

It is easy to see that G_{nc} is 2-connected because it contains no cut vertices. Additionally, G_{nc} remained bipartite after each step in the construction, and the bipartition of $V(G_{1c})$ is shown in Figure 3 by the vertices that are circled and those that are not. Lastly, $4 \times G_{nc}$ naturally contains a spanning subgraph G_{nc} which is a connected graph such that each vertex has even degree at most 4, so by definition G_{nc} has a 2-walk.

Now let us show that the crown graph G_{nc} is not prism-hamiltonian. Equivalently, let us show that $G_{nc} \square K_2$ does not contain a Hamiltonian cycle. For contradiction, assume that $G_{nc} \square K_2$ contains a Hamiltonian cycle C. Our proof will attempt to force the edges contained in the cycle C by proving lemmas related to edges in $E(C)$ which have endpoints in the tough-set of the graph.

Let V and V' be the disjoint vertex sets of the two copies of G_{nc} in $G_{nc}\Box K_2$, and for every v in V, let v' be the corresponding vertex in V'. By Lemma 3.1 we know $|E(C) \cap \{a_jb_j, a'_jb'_j, h_jk_j, h'_jk'_j\}| = 2$ for each subgraph H_j of G_{nc}. Then Lemma 3.3 states that $|E(C) \cap \{a_jb_j, a'_jb'_j\}| = |E(C) \cap \{h_jk_j, h'_jk'_j\}| = 1$ for each subgraph H_j. Let Q_j be the path of C through $H_j\Box K_2$. According to Lemma 3.4, Q_j either contains a_jb_j and $h'_jk'_j$ or $a'_jb'_j$ and h_jk_j for each j. Without loss of generality, assume Q_0 contains $a_0b_0 = x_0b_0$ and $h'_0k'_0 = h'_0x'_1$. Now, if the terminal edges of Q_1 are $a_1b_1 = x_0b_1$ and $h'_1k'_1 = h_1x_1$ then $Q_0 \cup Q_1$ forms a closed cycle and hence is not in a Hamiltonian cycle of G. Therefore, $a'_1b'_1 = x'_0b'_1$ and $h_1k_1 = h_1x_1$ must be the terminal edges of Q_1. Similarly, for each $i = 0, 1, \ldots, 2n-1$ we may assume (using the symmetry between H_{2i} and H_{2i+1}) that Q_{2i} has terminal edges $a_{2i}b_{2i} = x_ib_{2i}$ and $h'_{2i}k'_{2i} = h'_{2i}x'_{i+1}$ while Q_{2i+1} has terminal edges $a'_{2i+1}b'_{2i+1} = x'_ib'_{2i+1}$ and $h_{2i+1}k_{2i+1} = h_{2i+1}x_{i+1}$. However, now $Q_0 \cup Q_3 \cup Q_4 \cup Q_7 \cup Q_8 \cdots \cup Q_{4n-4} \cup Q_{4n-1}$ gives us a simple cycle which does not contain all of the vertices of $G_{nc}\Box K_2$ and we have a contradiction. Therefore, our initial assumption must be false and G_{nc} is not prism-hamiltonian. □

3 Supporting lemmas

The following lemmas exist to support our proof that the crown graph G_{nc} is not prism-hamiltonian. Lemma 3.1 determines how many edges connected to vertices in the cut-set $X \cup X'$ from the bipartition (X, Y) of G_n belong to $E(C)$.

Lemma 3.1. *For some $n \in \mathbb{N}^+$, let G_{nc} be the crown graph constructed in Theorem 1.3. Assume $G_{nc}\Box K_2$ contains a Hamiltonian cycle C. For each $j \in \{0, 1, \ldots, 4n-1\}$ we have $|E(C) \cap S_1^j| = 2$ where $S_1^j = \{a_jb_j, a'_jb'_j, h_jk_j, h'_jk'_j\}$.*

Proof. We first show that $|S_1^j \cap E(C)| \geq 2$ for every $j \in \{0, 1, \ldots, 4n-1\}$. For any j, consider the partition $V(G_{nc}) = V_1 \cup V_2$, where $V_1 = V(H_j\Box K_2 - \{a_j, a'_j, k_j, k'_j\})$ and $V_2 = V(G_{nc}\Box K_2) \setminus V_1$. Since the edges in S_1^j are the only edges in G_{nc} that connect V_1 and V_2, we know that at least two of them must be in the Hamiltonian cycle C.

