AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **95(1)** (2026), Pages 172–189

On dominator coloring in Mycielskian graphs

Yue Lyu Chonglong Wang Chunxiang Wang*

School of Mathematics and Statistics, and Hubei Key Lab–Math. Sci.
Central China Normal University
Wuhan 430079, China
`ylyu@mails.ccnu.edu.cn` `wcxiang@mail.ccnu.edu.cn`*
`wangchonglong123@foxmail.com`

Abstract

A dominator coloring (DC) of a graph G is a proper coloring in which every vertex dominates all vertices of at least one color class. Let $\chi_d(G)$ denote the minimum number of colors in a DC of G. Arumugam et al. in 2012 proved that $\chi_d(G) + 1 \leq \chi_d(\mu(G)) \leq \chi_d(G) + 2$. Building on this, Abid and Rao in 2019 characterized graphs satisfying $\chi_d(\mu(G)) = \chi_d(G)+1$, and Kalarkop and Rangarajan (2021) claimed that $\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$ for $n \geq 8$. In this paper, by determining the exact values of $\chi_d(\mu(C_8))$ and $\chi_d(\mu(C_9))$, we provide counterexamples to the above two results. Motivated by this, we completely characterize when $\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$, namely, if and only if $n \equiv 3 \pmod 6$ with $n \neq 3$, or $n \equiv 5 \pmod 6$. In addition, we determine the exact values of $\chi_d(\mu(P_n))$ for all paths P_n. Our results correct earlier claims and provide a complete description of dominator colorings under the Mycielskian construction for these graph classes.

1 Introduction

All graphs considered in this paper are simple, finite, and undirected. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ denote the degree and neighborhood of v, respectively, and let $N_G[v] = N_G(v) \cup \{v\}$. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degrees of G.

A *proper coloring* of a graph G is a mapping $f : V(G) \to \{1, \ldots, k\}$ such that any two adjacent vertices receive different colors. A graph G is *k-colorable* if it admits a proper coloring using at most k colors. The *chromatic number* $\chi(G)$ is the minimum k such that G is k-colorable. For a coloring f, the set $V_i = \{v \in V(G) : f(v) = i\}$

---

[*] Corresponding author.

ISSN: 2202-3518

($1 \leq i \leq k$) is called the *color class* of color i. For a coloring φ of G and a subgraph $H \subseteq G$, we denote by $\varphi\mid_H$ the *restriction* of φ to $V(H)$.

A *dominator coloring* of a graph G is a proper coloring in which every vertex dominates all vertices of at least one color class. Let φ be a coloring of G and let V_c denote the color class of color c. When no confusion arises, we say that a vertex v *dominates color* c if v dominates every vertex in V_c. The *dominator chromatic number* $\chi_d(G)$ is the minimum number of colors in a dominator coloring of G. For simplicity, we denote a dominator coloring φ of cardinality $\chi_d(G)$ by $\chi_d(G)$-DC φ.

The concept of dominator coloring was introduced by Gera et al. [9] and has since been studied in various contexts; see, for example, [2, 5, 6, 8, 14]. In recent years, this topic has attracted increasing attention due to its connections with domination theory and graph coloring, as well as its relevance to monitoring and control problems in complex networks such as social and biological systems [11, 10].

From a computational perspective, the dominator coloring problem is computationally difficult. In particular, for fixed $k \geq 4$, it is NP-complete to decide whether a graph admits a dominator coloring with at most k colors [6]. Moreover, determining the dominator chromatic number is NP-complete even for split graphs [3]. Research on dominator coloring has mainly focused on designing algorithms for computing such colorings and determining bounds or exact values of $\chi_d(G)$ for special graph classes. Recent works have investigated algorithmic aspects [3, 4] and established results for several graph families, including trees, generalized Petersen graphs, and graph products; see, e.g., [6, 13, 7, 12].

In this context, the Mycielskian construction is of particular interest. For a graph $G = (V, E)$, the *Mycielskian* $\mu(G)$ has vertex set $V \cup V' \cup \{u\}$ and edge set

$$E \cup \{xy' : x, y \in V,\ xy \in E\} \cup \{x'u : x' \in V'\},$$

where $V' = \{x' : x \in V\}$ is disjoint from V and $u \notin V(G)$. The vertices x and x' are called *twins*, and u is the *root* of $\mu(G)$. This construction is a classical method for increasing the chromatic number of a graph while preserving certain structural properties [15]. Therefore, studying dominator colorings of Mycielskian graphs is a natural and interesting direction.

Arumugam et al. [2] initiated the study of dominator colorings of Mycielskian graphs and established the following bounds.

Theorem 1.1. [2] *For any graph G,*

$$\chi_d(G) + 1 \leq \chi_d(\mu(G)) \leq \chi_d(G) + 2.$$

Moreover, if there exists a $\chi_d(G)$-dominator coloring of G in which every vertex v dominates a color class V_i with $v \notin V_i$, then $\chi_d(\mu(G)) = \chi_d(G) + 1$.

In [2], the authors posed the following open problems.

Questions.

1. Characterize graphs G for which $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1$.
2. For which cycles C_n does $\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$ hold?

Subsequent results were reported in [1, 16]. We adopt the terminology introduced in [1]. Let $\mathscr{C} = \{V_1, \ldots, V_k\}$ be a dominator coloring of G with $\chi_d(G) = k$. A color class V_i is called a *spare color class* if every vertex $v \in V(G)$ dominates some V_j with $j \neq i$. A vertex v is *solitary* if $\{v\} \in \mathscr{C}$ and $N(v)$ does not contain any color class.

Theorem 1.2. [1] *Given a graph G, $\chi_d(\mu(G)) = \chi_d(G) + 1$ if and only if for some $\chi_d(G)$-DC of G with $\mathscr{C} = \{V_1, \ldots, V_k\}$:*

(i) *each vertex v dominates some V_i with $v \notin V_i$;*
(ii) *there exists a solitary vertex v and $\mathscr{C}$ contains a spare color class V_i disjoint from $N(v)$.*

Since conditions (i) and (ii) cannot hold simultaneously, Theorem 1.2 shows that $\chi_d(\mu(G)) = \chi_d(G) + 1$ holds if and only if exactly one of (i) and (ii) is satisfied.

Theorem 1.3. [16] *$\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$ for $n \geq 8$.*

However, we show that

$$\chi_d(\mu(C_8)) = \chi_d(C_8) + 1 \quad \text{and} \quad \chi_d(\mu(C_9)) = \chi_d(C_9) + 2,$$

which contradicts Theorem 1.3. Moreover, we construct a dominator coloring of C_9 satisfying the conditions of Theorem 1.2, showing that this theorem does not hold in general.

Motivated by this discrepancy, we revisit the problem of determining $\chi_d(\mu(C_n))$. Our main result (Theorem 1.4) provides a complete characterization of the cases in which $\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$.

Theorem 1.4. *Let C_n be a cycle of order n. Then $\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$ if and only if $n \equiv 3 \pmod 6$ with $n \neq 3$, or $n \equiv 5 \pmod 6$.*

2 Preliminaries

The following theorems play a crucial role in our proof.

Theorem 2.1. [8] *For the cycle C_n, we have*

$$\chi_d(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & if\ n = 4 \\ \lceil \frac{n}{3} \rceil + 1 & if\ n = 5 \\ \lceil \frac{n}{3} \rceil + 2 & otherwise. \end{cases}$$

Theorem 2.2. [9] *For the path P_n, $n \geq 2$, we have*

$$\chi_d(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1 & \text{if } n = 2, 3, 4, 5, 7 \\ \lceil \frac{n}{3} \rceil + 2 & \text{otherwise.} \end{cases}$$

In order to prove Theorem 1.4, we first consider C_n where $3 \leq n \leq 5$.

Lemma 2.3. [2] *$\chi_d(\mu(C_5)) = \chi_d(C_5) + 2$.*

Lemma 2.4. *$\chi_d(\mu(C_3)) = \chi_d(C_3) + 1$, $\chi_d(\mu(C_4)) = \chi_d(C_4) + 1$.*

Proof. Let $C_3 = v_1v_2v_3v_1$ and $C_4 = v_1v_2v_3v_4v_1$. By Theorem 2.1, we know $\chi_d(C_4) = 2$, and it is easy to verify that $\chi_d(C_3) = 3$. Now we need to prove that $\chi_d(\mu(C_4)) = 3$ and $\chi_d(\mu(C_3)) = 4$. For C_4, we can color v_1, v_2, v_3, v_4 with $1, 2, 1, 2$, which gives a dominator coloring of C_4. Thus, by Theorem 1.1, $\chi_d(\mu(C_4)) = 3$ is verified. For C_3, we can color v_1, v_2, v_3 with $1, 2, 3$, which gives a dominator coloring of C_3. Thus, by Theorem 1.1, $\chi_d(\mu(C_3)) = 4$ is verified. □

We next present a sufficient condition for $\chi_d(\mu(G)) = \chi_d(G) + 1$, which is related to Question (1) in [2].

