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Variants of the Euler operator and Eulerian polynomials

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Abstract

In this paper, motivated by the definition of a non-commutative harmonic oscillator studied by Parmeggiani and Wakayama [*Proc. Natl. Acad. Sci. USA* 98 (2001), 26–30], we introduce some variants of the Euler operator with the aim of exploring the similarity between recurrence relations and differential operators. By using variants of the Euler operator, we establish a link between Eulerian polynomials of types A and B, and second-order Eulerian polynomials of types A and B as well as Narayana polynomials of types A and B.

1 Introduction

The operator $x\frac{\mathrm{d}}{\mathrm{d}x}$ is often called the *Euler operator*, since Leonhard Euler in 1736 initiated the study of the following series summation or successive differentiation (see [8]):

$$\sum_{k=0}^{\infty} k^n x^k = \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{1-x} = \frac{A_n(x)}{(1-x)^{n+1}}, \tag{1}$$

where $A_n(x)$ are known as the classical *Eulerian polynomials*. They satisfy the recursion

$$A_n(x) = nxA_{n-1}(x) + x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_{n-1}(x),\ A_0(x) = 1.$$

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Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, 2, \ldots, n\}$. For $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$, the index $i \in [n-1]$ is an *excedance* if $\pi(i) > i$. Let $\mathrm{exc}\,(\pi)$ be the number of excedances of π. A well known interpretation of Eulerian polynomials is given as follows:

$$A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)+1}.$$

The Eulerian polynomials and their generalizations have been widely studied because of their natural occurrence in many different contexts, see [8, 10, 13, 16] for instance.

Let Q be a second-order differential operator defined by

$$Q = \frac{1}{2}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\left(x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\right).$$

This system defined by Q is called the non-commutative harmonic oscillator. In [14], Parmeggiani and Wakayama found that Q defines a positive and self-adjoint operator in the Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^2$, which has a discrete spectrum $(0 <)\lambda_1 \leqslant \lambda_2 \leqslant \cdots \lambda_n \leqslant \cdots \to \infty$. The operator $x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2}$ is known as the Hermitian operator. It should be noted that

$$x^2 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right) - \left(x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}x\right).$$

Let $\{h, e^+, e^-\}$ be the standard basis of the complexified Lie algebra $\mathfrak{sl}_2$. An oscillator representation w of $\mathfrak{sl}_2$ can be defined by

$$w(h) = x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2},\ w(e^+) = \frac{i}{2}x^2,\ w(e^-) = \frac{i}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}.$$

Let $\pm[n] := [n] \cup \{\bar{1}, \ldots, \bar{n}\}$, where $\bar{i} = -i$. Let $\mathfrak{S}_n^B$ be the *hyperoctahedral group* of rank n. Elements of $\mathfrak{S}_n^B$ are permutations of $\pm[n]$ with the property that $\sigma(\bar{i}) = -\sigma(i)$ for all $i \in [n]$. Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathfrak{S}_n^B$. We say that $i \in [n]$ is a *weak excedance* of σ if $\sigma(i) = i$ or $\sigma(|\sigma(i)|) > \sigma(i)$ (see [2, p. 431]). Let $\mathrm{wexc}\,(\sigma)$ be the number of weak excedances of σ. The number of *descents* of σ is defined by

$$\mathrm{des}_B(\sigma) = \#\{i \in \{0, 1, \ldots, n-1\} \mid \sigma(i) > \sigma(i+1)\},\ \sigma(0) = 0.$$

It is well known that the statistics des_B and wexc have the same distribution over $\mathfrak{S}_n^B$ (see [2, Theorem 3.15]), and their common enumerative polynomial is the *type B Eulerian polynomial*:

$$B_n(x) = \sum_{\sigma\in\mathfrak{S}_n^B} x^{\mathrm{des}_B(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n^B} x^{\mathrm{wexc}\,(\sigma)}.$$

According to [2, Theorem 3.4], the *type B Eulerian polynomials* satisfy the recursion

$$B_n(x) = (1 + (2n-1)x)B_{n-1}(x) + 2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}B_{n-1}(x),\ B_0(x) = 1. \tag{2}$$

Let D denote the differential operator. Motivated by the definition of the harmonic oscillator Q, we note that $2^n(xD + \frac{1}{2})^n = (2xD + 1)^n$. The motivation of this paper is to explore the similarity between recurrence relations and differential operator. We first accidental discover the following result.