Now we can complete the proof. Define A^i and $A^{i'}$ as follows:

$$A^i = E(C) \cap \{x_ib_{2i}, x_ib_{2i+1}, x_ih_{2i-2}, x_ih_{2i-1}\}$$
$$A^{i'} = E(C) \cap \{x'_ib'_{2i}, x'_ib'_{2i+1}, x'_ih'_{2i-2}, x'_ih'_{2i-1}\}.$$

Observe that, since $a_0 = x_0$ is incident to exactly two edges in C, we have $|A^0| \leq 2$. Similarly, for each i we have $|A^i| \leq 2$ and $|A^{i'}| \leq 2$. The cardinality of a disjoint union is exactly equal to the sum of its pieces. Since a disjoint union of $A^0, A^1, \ldots, A^{2n-1}$ and $A^{0'}, A^{1'}, \ldots, A^{2n-1'}$ equals a disjoint union of $E(C) \cap S_1^0, E(C) \cap S_1^1, \ldots, E(C) \cap S_1^{4n-1}$, we get

$$8n \geq \sum_{i=0}^{2n-1} (|A^i| + |A^{i'}|) = \sum_{j=0}^{4n-1} |S_1^j \cap E(C)| \geq 8n.$$

From the point-wise inequality $|S_1^i \cap E(C)| \geq 2$ and $|A^i|, |A^{i'}| \leq 2$, we also know that $|A^i| = |A^{i'}| = 2$ for every $i \in \{0, 1, \ldots, 2n-1\}$ and $|S_1^j \cap E(C)| = 2$ for every $j \in \{0, 1, \ldots, 4n-1\}$. □

Lemma 3.2 determines which edge sets connected to vertices in the cut-seti $\{c_j, c_j', g_j, g_j'\}$ of the subgraph H_j of $G_{nc}\Box K_2$ belong to $E(C)$. With regards to the $K_{2,3}\Box K_2$ subgraph which is sandwiched between $\{c_j, c_j'\}$ and $\{g_j, g_j'\}$, the lemma concludes that the cycle must enter and exit this region on different levels of the prism, i.e., on different copies of G_{nc} in the prism. Equivalently, for any j in $\{0, 1, \ldots, 4n-1\}$, exactly one of b_jc_j or g_jh_j is in the cycle, and exactly one of $b_j'c_j'$ or $g_j'h_j'$ is in the cycle.

Lemma 3.2. *Suppose H is a subgraph of G as shown in Figure 2 such that only vertices a and k are adjacent to vertices outside H. Let C be a Hamiltonian cycle in $G\Box K_2$ such that $|E(C)\cap\{ab, a'b', hk, h'k'\}| = 2$. Then $|E(C)\cap S_2| = |E(C)\cap S_3| = 1$ where $S_2 = \{bc, gh\}$ and $S_3 = \{b'c', g'h'\}$.*

Proof. We first show that $|(S_2 \cup S_3)\cap E(C)| \geq 2$. To do so, consider the partition $V(H) = V_1 \cup V_2$, where $V_1 = \{c, c', d, d', e, e', f, f', g, g'\}$ and $V_2 = V(H)\setminus V_1$. Since the edges in $S_2 \cup S_3$ are the only edges in H that connect V_1 and V_2, we know that at least two of them must be in the Hamiltonian cycle C.

We now give an upper bound on $|(S_2 \cup S_3)\cap E(C)|$ by finding an upper bound on $|S_2 \cap E(C)|$ and $|S_3 \cap E(C)|$. For $v \in V(H)$, let E_v denote the set of edges of C incident with v, excluding vv'. Notice that the disjoint union of E_c and E_g equals the disjoint union of E_d, E_e, E_f, and $S_2 \cap E(C)$. Since every vertex in $V(C)$ is incident to exactly two edges in $E(C)$, we have $1 \leq |E_c|, |E_d|, |E_e|, |E_f|, |E_g| \leq 2$. Altogether, we have

$$4 \geq |E_c| + |E_g| \geq |E_d| + |E_e| + |E_f| + |S_2 \cap E(C)| \geq 3 + |S_2 \cap E(C)|,$$

which gives $|S_2 \cap E(C)| \leq 1$. By symmetry, we also have $|S_3 \cap E(C)| \leq 1$. Combined with the lower bound $|(S_2 \cup S_3)\cap E(C)| \geq 2$, we know that $|S_2 \cap E(C)| = |S_3 \cap E(C)| = 1$. □