Theorem 2.5. *Given a graph G, if for some $\chi_d(G)$-dominator coloring of G which has the property that for any solitary vertex v, there is a color class V_i, $i \in \{1, \ldots, \chi_d(G)\}$ such that $V_i \cap N[v] = \emptyset$ and for any $x \in V(G)$, $V_i \nsubseteq N[x]$, then $\chi_d(\mu(G)) = \chi_d(G) + 1$.*

Proof. Let φ be a $\chi_d(G)$-DC of G and $\mathscr{C} = \{V_1, V_2, \ldots, V_{\chi_d(G)}\}$, $V_S \triangleq \{v : \{v\} \in \mathscr{C}\}$. Then for any $v \in V_S$, there exists a color class V_i such that $V_i \cap N[v] = \emptyset$ and for any $x \in V(G)$, $V_i \nsubseteq N[x]$. Then we assign i to v's twin v'. For $w \in V \setminus V_S$, if $w \in V_j$ then we assign j to w'. Finally we assign $\chi_d(G) + 1$ to the root u. Then we get a coloring of $\mu(G)$ and it is easy to verify that coloring is a $(\chi_d(G) + 1)$-DC of $\mu(G)$. □

3 Counterexamples to Theorems 1.2 and 1.3

We first prove that $\chi_d(\mu(C_8)) = \chi_d(C_8) + 1$ and $\chi_d(\mu(C_9)) = \chi_d(C_9) + 2$, and then show that C_8 and C_9 serve as counterexamples to Theorems 1.3 and 1.2, respectively.

Theorem 3.1. *$\chi_d(\mu(C_8)) = \chi_d(C_8) + 1$.*

Proof. By Theorem 1.1, $\chi_d(C_8) + 1 \leq \chi_d(\mu(C_8)) \leq \chi_d(C_8) + 2$. Thus it suffices to construct a dominator coloring of $\mu(C_8)$ using $\chi_d(C_8) + 1$ colors.

Let $C_8 = v_1v_2 \ldots v_8v_1$. By Theorem 2.1, $\chi_d(C_8) = 5$. Define a coloring φ of $\mu(C_8)$ as follows: for $s \in \{1, 2, 3, 4\}$, let $\varphi(v_1) = \varphi(v_3) = \varphi(v_7) = \varphi(v'_{2s-1}) = 1$ and $\varphi(v_4) = \varphi(v_6) = \varphi(v'_{2s}) = 2$; moreover, let $\varphi(v_2) = 3$, $\varphi(v_5) = 4$, $\varphi(v_8) = 5$, and $\varphi(u) = 6$ (see Figure 2(c)).

Each vertex in $V' \cup \{u\}$ dominates color 6, vertices v_1, v_2, v_3 dominate color 3, vertices v_4, v_5, v_6 dominate color 4, and vertices v_7, v_8 dominate color 5. Hence φ is a dominator coloring of $\mu(C_8)$ with $6 = \chi_d(C_8) + 1$ colors. □

Theorem 3.2. $\chi_d(\mu(C_9)) = \chi_d(C_9) + 2$.

Proof. Let $C_9 = v_1 v_2 \dots v_9 v_1$. By Theorem 2.1, $\chi_d(C_9) = 5$. Suppose for contradiction that $\chi_d(\mu(C_9)) = 6$. Let φ be a 6-DC of $\mu(C_9)$. Then $|\varphi(C_9)| \leq 5$, since the root u dominates a color class. Give C_9 a clockwise orientation.

For $i \in \{0, 1, 2\}$, let x^i denote the number of colors appearing exactly i times in $\varphi|_{C_9}$, and let x^{3+} denote the number of colors appearing at least three times. Let C^{3+} denote the set of colors that appear at least three times in $\varphi|_{C_9}$. Since $\Delta(C_9) = 2$, a color appearing once in $\varphi\ |_{C_9}$ can be dominated by at most three vertices of C_9, appearing twice by at most one vertex, and appearing at least three times by none.

If a vertex $v \in C_9$ does not dominate any color class under $\varphi\ |_{C_9}$, then there exists a color $c \notin \varphi(C_9)$ such that $\varphi^{-1}(c) = \{v^{+\prime}, v^{-\prime}\}$, $\{v^{+\prime}\}$, or $\{v^{-\prime}\}$. Since the twin vertex v' must dominate some color under φ, there also exists another color $c' \notin \varphi(C_9)$ such that $\varphi^{-1}(c') = \{v'\}$ or $\{u\}$. Consequently, if there are t such vertices v, then $x^0 \geq \frac{t}{2} + 1$ when $\varphi^{-1}(c') = \{u\}$, and $x^0 \geq t$ otherwise. In particular, $x^0 \geq 2$ when $t = 1$.

We now distinguish two cases according to the value of $|\varphi(C_9)|$.

Case (i). $|\varphi(C_9)| = 5$.

Then $\varphi\ |_{C_9}$ is a 5-DC of C_9. Since $|C_9| = 9$, we obtain

$$\begin{cases} x^1 + x^2 + x^{3+} = 5, \\ x^1 + 2x^2 + |(\varphi\ |_{C_9})^{-1}(C^{3+})| = 9, \\ 3x^1 + x^2 \geq 9. \end{cases}$$

If $x^{3+} = 0$, then $|\varphi^{-1}(C^{3+})| = 0$, which contradicts the system.

If $x^{3+} = 1$, let $C^{3+} = \{c\}$. Consider a path P in C_9 whose endpoints x, y satisfy $\varphi(x) = \varphi(y) = c$. Since $\varphi\ |_{C_9}$ is a 5-DC, there exists at least one vertex $v \in V(P) \setminus \{x, y\}$ such that $\varphi(v)$ appears exactly once in $\varphi\ |_{C_9}$; otherwise a vertex in $N(x) \cap P$ would fail to dominate any color class. Hence $|(\varphi\ |_{C_9})^{-1}(C^{3+})| \leq x^1$, which again contradicts the system.

Thus $x^{3+} \geq 2$. Solving the system yields $x^1 = 3$, $x^2 = 0$, $x^{3+} = 2$ and hence $9 = 3x^1$. Without loss of generality, for $s \in \{0, 1, 2\}$ (with $v_0 = v_9$) let $\varphi(v_{3s+1}) = s + 1$, and $\{\varphi(v_{3s+2}), \varphi(v_{3s+3})\} = \{4, 5\}$. Since 3 is odd, we may assume $\varphi(v_2) = 4$ and $\varphi(v_9) = 5$.

For each $s \in \{0, 1, 2\}$ we have $N(v'_{3s+1}) = \{v_{3s}, v_{3s+2}, u\}$, so $\varphi^{-1}(6) = \{u\}$ or $\{v'_{3s+1}\}$. Hence $\varphi^{-1}(6) = \{u\}$, implying $\varphi(v'_1) \in \{1, 2, 3\}$. Consequently one of v_1, v_4, v_7 fails to dominate any color class, contradicting that φ is a 6-DC.

Case (ii). $|\varphi(C_9)| \le 4$.

Then $x^{3+} \ge 1$; otherwise, we have $x^1 + 2x^2 \le 8 < 9$, which yields a contradiction. Moreover, since $\chi_d(C_9) = 5$, we have $x^0 \ge 2$. Since φ is a 6-DC of $\mu(C_9)$, it follows that

$$\begin{cases} x^0 + x^1 + x^2 + x^{3+} = 6, \\ \max\{x^0, 2(x_0 - 1)\} + 3x^1 + x^2 \ge 9, \\ x^0 \ge 2, \\ x^{3+} \ge 1. \end{cases}$$

If $x^{3+} \ge 2$, the system has no solution. If $x^{3+} = 1$, the possible solutions are

$$(x^0, x^1, x^2) = (4, 1, 0), (2, 2, 1), (3, 2, 0), (2, 3, 0).$$

However, since φ is a proper coloring, we must have $2(x^1 + 2x^2) \ge 9$, which none of these solutions satisfies. This contradiction completes the proof. □

Next, we show that Theorems 1.3 and 1.2 do not hold.

For Theorem 1.3, Theorem 3.1 already provides a counterexample.