Proposition 1.1. *We have*

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\right)^n\frac{1}{1-x}=\frac{B_n(x)}{2^n(1-x)^{n+1}}, \tag{3}$$

where $B_n(x)$ *are the type* B *Eulerian polynomials.*

Proof. Notice that

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\right)\frac{1}{1-x}=\frac{1+x}{2(1-x)^2}.$$

So the result holds for $n=1$. Assume it holds for $n=m$, where $m\geqslant 1$. Then

$$\begin{aligned}\left(x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\right)^{m+1}\frac{1}{1-x}&=\left(x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\right)\frac{B_m(x)}{2^m(1-x)^{m+1}}\\&=x\frac{\mathrm{d}}{\mathrm{d}x}\frac{B_m(x)}{2^m(1-x)^{m+1}}+\frac{1}{2}\frac{B_m(x)}{2^m(1-x)^{m+1}}\\&=\frac{1}{2^m}\frac{(m+1)xB_m(x)+x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}B_m(x)}{(1-x)^{m+2}}+\frac{B_m(x)}{2^{m+1}(1-x)^{m+1}}\\&=\frac{(1+(2m+1)x)B_m(x)+2x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}B_m(x)}{2^{m+1}(1-x)^{m+2}}.\end{aligned}$$

It follows from (2) that the result holds for $n=m+1$. This completes the proof. □

Using the binomial expansion formula, we see that

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\right)^n\frac{1}{1-x}=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{2}\right)^{n-k}\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^k\frac{1}{1-x}.$$

It follows from (1) and Proposition 1.1 that

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}2^k(1-x)^{n-k}A_k(x).$$

This paper is organized as follows. In the next section, we consider some generalizations of (1) and (3). In particular, a unified extension of (1) and (3) is given by Theorem 2.6:

$$\left((1+q)x\frac{\mathrm{d}}{\mathrm{d}x}+1\right)^n\frac{1}{1-x}=\frac{B_n(x,q)}{(1-x)^{n+1}},$$

where x marks the number of type B descents of signed permutations in the hyperoctahedral group $\mathfrak{S}_n^B$ and q marks that of negative entries. When $n\geqslant 1$ and $q=0$, it reduces to

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}+1\right)^n\frac{1}{1-x}=\frac{A_n(x)}{x(1-x)^{n+1}}.$$

In Section 3, we consider variants of the Euler operator. In particular, a variant of the Euler operator can be used to generate second-order q-Eulerian polynomials of type B.

2 Generalizations of (1) and (3)

Let k be a positive integer. In [15], Savage-Viswanathan introduced the $1/k$-*Eulerian polynomials* $A_n^{(k)}(x)$, which can be defined by any of the following relations:

$$\sum_{n=0}^{\infty} A_n^{(k)}(x)\frac{z^n}{n!} = \left(\frac{1-x}{\mathrm{e}^{kz(x-1)}-x}\right)^{\frac{1}{k}},$$

$$\sum_{t\geqslant 0}\binom{t-1+\frac{1}{k}}{t}(kt)^n x^t = \frac{x^n A_n^{(k)}(1/x)}{(1-x)^{n+\frac{1}{k}}}. \tag{4}$$

In particular, $A_n(x) = x^n A_n^{(1)}(1/x)$. The $1/k$-Eulerian polynomials have been extensively studied in recent years, see [8, 11]. By (4), we find that

$$\left(kx\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{(1-x)^{1/k}} = \frac{x^n A_n^{(k)}(1/x)}{(1-x)^{n+\frac{1}{k}}}. \tag{5}$$

Let $\operatorname{cyc}(\pi)$ be the number of cycles of π. It is now well known (see [16]) that

$$A_n^{(k)}(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{exc}(\pi)} k^{n-\operatorname{cyc}(\pi)}.$$

By [3, Proposition 7.2], it is easy to verify that

$$A_{n+1}^{(k)}(x) = (1+knx)A_n^{(k)}(x) + kx(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_n^{(k)}(x),\ A_0^{(k)}(x) = 1. \tag{6}$$