Informally, we will refer to each subgraph $(H_j - \{a_j, k_j\})\Box K_2$ of $G_{nc}\Box K_2$ as a branch. Lemma 3.3 tightens Lemma 3.1 by showing that for each branch, the Hamiltonian cycle does not enter and exit the branch at the same end. If it enters a branch with edge a_jb_j or $a_j'b_j'$, it must exit the branch through edge h_jk_j or $h_j'k_j'$, and vice-versa. That is, for each j, exactly one of a_jb_j or $a_j'b_j'$ is in the cycle, and exactly one of h_jk_j or $h_j'k_j'$ is in the cycle C.

Lemma 3.3. *Suppose H is a subgraph of G as shown in Figure 2 such that only a and k are adjacent to vertices outside H. Let C be a Hamiltonian cycle in $G\Box K_2$ such that $|E(C)\cap\{ab, a'b', hk, h'k'\}| = 2$. Then $|E(C)\cap\{ab, a'b'\}| = |E(C)\cap\{hk, h'k'\}| = 1$.*

Proof. Assume for contradiction that we do not have $|E(C)\cap\{ab, a'b'\}| = |E(C)\cap\{hk, h'k'\}| = 1$. By assumption, we know $|E(C)\cap\{ab, a'b', hk, h'k'\}| = 2$, so without loss of generality we can assume that $|\{ab, a'b'\}\cap E(C)| = 2$ and $|\{kh, k'h'\}\cap E(C)| = 0$. Now in order for C to traverse h and h', we must have $\{gh, hh', h'g'\} \subseteq E(C)$. By Lemma 3.2 we also have $|(S_2 \cup S_3)\cap E(C)| = 2$, so we know $|\{bc, b'c'\}\cap E(C)| = 0$. Therefore, we have $\{bc, b'c', kh, k'h'\}\cap E(C) = \emptyset$. However, removing every edge in

$\{bc, b'c', kh, k'h'\}$ from $E(H\Box K_2)$ would make $G\Box K_2$ disconnected. Thus, we have a contradiction because the Hamiltonian cycle C must contain at least two edges among $bc, b'c', kh, k'h'$. Therefore, our initial assumption was false and $|E(C) \cap \{ab, a'b'\}| = |E(C) \cap \{hk, h'k'\}| = 1$. $\square$

Finally, Lemma 3.4 will be used to prove that if a Hamiltonian cycle C exists in $G_{nc}\Box K_2$ then it must enter and exit each branch on a different copy of G_{nc}. It extends Lemma 3.3 by showing either $|E(C) \cap \{a_j b_j, h'_j k'_j\}| = 1$ or $|E(C) \cap \{a'_j b'_j, h_j k_j\}| = 1$.

Lemma 3.4. *Suppose H is a subgraph of G as shown in Figure 2 such that only a and k are adjacent to vertices outside H. Let C be a Hamiltonian cycle in $G\Box K_2$ such that $|E(C)\cap\{ab, a'b', hk, h'k'\}| = 2$. Then the edges in $E(C)\cap E(H\Box K_2) - \{aa', kk'\}$ form a subgraph Q which is a path whose terminal edges are either ab and $h'k'$ or $a'b'$ and hk.*

Proof. Since $|E(C)\cap\{ab, a'b', hk, h'k'\}| = 2$, the cycle C enters the subgraph $H\Box K_2 - \{a, a', k, k'\}$ once and leaves it once, so Q is a path. By Lemma 3.3, we know that Q includes exactly one of the edges ab and $a'b'$. Because H and its prism copy are identical, we can assume without loss of generality that ab is contained in this path and $a'b'$ is not. Now in order for this segment of the Hamiltonian cycle C to traverse b', the path must include edges bb' and $b'c'$. Then, by Lemma 3.2, the edge $g'h'$ is not in this path, so in order to traverse h' without using the edge $g'h'$ the path must include edges hh' and $h'k'$. Hence, if ab is a terminal edge of Q then $h'k'$ is the other terminal edge. By symmetry (switching the two copies of G), if $a'b'$ is a terminal edge then hk is the other terminal edge. $\square$

4 Outlook

We have successfully shown that the family of crown graphs is bipartite, 2-connected, and $\frac{1}{2}$-tough yet not prism-hamiltonian. A natural response would be to find the maximum toughness of a bipartite graph that is not prism-hamiltonian. That is, does there exists some toughness t such that all 2-connected bipartite graphs with $\tau(G) \geq t$ must be prism-hamiltonian? One approach could be to find sufficient toughness conditions that imply prism-hamiltonicity of a bipartite graph.