For Theorem 1.2, let $C_9 = v_1 v_2 \dots v_9 v_1$. By Theorem 2.1, $\chi_d(C_9) = 5$. Define a proper 5-DC $\mathscr{C} = \{V_1, V_2, \dots, V_5\}$ of C_9 as follows: $V_1 = \{v_1, v_3, v_7\}$, $V_2 = \{v_4, v_6, v_9\}$, $V_3 = \{v_2\}$, $V_4 = \{v_5\}$, and $V_5 = \{v_8\}$ (see Figure 1). Since $\Delta(C_9) = 2$ and $|V_1| = |V_2| = 3 > 2$, both V_1 and V_2 are spare color classes. Since $\{v_2\} = V_3$ and $N(v_2) \subseteq V_1$, the vertex v_2 is solitary. Moreover, $N(v_2) \cap V_2 = \emptyset$. Thus the conditions of Theorem 1.2 are satisfied, implying $\chi_d(\mu(C_9)) = \chi_d(C_9) + 1$, which contradicts Theorem 3.2.

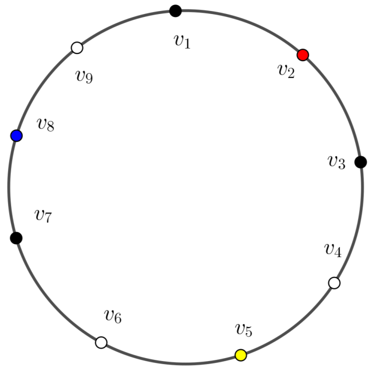

Figure 1: Counterexample of Theorem 1.2

Remark. By Theorem 1.4, we can construct families of counterexamples to Theorems 1.3 and 1.2.

Let $k \ge 1$, by Theorem 2.1, $\chi_d(C_{6k+3}) = 2k + 3$. Define a $\chi_d(C_{6k+3})$-DC coloring φ of C_{6k+3} as follows. For each $m \in \{1, 2, \dots, k\}$, let $\varphi(v_{6m-5}) = \varphi(v_{6m-3}) = \varphi(v_{6k+1}) = 1$ and $\varphi(v_{6m-2}) = \varphi(v_{6m}) = \varphi(v_{6k+3}) = 2$, and assign distinct colors to all remaining vertices. Let $\mathscr{C} = \{V_c : c \in \{1, \dots, 2k+3\}\}$ denote the induced color classes, where $V_c = \varphi^{-1}(c)$.

Observe that v_2 is a solitary vertex and that $\mathscr{C}$ contains a spare color class V_2 disjoint from $N(v_2)$. Hence φ is a DC of C_{6k+3} satisfying condition (ii) of Theorem 1.2. However, by Theorem 1.4, $\chi_d(\mu(C_{6k+3})) = \chi_d(C_{6k+3}) + 2$, contradicting the conclusion of Theorem 1.2. Thus C_{6k+3}, for $k \geq 1$, forms a family of counterexamples to Theorem 1.2.

Moreover, by Theorem 1.4, $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1$ for all $n \equiv 0, 1, 2, 4 \pmod 6$. Hence the cycles C_n with $n \geq 8$ and $n \equiv 0, 1, 2, 4 \pmod 6$ form a family of counterexamples to Theorem 1.3.

4 Main results

Theorems 1.1 and 1.4 together completely answer Question (2) posed in [2] and provide a complete characterization of $\chi_d(\mu(C_n))$ for all $n \geq 3$.

Corollary 4.1. *For the cycle C_n, we have*

$$\chi_d(\mu(C_n)) = \begin{cases} \chi_d(C_n) + 1 & \textit{if } n = 3 \textit{ or } n \equiv 0, 1, 2, 4 \pmod 6 \\ \chi_d(C_n) + 2 & \textit{otherwise.} \end{cases}$$

By Lemmas 2.3 and 2.4, it suffices to consider the case $n \geq 6$.

Let G be a graph and φ be a vertex coloring of $\mu(G)$ using a coloring set $\mathcal{C}$, in other words, φ is a function $\varphi : V(\mu(C_n)) \to \mathcal{C}$. The coloring φ naturally induces a partition of $V(\mu(C_n))$ into color classes $\mathscr{C} = \{V_c : c \in \mathcal{C}\}$, where $V_c = \varphi^{-1}(c)$ for each $c \in \mathcal{C}$. Define $C'^1 = \{c \in \mathcal{C} : |\varphi^{-1}(c)| = 1\}$.

Lemma 4.2. *Let C_n be a cycle of order n, $n \geq 6$. If $n \equiv 0, 1, 2, 4 \pmod 6$, then $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1$.*

Proof. By Theorem 1.1, we know $\chi_d(C_n) + 1 \leq \chi_d(\mu(C_n)) \leq \chi_d(C_n) + 2$. So, it suffices to show a dominator coloring of $\mu(C_n)$ utilizing precisely $\chi_d(C_n) + 1$ colors.

Case (i). $n \equiv 0 \pmod 6$. Without loss of generality, let $n = 6k$, $k \geq 1$ and $C_n = v_1 v_2 \dots v_{6k} v_1$. By Theorem 2.1, $\chi_d(C_n) = 2k + 2$, so we need to prove $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1 = 2k + 3$. Let φ be a proper coloring of $\mu(C_n)$ as follows:

For each $m \in \{1, 2, \dots, k\}$: $\varphi(v_{6m-5}) = \varphi(v_{6m-3}) = 1$, $\varphi(v_{6m-2}) = \varphi(v_{6m}) = 2$;

For each $s \in \{1, 2, \dots, 3k\}$: $\varphi(v'_{2s-1}) = 1$ and $\varphi(v'_{2s}) = 2$;

Assign each of the remaining vertices (including the root u) one color in C'^1. It is easy to verify that φ is a dominator coloring of $\mu(C_n)$. (See Figure 2(a).)

Case (ii). $n \equiv 1 \pmod 6$. Without loss of generality, let $n = 6k + 1$, $k \geq 1$ and $C_n = v_1 v_2 \dots v_{6k+1} v_1$. By Theorem 2.1, $\chi_d(C_n) = 2k + 3$, so we need to prove $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1 = 2k + 4$. Let φ be a proper coloring of $\mu(C_n)$ as follows:

For each $m \in \{1, 2, \ldots, k-1\}$: $\varphi(v_{6m-5}) = \varphi(v_{6m-3}) = \varphi(v_{6k-5}) = \varphi(v_{6k-3}) = 1$, $\varphi(v_{6m-2}) = \varphi(v_{6m}) = \varphi(v_{6k-2}) = \varphi(v_{6k+1}) = 2$;

For each $s \in \{1, 2, \ldots, 3k-1\}$: $\varphi(v'_{2s-1}) = \varphi(v'_{6k-1}) = \varphi(v'_{6k}) = 1$ and $\varphi(v'_{2s}) = \varphi(v'_{6k+1}) = 2$;

Assign each of the remaining vertices (including the root u) one color in C'^1. It is easy to verify that φ is a dominator coloring of $\mu(C_n)$. (See Figure 2(b).)

Case (iii). $n \equiv 2 \pmod 6$. Without loss of generality, let $n = 6k+2$, $k \geq 1$ and $C_n = v_1 v_2 \ldots v_{6k+2} v_1$. By Theorem 2.1, $\chi_d(C_n) = 2k+3$, so we need to prove $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1 = 2k+4$.

Define a proper coloring φ of $\mu(C_n)$ as follows:

For each $m \in \{1, 2, \ldots, k\}$: $\varphi(v_{6m-5}) = \varphi(v_{6m-3}) = 1$, $\varphi(v_{6m-2}) = \varphi(v_{6m}) = 2$;

For the last two vertices: $\varphi(v_{6k+1}) = 1$, assign the vertex v_{6k+2} one color in C'^1;

For each $s \in \{1, 2, \ldots, 3k+1\}$: $\varphi(v'_{2s-1}) = 1$ and $\varphi(v'_{2s}) = 2$;

Assign each of the remaining vertices (including the root u) one color in C^1. It is easy to verify that φ is a dominator coloring of $\mu(C_n)$. (See Figure 2(c).)

Case (iv). $n \equiv 4 \pmod 6$. Without loss of generality, let $n = 6k+4$, $k \geq 1$ and $C_n = v_1 v_2 \ldots v_{6k+4} v_1$. By Theorem 2.1, $\chi_d(C_n) = 2k+4$, so we need to prove $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1 = 2k+5$.