When $k=2$, then (5) reduces to

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{\sqrt{1-x}} = \frac{x^n A_n^{(2)}(1/x)}{2^n(1-x)^{n+\frac{1}{2}}}.$$

In particular, we have

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)\frac{1}{\sqrt{1-x}} = \frac{x}{2(1-x)^{\frac{3}{2}}},\quad \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^2\frac{1}{\sqrt{1-x}} = \frac{2x+x^2}{4(1-x)^{\frac{5}{2}}}.$$

Motivated by Proposition 1.1, we notice that

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\right)\frac{1}{\sqrt{1-x}} = \frac{1}{2(1-x)^{\frac{3}{2}}},\quad \left(x\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\right)^2\frac{1}{\sqrt{1-x}} = \frac{1+2x}{4(1-x)^{\frac{5}{2}}}.$$

Using (6), it is routine to verify the following result.

Proposition 2.1. *Let* $A_n^{(k)}(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\operatorname{exc}(\pi)} k^{n-\operatorname{cyc}(\pi)}$. *We have*

$$\left(kx\frac{\mathrm{d}}{\mathrm{d}x}+1\right)^n \frac{1}{(1-x)^{1/k}} = \frac{A_n^{(k)}(x)}{(1-x)^{n+\frac{1}{k}}}.$$

We define the *type B 1/k-Eulerian polynomials* as follows:

$$B_n^{(k)}(x) = \sum_{\sigma \in \mathfrak{S}_n^B} x^{\mathrm{wexc}\,(\sigma)} k^{n-\mathrm{cyc}\,(\sigma)}.$$

Lemma 2.2. *We have*

$$B_{n+1}^{(k)}(x) = (1 + x + 2nkx)B_n^{(k)}(x) + 2kx(1 - x)\frac{\mathrm{d}}{\mathrm{d}x}B_n^{(k)}(x), \; B_0^{(k)}(x) = 1. \tag{7}$$

Proof. In the following discussion, we always write a signed permutation, by its standard cycle form, in which each cycle has its smallest (in absolute value) element first and the cycles are written in increasing order of the absolute value of their first elements. For $\sigma \in \mathfrak{S}_n^B$, if $n + 1$ or $\overline{n+1}$ forms a new cycle, then we get the term $(1+x)B_n^{(k)}(x)$. Let $\sigma_{(i)}$ be obtained from σ by inserting $n+1$ or $\overline{n+1}$ into the cycle with $\sigma(i)$ and right after $\sigma(i)$. Note that

$$\mathrm{exc}\,(\sigma_{(i)}) = \begin{cases} \mathrm{exc}\,(\sigma), & \text{if } \sigma(i) < \sigma(|\sigma(i)|); \\ \mathrm{exc}\,(\sigma) + 1, & \text{if } \sigma(i) = i \text{ or } \sigma(i) = \bar{i}; \\ \mathrm{exc}\,(\sigma) + 1, & \text{if } \sigma(i) > \sigma(|\sigma(i)|). \end{cases}$$

Let single $(\sigma) = \#\{i \in [n] : \sigma(i) = \bar{i}\}$. We conclude that

$$\begin{aligned} B_{n+1}^{(k)}(x) = (1 + x)B_n^{(k)}(x) + \sum_{\sigma \in \mathfrak{S}_n^B} \left(2\,\mathrm{wexc}\,(\sigma)x^{\mathrm{wexc}\,(\sigma)}k^{n+1-\mathrm{cyc}\,(\sigma)}\right) + \\ \sum_{\sigma \in \mathfrak{S}_n^B} \left(2\,\mathrm{single}\,(\sigma)x^{\mathrm{wexc}\,(\sigma)+1}k^{n+1-\mathrm{cyc}\,(\sigma)}\right) + \\ \sum_{\sigma \in \mathfrak{S}_n^B} \left(2(n - \mathrm{wexc}\,(\sigma) - \mathrm{single}\,(\sigma))x^{\mathrm{wexc}\,(\sigma)+1}k^{n+1-\mathrm{cyc}\,(\sigma)}\right) \\ = (1+x)B_n^{(k)}(x)+2kx\frac{\mathrm{d}}{\mathrm{d}x}B_n^{(k)}(x)+2kx\sum_{\sigma \in \mathfrak{S}_n^B}(n-\mathrm{wexc}\,(\sigma))x^{\mathrm{wexc}\,(\sigma)}k^{n-\mathrm{cyc}\,(\sigma)} \\ = (1+x)B_n^{(k)}(x)+2kx\frac{\mathrm{d}}{\mathrm{d}x}B_n^{(k)}(x)+2nkxB_n^{(k)}(x) - 2kx^2\frac{\mathrm{d}}{\mathrm{d}x}B_n^{(k)}(x), \end{aligned}$$

and (7) follows. This completes the proof. □

By Lemma 2.2, we find the following generalization of (3), and the proof is omitted for simplicity.