For instance, the crown graphs were inspired in part by searching for graphs without a spanning bipartite 2-cactus. A graph G is a 2-*cactus* if every block of G is a cycle or K_2 and every vertex $v \in V(G)$ is contained in at most two blocks of G where a block is a maximal subgraph without a cut-vertex. A 2-cactus is bipartite if the graph is bipartite or, equivalently, the cycles are all even or K_2. Note there is some disparity in the literature when defining cacti; a bipartite 2-cactus is refereed to as a 'SBEP subgraph graph' in [4], a 'good even cactus' in [8] and [10], an 'even 2-cactus' in [9], a 'bipartite cactus' in [11], and a 'good cactus' in [12]. We choose to use the terminology '2-cactus', matching the notation used in [9] and allowing for a more general t-cactus where each vertex belongs to at most t blocks. We also refer to cacti as being bipartite rather than even because we find this is more explicit.

Importantly, the following result has been proven in [4] and [11], and it has been implied in several earlier papers.

Theorem 4.1. *Every graph that has a bipartite* 2*-cactus as a spanning subgraph is prism-hamiltonian.*

Thus, research has been done to find classes of graphs which contain spanning bipartite 2-cacti subgraphs because these graphs are necessarily prism-hamiltonian.

In [10], Špacapan asked whether the converse of Theorem 4.1 was true for 3-connected planar graphs. Lo [8] answered in the negative with an infinite class of 3-connected planar graphs which have no spanning bipartite 2-cactus but are still prism-hamiltonian. Kaiser et al. [6] provided graphs which have $\tau(G) = (\frac{9}{8} - \epsilon)$, are 2-connected for $\epsilon < \frac{5}{8}$, are not prism-hamiltonian, and hence do not contain spanning bipartite 2-cacti. Could we also find bipartite graphs which have a toughness greater than one-half and are not prism-hamiltonian by searching for graphs which do not contain a spanning bipartite 2-cactus?

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References

[1] D. Bauer, H. Broersma and E. Schmeichel, Toughness in graphs—a survey, *Graphs Combin.* 22 (2006), 1–35.

[2] D. Bauer, H. Broersma and H. J. Veldman, Not every 2-tough graph is hamiltonian, *Discrete Appl. Math.* 99 (2000), 317–321.

[3] V. Chvátal, Tough graphs and hamiltonian circuits *Discrete Math.* 5 (1973), 215–228.

[4] M. N. Ellingham, P. Salehi Nowbandegani and S. Shan, Toughness and prism-hamiltonicity of P_4-free graphs, *Discrete Appl. Math.* 284 (2020), 201–206.

[5] M. N. Ellingham and X. Zha, Toughness, trees, and walks, *J. Graph Theory* 33 (2000), 125–137.

[6] T. Kaiser, Z. Ryjáček, D. Král, M. Rosenfeld and H.-J. Voss, Hamiltonian cycles in prisms over Graphs, *J. Graph Theory* 56 (2007), 249–269.

[7] B. Jackson and N. C. Wormald, k-walks of graphs, *Australas. J. Combin.* 2 (1990) 135–146.

[8] O.-H. S. Lo, On the hamiltonian property hierarchy of 3-connected planar graphs. *Electron. J. Combin.* 29 (4) (2022), #P4.18, 14 pp.

[9] P. Salehi Nowbandegani, *Prism-Hamiltonicity and Hamiltonicity of Graph Products*, PhD Thesis, Vanderbilt University, 2020.

[10] S. Špacapan, A counterexample to prism-hamiltonicity of 3-connected planar graphs, *J. Combin. Theory Ser. B* 146 (2021), 364–371.

[11] S. Špacapan, Polyhedra without cubic vertices are prism-hamiltonian, *J. Graph Theory* 106 (2024), 380–406.

[12] S. Špacapan and P. Horak, On prism-hamiltonian bipartite graphs, *Australas. J. Combin.* 88 (2024), 194–203.

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