Define a proper coloring φ of $\mu(C_n)$ as follows:

For each $m \in \{1, 2, \ldots, k\}$: $\varphi(v_{6m-5}) = \varphi(v_{6m-3}) = \varphi(v_{6k+1}) = 1$, $\varphi(v_{6m-2}) = \varphi(v_{6m}) = \varphi(v_{6k+4}) = 2$;

For each $s \in \{1, 2, \ldots, 3k+2\}$: $\varphi(v'_{2s-1}) = 1$ and $\varphi(v'_{2s}) = 2$;

Assign each of the remaining vertices (including the root u) one color in C'^1. It is easy to verify that φ is a dominator coloring of $\mu(C_n)$. (See Figure 2(d).) ☐

5 Proof of Theorem 1.4

Proof. Let φ be a $\chi_d(\mu(C_n))$-DC coloring of $\mu(C_n)$, and orient C_n clockwise.

First, classify $\varphi|_{C_n}(V(C_n))$ as: $C^1 = \{c \in \varphi|_{C_n}(V(C_n)) \mid |(\varphi|_{C_n})^{-1}(c)| = 1\}$, $C^2 = \{c \in \varphi|_{C_n}(V(C_n)) \mid |(\varphi|_{C_n})^{-1}(c)| = 2\}$, $C^{3+} = \{c \in \varphi|_{C_n}(V(C_n)) \mid |(\varphi|_{C_n})^{-1}(c)| \geq 3\}$. Correspondingly, partition $V(C_n)$ as: $A^1 = \{v \in V_c \mid c \in C^1\}$, $A^2 = \{v \in V_c \mid c \in C^2\}$, $A^3 = \{v \in V_c \mid c \in C^{3+}\}$. Clearly, $|A^1| = |C^1|$, $|A^2| = 2|C^2|$, and for any $c \in C^2$, the color class V_c can be dominated by at most one vertex in $V(C_n)$ under $\varphi|_{C_n}$. Finally, analogously, define C'^i and A'^i for φ on $\mu(C_n)$.

Recalling Lemmas 2.3, 2.4, 4.2 and Theorem 1.1, we only need to prove Lemmas 5.2 and 5.3. Before proving the two lemmas, we first prove Claim 1.

Claim 1. *For $n \equiv 3, 5 \pmod 6$, $n \geq 9$, if $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1$, then there exists a $(\chi_d(C_n) + 1)$-DC ϕ such that $\phi|_{C_n}$ is a DC of C_n.*

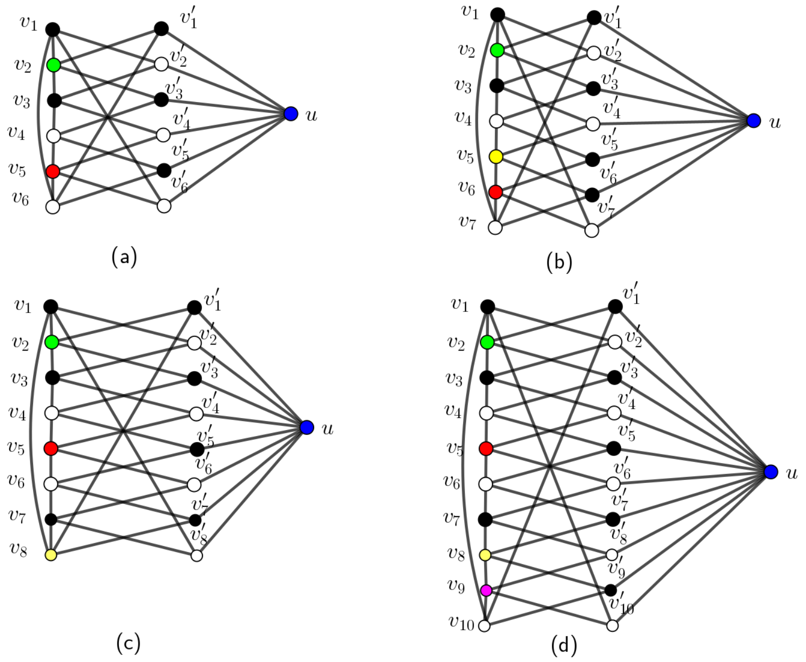

Figure 2: Examples of $\mu(C_6), \mu(C_7), \mu(C_8)$ and $\mu(C_{10})$

Proof of Claim 1. Let $C_n = v_1v_2 \dots v_nv_1$, $n \equiv 3, 5 \pmod 6$. On the contrary, for any $(\chi_d(C_n) + 1)$-DC φ of $\mu(C_n)$, $\varphi\mid_{C_n}$ is not a DC of C_n. Let $D^0_{\varphi|_{C_n}} = \{v : v$ does not dominate any color class in $\mathscr{C}$ under the coloring $\varphi\mid_{C_n}\}$. Then there exists $v \in D^0_{\varphi|_{C_n}}$. Now we analyze the properties of $v \in D^0_{\varphi|_{C_n}}$.

Since $v \in D^0_{\varphi|_{C_n}}$ and φ is a DC of $\mu(C_n)$, v must dominate one color class in $\mathscr{C}$ under the coloring φ. Without loss of generality, assume the color of this color class is c_v, then $c_v \notin \varphi\mid_{C_n}(V(C_n))$. Otherwise, $v \notin D^0_{\varphi|_{C_n}}$. We define *new-colors* as those colors that appear only in V'. Thus, we obtain the following properties:

Property 1. For any $v \in D^0_{\varphi|_{C_n}}$, v must dominate a new-color c_v, and $v^-, v, v^+ \notin A^1$. Furthermore, if v^- (or v^+) belongs to A^2, then $\varphi(v^+) \neq \varphi(v^-)$.

Now consider the coloring of the neighbors of $v \in D^0_{\varphi|_{C_n}}$ in V'. Since φ is a dominator coloring of $\mu(C_n)$, we have the following:

Property 2. For any $v \in D^0_{\varphi|_{C_n}}$, the set $N_{\mu(C_n)}(v) \cap V'$ must satisfy one of the following:

1. $N_{\mu(C_n)}(v) \cap V'$ contains one color $c'_1 \in C'^1$;
2. $\varphi(v^{+\prime}) = \varphi(v^{-\prime}) = c'_2 \in C'^2$.

Clearly, c'_1 and c'_2 are new-colors. Next, we proceed by considering the value of $\varphi(u)$:

Case 1. $\varphi(u) \in C'^1$.

Firstly, we define two transformation methods as follows:

Definition 5.1. Two transformation methods for $v \in D^0_{\varphi|_{C_n}}$:

Method 1. In the first case of Property 2, without loss of generality, assume that $v^{-\prime}$ is assigned the color $c_1' \in C'^1$. We exchange the color of $v^{-\prime}$ with that of v^- to obtain a new coloring φ_{sw} of $\mu(C_n)$.

Method 2. In the second case of Property 2, we obtain a coloring φ_{sw} of $\mu(C_n)$ by swapping the color of $v^{+\prime}$ with that of v^+, and the color of $v^{-\prime}$ with that of v^-.

By Property 1, we have $v^+, v^- \notin A^1$. After the above transformation, the set of colors assigned to the neighbors of each vertex in V remains unchanged. That is, for every $v \in V$, we have

$$\varphi(N_{\mu(C_n)}(v)) = \varphi_{\text{sw}}(N_{\mu(C_n)}(v)).$$

Therefore, for each vertex in V, if it dominates some colors under φ, then it always dominates the color under φ_{sw}.

Since $\varphi(u) \in C'^1$, it follows that $\varphi(u)$ is always dominated by every vertex in $u \cup V'$. Together with them, we conclude that the transformed coloring φ_{sw} is a $(\chi_d(C_n) + 1)$-DC of $\mu(C_n)$.

Furthermore, since $v \in D^0_{\varphi|_{C_n}}$, it follows from Property 1 that the transformed coloring φ_{sw} uses strictly more colors on C_n than on φ, specifically, at least one additional color.

By the minimality of $\chi_d(C_n)$, as long as $|\varphi_{\text{sw}}(V(C_n))| < \chi_d(C_n)$, there exists a vertex $v \in D^0_\varphi$. Therefore, by repeatedly applying the transformation process, we eventually obtain a $(\chi_d(C_n) + 1)$-DC φ_0 of $\mu(C_n)$ such that $|\varphi_0(V(C_n))| \geq \chi_d(C_n)$.

Since $\varphi(u) \in C'^1$ and $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1$, we have $|\varphi_0(V(C_n))| \leq \chi_d(C_n) + 1 - |\{\varphi(u)\}| = \chi_d(C_n)$. Thus, $|\varphi_0(V(C_n))| = \chi_d(C_n)$. Moreover, if there exists a vertex $v \in D^0_{\varphi|_{C_n}}$ in V, then by Property 2, a new-color would appear in V', which leads to a contradiction. This implies that $\varphi_0 |_{C_n}$ is a $\chi_d(C_n)$-DC of C_n.

Therefore, let $\phi = \varphi_0$. Then ϕ is a $(\chi_d(C_n) + 1)$-DC of $\mu(C_n)$, and $\phi |_{C_n}$ is a $\chi_d(C_n)$-DC of C_n.

Case 2. $\varphi(u) \notin C'^1$.

Let $n = 6k + a$, $a \in \{3, 5\}$, $k \geq 1$. If $|D^0_{\varphi|_{C_n}}| = 0$, then let $\phi = \varphi$, and the conclusion follows. Now consider the case where $|D^0_{\varphi|_{C_n}}| \neq 0$.

Claim 1.1. If $v \in D^0_{\varphi|_{C_n}}$, its twin v' must dominate the color $\varphi(v')$. In other words, its twin $v' \in A'^1$.

Proof of Claim 1.1. Let $v \in D^0_{\varphi|_{C_n}}$, and suppose that under φ, the vertex v' dominates the color c. By the contrary, we assume $c \neq \varphi(v')$. If $c \neq \varphi(u)$, then $N_{\mu(C_n)}[v'] \setminus \{u, v'\} = N_{C_n}(v)$. That is, v dominates the color c under $\varphi |_{C_n}$, which contradicts the assumption that $v \in D^0_{\varphi|_{C_n}}$. So $c = \varphi(u)$. Then, since $\varphi(u) \notin C'^1$, at least one of v^+ or v^- must be colored with $\varphi(u)$. Moreover, since v' dominates the color c under φ, no other vertex in $V \setminus \{v^-, v^+\}$ is colored with $\varphi(u)$. Then $\varphi(u)$ appears in the coloring of C_n, and either $\varphi(u) \in C^1$, in which case $v^- \in A^1$ or $v^+ \in A^1$, or $\varphi(u) \in C^2$, in which case $\varphi(v^-) = \varphi(v^+) = \varphi(u)$. By Property 1, this contradicts the assumption that $v \in D^0_{\varphi|_{C_n}}$. □

We now consider the dominator properties of vertices in the cycle graph C_n. With respect to a coloring function $\varphi \mid_{C_n}$, the vertex set $V(C_n)$ can be partitioned into dominator ($v \notin D^0_{\varphi|_{C_n}}$) and non-dominator ($v \in D^0_{\varphi|_{C_n}}$) vertices .

We say that a color $c \in \varphi \mid_{C_n} (V(C_n))$ is *dominator-effective* under $\varphi \mid_{C_n}$ if there exists a vertex v such that $(\varphi \mid_{C_n})^{-1}(c) \subseteq N_{C_n}[v]$. So, this type of color appears either in C^1 or in C^2 by $\Delta(C_n) = 2$.

Let:

- x^0 be the number of new-colors.
- x^1 be the number of dominator-effective colors under $\varphi \mid_{C_n}$ from C^1,
- x^2 be the number of dominator-effective colors under $\varphi \mid_{C_n}$ from C^2,
- z be the number of remaining colors.

We observe the following.

- If C_n has a $v \in D^0_{\varphi|_{C_n}}$, then its twin v' must be assigned a new-color of C'^1 (since $v' \in A'^1$). So, each new-color in V' can correspond to at most one non-dominator vertex in C_n.
- Each color in C^1 can be dominated by at most three vertices under $\varphi \mid_{C_n}$, and each color in C^2 can be dominated by at most one vertex under $\varphi \mid_{C_n}$.

Hence, we obtain the following equations:

$$x^0 + x^1 + x^2 + z = \chi_d(C_n) + 1 \tag{1}$$

$$x^0 + 3x^1 + x^2 \geq n \tag{2}$$

We say, $|D^0_{\varphi|_{C_n}}| = 0$ when $x^0 \leq 1$. Otherwise, by Claim 1.1, if $|D^0_{\varphi|_{C_n}}| \neq 0$, then $x^0 \geq 1$, so assume that $x^0 = 1$ and $|D^0_{\varphi|_{C_n}}| = 1$. Let $\{v\} = D^0_{\varphi|_{C_n}}$ and v's twin $v' \in A'^1 \cap V'$ must be colored with the new-color. However, since there must exist another new-color in $N_{\mu(C_n)}(v) \cap V'$ by Property 2, this contradicts the fact that $x^0 = 1$.

So we only need to consider the case that $x^0 \geq 2$. Moreover, $x^0 + x^2 \geq 2$.

Recall that $k \geq 1$.

Subcase 2.1. $n = 6k + 3$.

At that time $\chi_d(C_n) + 1 = 2k + 4$.

Subcase 2.1.1. If $z = 0$, then any color is either a new-color in V' or a dominator-effective colors in C^1 and C^2. Therefore, all the vertices in V are assigned the dominator-effective colors in C^1 and C^2. So we have $6k+3 = n = |A^1|+|A^2|+|A^3| =$

$|C^1| + 2|C^2| = x^1 + 2x^2$. From (1), we obtain $x^0 + x^2 + x^1 = 2k + 4$, and from (2) , we obtain $x^0 + x^2 + x^1 + 2x^1 \geq 6k + 3 \Rightarrow x^1 \geq 2k$. Since $2(x^1 + x^2) - x^1 = x^1 + 2x^2 = 6k + 3 \Rightarrow 2(x^1 + x^2) = 6k + 3 + x^1 \geq 8k + 3 \Rightarrow x^1 + x^2 \geq 4k + 2$. Thus, we have $x^0 = 2k + 4 - (x^1 + x^2) \leq -2k + 2$, contradicting $x^0 \geq 2$.

Subcase 2.1.2. If $z \geq 2$, then $x^0 + x^2 + x^1 = 2k + 4 - z \leq 2k + 2$. From (2), we get

$$x^0 + x^2 + x^1 + 2x^1 \geq 6k + 3 \;\Rightarrow\; x^1 \geq 2k + \frac{1}{2} \;\Rightarrow\; x^1 \geq 2k + 1.$$

This implies $x^0 + x^2 \leq 2k + 2 - x^1 \leq 1$, contradicting $x^0 + x^2 \geq 2$.

Subcase 2.1.3. If $z = 1$. Then $x^0 + x^2 + x^1 = 2k + 3$, and again we have $x^1 \geq 2k$ from (2). Because $x^0 \geq 2$, we have $2k + 1 \geq x^1 \geq 2k$.

Now, C_n is colored with $(2k + 4 - x^0)$ colors, where x^1 colors are taken from C^1, x^2 colors are taken from C^2, and there is an additional non-dominator-effective color. Therefore, with these colors, a proper coloring can dye at most $2(x^1 + 2x^2)$ vertices. If $x^1 = 2k + 1$ then $x^0 + x^2 = 2$, which forces $x^0 = 2$, $x^2 = 0$. Thus at most $2(2k + 1) = 4k + 2 < 6k + 3$ vertices can be colored, a contradiction. If $x^1 = 2k$, then $x^0 + x^2 = 3$. In this case, either $x^0 = 3$ and $x^2 = 0$, which gives at most $4k$ colored vertices, or $x^0 = 2$ and $x^2 = 1$, which gives at most $2(2k + 2)$ colored vertices. Both bounds are smaller than $6k + 3$, again yielding a contradiction.

Subase 2.2. $n = 6k + 5$.

At that time $\chi_d(C_n) + 1 = 2k + 5$.

Subcase 2.2.1. If $z = 0$, then $2k + 5$ colors are either a new-color in V' or a dominator-effective color in C^1 or C^2. Therefore, all vertex in V are assigned the dominator-effective colors in C^1 and C^2. So we have $6k+5 = n = |A^1|+|A^2|+|A^3| = |C^1| + 2|C^2| = x^1 + 2x^2$. From (1) $x^0 + x^2 + x^1 = 2k + 5$, and from (2), we obtain $x^0 + x^2 + x^1 + 2x^1 \geq 6k + 5 \Rightarrow x^1 \geq 2k$. Since $2(x^1 + x^2) - x^1 = x^1 + 2x^2 = 6k + 5 \Rightarrow 2(x^1 + x^2) = 6k + 3 + x^1 \geq 8k + 5 \Rightarrow x^1 + x^2 \geq 4k + 3$. Furthermore, we have $x^0 = 2k + 5 - (x^2 + x^1) \leq -2k + 2$, contradicting $x^0 \geq 2$.

Subcase 2.2.2. If $z \geq 3$, then $x^0 + x^2 + x^1 = 2k + 5 - z \leq 2k + 2$. Then from (2) $x^0 + x^2 + x^1 + 2x^1 \geq 6k + 5 \Rightarrow x^1 \geq 2k + 2$. Then $x^0 + x^2 \leq 2k + 2 - x^1 \leq 0$, contradiction.

Subcase 2.2.3. If $z = 1$, then $x^0 + x^2 + x^1 = 2k + 4$, and $x^0 + x^2 + 3x^1 \geq 6k + 5$, which implies $x^1 \geq 2k + 1$. Since $x^0 \geq 2$, we have $2k + 2 \geq x^1 \geq 2k + 1$.

Now, C_n is colored with $2k + 5 - x^0$ colors, where x^1 colors are taken from C^1, x^2 colors are taken from C^2, together with one additional non-dominator-effective color. Therefore, with these colors, a proper coloring can dye at most $2(x^1 + 2x^2)$ vertices. If $x^1 = 2k + 2$, then $x^0 + x^2 = 2$. Since $x^0 \geq 2$, we must have $x^0 = 2$ and $x^2 = 0$. Thus at most $2(2k + 2) = 4k + 4 < 6k + 5$ vertices can be colored, a contradiction. If $x^1 = 2k + 1$, then $x^0 + x^2 = 3$. If $x^2 = 0$, then at most $2(2k + 1) = 4k + 2$ vertices can be colored. If $x^2 = 1$, then at most $2(2k + 3) = 4k + 6$ vertices can be colored. In both cases the number of colored vertices is less than $6k + 5$, again a contradiction.

Subcase 2.2.4. If $z = 2$, let the two non-dominator-effective colors be $\{c, d\}$. And we also have $x^0 + x^2 + x^1 = 2k + 3$ and $x^0 + x^2 + x^1 + 2x^1 \geq 6k + 5$ which implies that

$x^1 \geq 2k{+}1$. By $x^0 \geq 2$, we obtain $x^0 = 2$, $x^2 = 0$ and $x^1 = 2k{+}1$. Now, C_n is colored with $(2k+5-x^0) = 2k+3$ colors, where x^1 colors are taken from C^1 and two colors are taken from $\{c, d\}$. Moreover, by Claim 1.1, $|D^0_{\varphi|_{C_n}}| \leq x^0 = 2$. If $|D^0_{\varphi|_{C_n}}| = 1$, then the total number of vertices of C_n is at most $3(2k+1)+1 = 6k+4 < 6k+5$, a contradiction.

Thus $|D^0_{\varphi|_{C_n}}| = 2$. Since $x^0 = 2$, by Claim 1.1 and Property 2, there exists $v_1 \in D^0_{\varphi|_{C_n}}$ and $v_2 \in D^0_{\varphi|_{C_n}}$ in clockwise, where $v_1v_2 \in E(C_n)$ and their twins are colored with the two new-colors.

Since $v_1, v_2 \in D^0_{\varphi|_{C_n}}$ and $v_1^-, v_2^+ \notin D^0_{\varphi|_{C_n}}$ which means $v_1, v_2, v_1^-, v_2^+ \notin A^1$. Moreover, $v_1^{-2}, v_2^{+2} \in A^1$. Otherwise, if $v_1^{-2} \notin A^1$ ($v_2^{+2} \notin A^1$) then $\varphi\mid_{C_n}(v_1^{-2}) = \varphi\mid_{C_n}(v_1) \in A^2$ ($\varphi\mid_{C_n}(v_2^{+2}) = \varphi\mid_{C_n}(v_2) \in A^2$), contradicting $x^2 = 0$. Let $P = C_n \setminus \{v_1, v_2\}$, then P is a path of length $n-2 = 6k+3 = 3x^1$ and $\varphi\mid_P$ is a $(2k+3)$-DC of P that colored with $2k+3$ colors, where $2k+1$ colors are taken from C^1 and two colors are taken from $\{c, d\}$. Since $\varphi(u) \notin C'^1$, we have $\varphi(u) \in \{c, d\}$. Without loss of generality, assume $\varphi(u) = d$. Then for every $v' \in V' \backslash \{v_1', v_2'\}$ we have $\varphi(v') = c$, which is impossible.

Thus, $|D^0_{\varphi|_{C_n}}| = 0$, we let $\phi = \varphi$.

In conclusion, as long as there exists a $(\chi_d(C_n)+1)$-DC of $\mu(C_n)$, based on that coloring (regardless of its specific form), we can construct a $(\chi_d(C_n)+1)$-DC ϕ of $\mu(C_n)$ such that $\phi\mid_{C_n}$ is a DC of C_n. The proof Claim 1 is complete. □

Lemma 5.2. *If* $n \equiv 3 \pmod 6$, $n > 3$, *then* $\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$.

Proof. On the contrary, by Theorems 1.1 and 3.2, we suppose that $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1$ and $n > 9$. Let $n = 6k+3$ with $k \geq 2$, $C_n = v_1v_2 \cdots v_{6k+3}v_1$. Then $\chi_d(C_n) = 2k+3$ and $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1 = 2k+4$. By Claim 1, we may assume that φ is a $(\chi_d(C_n)+1)$-DC of $\mu(C_n)$ such that $\varphi|_{C_n}$ is a DC of C_n. Then $|\varphi\mid_{C_n}(V(C_n))| = \chi_d(C_n)$, otherwise u can not dominate any color class. We first consider the property of $\chi_d(C_n)$-DC $\varphi\mid_{C_n}$.

Claim 2. *If* $n \equiv 3 \pmod 6$, *then* $|C^{3+}| \geq 2$.

Proof. We can easily obtain $|C^{3+}| \neq 0$, since if $|C^{3+}| = 0$ then we have $2 \times (2k+3) < 6k+3$. Now we suppose that $|C^{3+}| = 1$. Let $C^{3+} = \{c\}$. Consider a path $|P| \geq 3$ in C_n with two different ends $x, y \in A^3$, then $V(P) \setminus \{x, y\} \cap A^1 \neq \emptyset$, otherwise $v \in N(x) \cap P$ does not dominate any color class under $\varphi\mid_{C_n}$, which is a contradiction. Thus, $|A^3| = |(\varphi\mid_{C_n})^{-1}(c)| \leq |A^1| = |C^1|$. Let $|C^1| = m$, then $|C^2| = \chi_d(C_n) - |C^1| - |C^{3+}| = 2k+3-m-1 = 2k+2-m$. We have $n = |A^1| + |A^2| + |A^3| \leq 1 \times |C^1| + 2 \times |C^2| + 1 \times |C^1| = m + 2 \times (2k+2-m) + m = 4k{+}4 < 6k{+}3$, which is also a contradiction. So, $|C^{3+}| \geq 2$, where $n \equiv 3 \pmod 6$. □

Let $c_3 \in C^{3+}$.

Claim 3. *There does not exist a vertex* $v \in V(C_n)$*, such that* v *only dominates the color* $c \in C^2$ *in* $\varphi\mid_{C_n}$.

Proof. On the contrary, without loss of generality, let $v \in V(C_n)$ only dominate a color $c \in C^2$, then $v \notin A^1$, $\varphi\mid_{C_n}(v^+) = \varphi\mid_{C_n}(v^-) = c$ and at least one of $v^{+2}, v^{-2} \in A^1$ (it can be easily verified by considering $v \in A^2$ or $v \in A^3$).

Case (i). Both $v^{+2}, v^{-2} \in A^1$. We can obtain a C_{6k+1} by deleting v, v^- and adding an edge $v^{-2}v^+$. After that we can get a dominator coloring ψ by changing the color on v^+ to c_3 in $\varphi\mid_{C_{6k+1}}$, since $N_{C_{6k+1}}(v^+) \subseteq A^1$. But at this time, the number of colors under ψ is $(2k+3-1) < 2k+3 = \chi_d(C_{6k+1})$, which is a contradiction by Theorem 2.1.(see Figure 3)

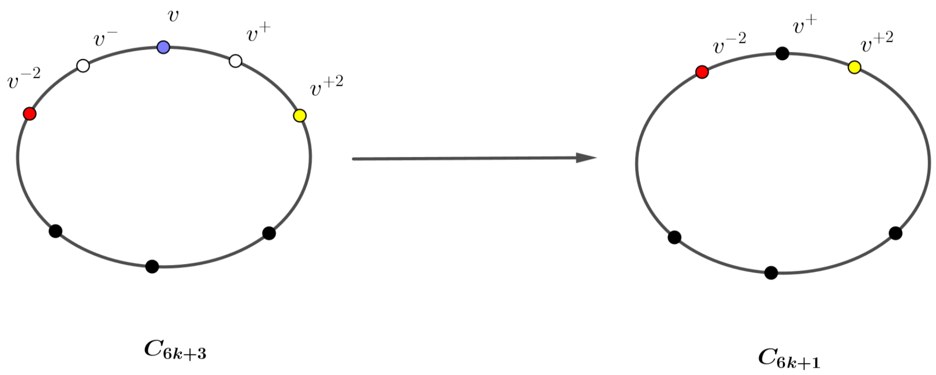

Figure 3: Case (i) in Claim 3

Case (ii). Only one of $v^{+2}, v^{-2} \in A^1$. Suppose $v^{-2} \in A^1$, then $c \neq \varphi\mid_{C_n}(v^{+2}) = \varphi(v)\mid_{C_n} \in C^2$, $v^{+3} \in A^1$ (by the dominating property of v^+ and v^{+2} respectively). We obtain a C_{6k} by deleting v, v^-, v^+ and adding an edge $v^{-2}v^{+2}$. After that we can get a dominator coloring ψ by changing the color on v^{+2} to c_3 in $\varphi|_{C_{6k}}$, since $N_{C_{6k}}(v^{+2}) \subseteq A^1$. But at this time the number of colors under ψ is $(2k+3-2) < 2k+2 = \chi_d(C_{6k})$, which is a contradiction by Theorem 2.1.(see Figure 4) □

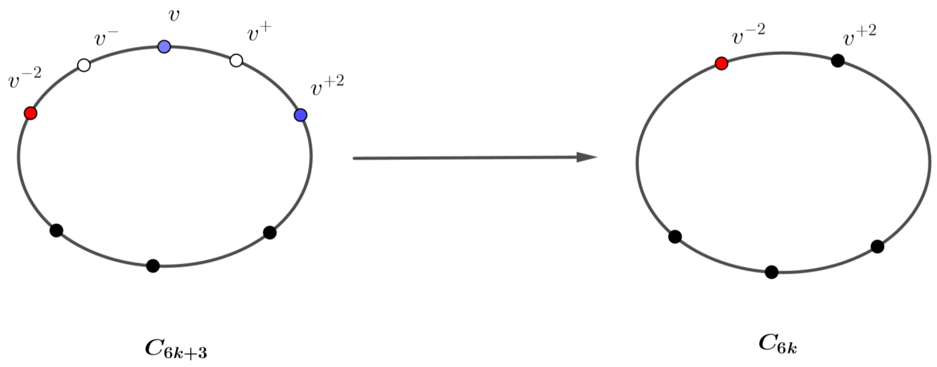

Figure 4: Case (ii) in Claim 3

So, we do not have a $v \in V(C_n)$ that only dominates a color class with size two. Since $\Delta(C_n) = 2$, the color class with size 1 can be dominated by at most 3 vertices. Since we have $6k+3$ vertices, $|C^1| \geq 2k+1$. We recall that $|\varphi |_{C_n} (V(C_n))| = \chi_d(C_n) = 2k+3$ and $|C^{3+}| \geq 2$. Consequently, we have $|C^1| = 2k+1$, $|C^2| = 0$ and $|C^{3+}| = 2$. Let $\varphi |_{C_n} (V(C_n)) = \{1, 2, \ldots, 2k+3\}$, $C^{3+} = \{1, 2\}$ and $C^1 = \{3, 4, \ldots, 2k+3\}$. Since $n = 6k+3 = 3 \times (2k+1) = 3|A^1|$, $k \geq 1$, no matter how we color C_n, there exists a vertex $v \in A^1$ under $\varphi |_{C_n}$ where $\{1, 2\} \subseteq \varphi |_{C_n} (N_{C_n}(v))$. Suppose $\varphi |_{C_n} (v^-) = 1$, $\varphi |_{C_n} (v^+) = 2$, and we have $\varphi |_{C_n} (v^{-2}) = 2$, $\varphi |_{C_n} (v^{+2}) = 1$, $v^{+3} \in A^1$ and $v^{-3} \in A^1$. At that time v' must be colored by a new color $2k+4$, $\varphi(v^{-3\prime}) \in \{2k+4, 1\}$, $\varphi(v^{+3\prime}) \in \{2k+4, 2\}$. If $v^{+3\prime}$ or $v^{-3\prime}$ is colored $2k+4$ in φ, then v' cannot dominate any color class, leading to contradiction. So, $\varphi(v^{-3\prime}) = 1$ and $\varphi(v^{+3\prime}) = 2$. Then u cannot be colored 1, 2, $2k+4$ by the colorable property of φ and $\varphi(u) \notin \{3, 4, \ldots, 2k+3\}$ by the dominating property of φ, which is a contradiction. ☐

By Lemma 2.4, we have $\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$ when $n \equiv 3 \pmod 6$ except $n = 3$. Now we only need to consider $n \equiv 5 \pmod 6$.

Lemma 5.3. *If* $n \equiv 5 \pmod 6$*,* $n > 5$*, then* $\chi_d(\mu(C_n)) = \chi_d(C_n) + 2$.

Proof. On the contrary, by Theorem 1.1, we suppose that $\chi_d(\mu(C_n)) = \chi_d(C_n) + 1$. Let $n = 6k+5$, $k \geq 1$. By Claim 1 we can assume that φ is $(\chi_d(C_n)+1)$-DC of $\mu(C_n)$ such that $\varphi |_{C_n}$ is a DC of C_n. Then $|\varphi |_{C_n} (V(C_n))| = \chi_d(C_n)$, otherwise suppose that $|\varphi |_{C_n} (V(C_n))| = \chi_d(C_n) + 1 = \chi_d(\mu(C_n))$, u cannot dominate any color class. By Theorem 2.1 we can obtain $\chi_d(C_n) = 2k+4$.

Claim 4. *If* $n \equiv 5 \pmod 6$*, then* $|C^{3+}| \geq 2$.

Proof. Since $2 \times (2k+4) < 6k+5$ when $k \geq 2$ and the case $k = 1$ is easy to verify that $|C^{3+}| \neq 0$. So we have $|C^{3+}| \neq 0$. We suppose that $|C^{3+}| = 1$ and $\{c\} = C^{3+}$. Consider a path P, $|P| \geq 3$ and $P \subseteq C_n$ with two different ends $x, y \in A^3$, then $V(P) \setminus \{x, y\} \cap A^1 \neq \emptyset$, otherwise $v \in N(x) \cap P$ does not dominate any color class under $\varphi |_{C_n}$, which is a contradiction. Thus, $|A^3| = |(\varphi |_{C_n})^{-1}(c)| \leq |A^1| = |C^1|$. Let $|C^1| = m$, then $|C^2| = \chi_d(C_n) - |C^1| - |C^{3+}| = 2k+4-m-1 = 2k+3-m$. We have $n = |A^1|+|A^2|+|A^3| \leq 1 \times |C^1| + 2 \times |C^2| + 1 \times |C^1| = m + 2 \times (2k+3-m) + m = 4k+6 < 6k+5$, which is also a contradiction. So, $|C^{3+}| \geq 2$, where $n \equiv 5 \pmod 6$. ☐

By Claim 4, without loss of generality, we denote two different colors $c_3^1, c_3^2 \in C^{3+}$.

Claim 5. *If* $n \equiv 5 \pmod 6$*, then there does not exist a* $v \in V(C_n)$*, such that* v *only dominates a color* $c \in C^2$*.*

Proof. On the contrary, without loss of generality, let $v \in V(C_n)$ only dominate a color $c \in C^2$. The same as Claim 3, we have $\varphi |_{C_n} (v^+) = \varphi |_{C_n} (v^-) = c$ and at least one of $v^{+2}, v^{-2} \in A^1$.

Case (i). Both $v^{+2}, v^{-2} \in A^1$. We obtain a C_{6k+4} by deleting v^- and adding an edge $v^{-2}v$. After that we can get a proper coloring ψ by changing the color on v, v^+

to c_3^1, c_3^2 in $\varphi|_{C_{6k+4}}$, respectively. Since $N_{C_{6k+4}}(v^+) \cap A^1 \neq \emptyset$, $N_{C_{6k+4}}(v) \cap A^1 \neq \emptyset$, $c_3^1 \neq c_3^2$, ψ is a dominator coloring of C_{6k+4}. But at this time, the number of colors under ψ is $(2k+4-1) < 2k+4 = \chi_d(C_{6k+4})$, which leads to a contradiction.(see Figure 5)

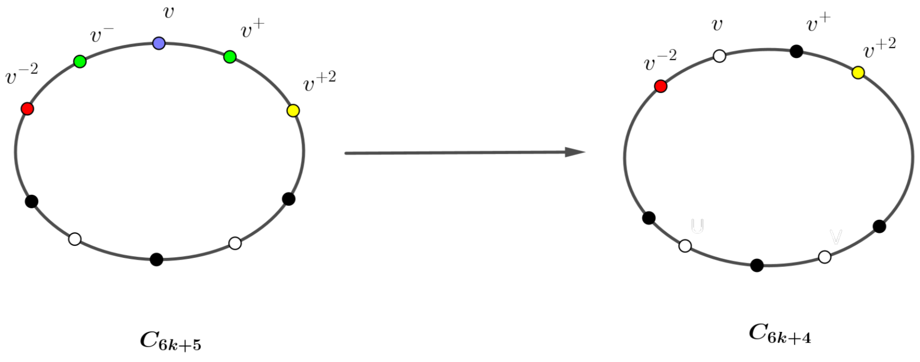

Figure 5: Case (i) in Claim 5

Case (ii). Only one of $v^{+2}, v^{-2} \in A^1$. Suppose $v^{-2} \in A^1$, then $c \neq \varphi\ |_{C_n}\ (v^{+2}) = \varphi\ |_{C_n}\ (v) \in C^2$, $v^{+3} \in A^1$, otherwise v^+ or v^{+2} will not dominate any colors respectively. We obtain a C_{6k+2} by deleting v, v^-, v^+ and adding an edge $v^{-2}v^{+2}$. After that we can get a dominator coloring ψ by changing the color on v^{+2} to c_3^1 in $\varphi|_{C_{6k+2}}$, since $N_{C_{6k+2}}(v^{+2}) \in A^1$. But at this time the number of colors under ψ is $(2k+4-2) < 2k+3 = \chi_d(C_{6k+2})$, which leads to a contradiction.(see Figure 6) □

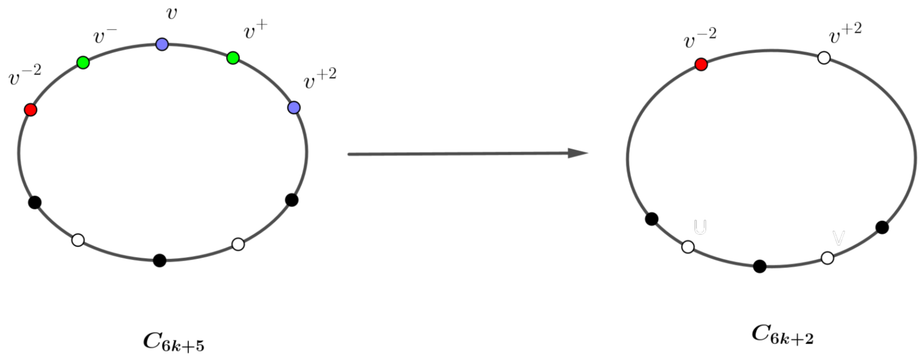

Figure 6: Case (ii) in Claim 5

Similarly, we do not have a $v \in V(C_n)$ that only dominates a color class with size two.

Since we have $6k+5$ vertices and $\chi_d(C_n) = 2k+4$, then $2k+4 \geq |C^1| \geq \lceil \frac{6k+5}{3} \rceil = 2k+2$. By Claim 4, $|C^{3^+}| \geq 2$, so $|C^2| = 0$ and $|C^{3^+}| = 2$. Let $C^{3+} = \{1, 2\}$,

$C^1 = \{3, 4, \ldots, 2k+4\}$ and $2k+5 \notin \varphi\mid_{C_n}(V(C_n))$. Then $\varphi(u) \in \{1, 2, 2k+5\}$, and for any $x' \in V'$, $\varphi(x') \in \{1, 2, 2k+5\}$.

Since $n = 6k+5 = 3 \times (2k+2) - 1$, no matter how we color C_n, there exists a vertex $v \in A^1$ under $\varphi\mid_{C_n}$ in C_n where $\{1, 2\} \subseteq \varphi\mid_{C_n}(N(v))$. We suppose $\varphi\mid_{C_n}(v^-) = 1$, $\varphi\mid_{C_n}(v^+) = 2$. At that time v' must be colored $2k+5$, and for any $x' \in V' \setminus \{v'\}$, $\varphi(x') \neq 2k+5$ otherwise v' cannot dominate any color class. Now $\varphi(u) \in \{1, 2\}$, assume $\varphi(u) = 1$, then for any vertex $x' \in V' \setminus \{v'\}$ we have $\varphi(x') = 2$, which is impossible.

Now we have considered all the cases, so Theorem 1.4 is correct. □

By Theorem 2.5 Theorem 1.1 and Theorem 2.2, it is easy to verify that $\chi_d(\mu(P_n)) = \chi_d(P_n) + 1$ when $n \neq 5, 7$.

Then using the same method we used before, we can also verify that $\chi_d(\mu(P_n)) = \chi_d(P_n) + 2$ when $n \in \{5, 7\}$. So we have:

Corollary 5.4. *For the path P_n, we have*

$$\chi_d(\mu(P_n)) = \begin{cases} \chi_d(P_n) + 2 & \textit{if } n = 5 \textit{ or } 7 \\ \chi_d(P_n) + 1 & \textit{otherwise.} \end{cases}$$

6 Conclusion and scope

The following are some interesting problems for further investigation.

Problem 6.1. Characterize graphs G, if $\chi_d(\mu(G)) = \chi_d(G) + 1$, then there exists a $(\chi_d(G)+1)$-DC ϕ such that $\phi\mid_G$ is a DC of G.

Problem 6.2. Iteratively applying the Mycielskian operator k times for a graph G, we get iterated Mycielskian $\mu^k(G)$ of G. That is, $\mu^k = \mu(\mu^{k-1}(G))$. Is it true that $\chi_d(G) + 2k - 1 \leq \chi_d(\mu^k(G)) \leq \chi(G) + 2k$?

Problem 6.3. Characterize graphs G for which $\chi_d(\mu(G)) = \chi_d(G) + 1$.

Acknowledgements

We sincerely acknowledge the partial support for this work from the National Natural Science Foundation of China under Grants 12171190 and 12061039, and the Fundamental Research Funds for the Central Universities under Grant No. CCNU25JCPT031.

References

[1] A. M. Abid and T. R. Ramesh Rao, Dominator coloring of Mycielskian graphs, *Australas. J. Combin.* 73(2) (2019), 274–279.

[2] S. Arumugam, J. S. Bagga and K. R. Chandrasekar, On dominator colorings in graphs, *Proc. Indian Acad. Sci. Math. Sci.* 122(4) (2012), 561–571.

[3] S. Arumugam, K. R. Chandrasekar, N. Misra, G. Philip and S. Saurabh, Algorithmic aspects of dominator colorings in graphs, *Combin. Algorithms* 7056 (2011), 19–30.

[4] A. Banik, P. N. Kasthurirangan and V. Raman, Dominator coloring and CD coloring in almost cluster graphs, *J. Comput. System Sci.* 150 (2025), Paper No. 103633, 20 pp.

[5] H. Boumediene Merouane and M. Chellali, On the dominator colorings in trees, *Discuss. Math. Graph Theory* 32(4) (2012), 677–683.

[6] M. Chellali and F. Maffray, Dominator colorings in some classes of graphs, *Graphs Combin.* 28(1) (2012), 97–107.

[7] Q. Chen, Dominator colorings of Cartesian product of graphs, *Util. Math.* 109 (2018), 155–172.

[8] R. M. Gera, On dominator colorings in graphs, *Graph Theory Notes N.Y.* 52 (2007), 25–30.

[9] R. M. Gera, C. W. Rasmussen and S. B. Horton, Dominator colorings and safe clique partitions, *Congr. Numer.* 181 (2006), 19–32.

[10] S. K. Grady, F. N. Abu-Khzam, R. D. Hagan et al., Domination based classification algorithms for the controllability analysis of biological interaction networks, *Sci. Rep.* 12 (2022), 11897.

[11] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of domination in graphs*, Monographs and Textbooks in Pure and Applied Mathematics, 208, Dekker, New York, 1998.

[12] M. Li, S. Zhang and C. Ye, Dominated coloring in product graphs, *J. Comb. Optim.* 46(4) (2023), Paper No. 24, 11 pp.

[13] J. Maria Jeyaseeli, N. Movarraei and S. Arumugam, Dominator coloring of generalized Petersen graphs, in *Theoretical Computer Science and Discrete Mathematics*, Lec. Notes in Comput. Sci. 10398 (2017), 144–151.

[14] L. Martin and D. Sindhuja, On dominator chromatic number of graphs, *Int. J. Math. Sci. Eng. Appl.* 5(6) (2011), 191–198.

[15] J. Mycielski, Sur le coloriage des graphs, *Colloq. Math.* 3 (1955), 161–162.

[16] R. Rangarajan and D. A. Kalarkop, A note on dominator chromatic number of graphs, *Montes Taurus J. Pure Appl. Math.* 3(2) (2021), 1–7.

(Received 22 Sep 2025; revised 16 Mar 2026)