Theorem 2.3. *We have*

$$\left(kx\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\right)^n \frac{1}{(1 - x)^{1/k}} = \frac{B_n^{(k)}(x)}{2^n(1 - x)^{n+\frac{1}{k}}}.$$

Corollary 2.4. *We have*

$$2^n A_n^{(k)}(x) = \sum_{i=0}^{n} \binom{n}{i}(1 - x)^{n-i}B_i^{(k)}(x).$$

Proof. Comparing Proposition 2.1 and Theorem 2.3,

$$\left(kx\frac{\mathrm{d}}{\mathrm{d}x}+1\right)^n\frac{1}{(1-x)^{1/k}}=\sum_{i=0}^{n}\binom{n}{i}\frac{1}{2^{n-i}}\left(kx\frac{\mathrm{d}}{\mathrm{d}x}+\frac{1}{2}\right)^i\frac{1}{(1-x)^{1/k}}$$
$$=\sum_{i=0}^{n}\binom{n}{i}\frac{1}{2^{n-i}}\frac{B_i^{(k)}(x)}{2^i(1-x)^{i+\frac{1}{k}}},$$

which yields the desired result. This completes the proof. □

A *context-free grammar* G over an alphabet V is defined as a set of substitution rules replacing a letter in V by a formal function over V. As usual, the formal function may be a polynomial or a Laurent polynomial. The formal derivative D_G with respect to G satisfies the derivation rules: $D_G(u+v)=D_G(u)+D_G(v),\ D_G(uv)=D_G(u)v+uD_G(v)$. So the *Leibniz rule* holds:

$$D_G^n(uv)=\sum_{k=0}^{n}\binom{n}{k}D_G^k(u)D_G^{n-k}(v).$$

Recently, context-free grammars have been widely used, see [5, 11] for instance.

Let $f(x)=\sum_{i=0}^n f_ix^i$ be a symmetric polynomial. We say that $f(x)$ is γ*-positive* if

$$f(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\gamma_kx^k(1+x)^{n-2k},$$

where $\gamma_k\geqslant 0$ for all $0\leqslant k\leqslant\lfloor n/2\rfloor$.

Proposition 2.5. *The polynomial $B_n^{(k)}(x)$ is γ-positive for any $n\geqslant 1$.*

Proof. Consider $G=\{x\to xy^{2k},\ y\to x^{2k}y\}$. Note that

$$D_G(xy)=xy(x^{2k}+y^{2k})=xy^{2k+1}\left(1+\frac{x^{2k}}{y^{2k}}\right).$$

Using (7), the following grammatical description of $B_n^{(k)}(x)$ follows by induction:

$$D_G^n(xy)=xy^{2nk+1}B_n^{(k)}\left(\frac{x^{2k}}{y^{2k}}\right). \tag{8}$$

Setting $u=xy,\ v=x^{2k}+y^{2k}$ and $w=x^{2k}y^{2k}$, we see that

$$D_G(u)=uv,\ D_G(v)=4kw,\ D_G(w)=2kvw.$$

Note that $D_G^2(u)=u(v^2+4kw)$. By induction, we find that there exist nonnegative integers $\gamma_{n,i}(k)$ such that

$$D_G^n(u)=u\sum_{i\geqslant 0}\gamma_{n,i}(k)w^iv^{n-2i},$$

where $\gamma_{n,i}(k)$ satisfy the recursion

$$\gamma_{n+1,i}(k) = (1 + 2ki)\gamma_{n,i}(k) + 4k(n - 2i + 2)\gamma_{n,i-1}(k),$$

with $\gamma_{0,0}(k) = 1$ and $\gamma_{0,i}(k) = 0$ for $i \neq 0$. By (8), we find that

$$B_n^{(k)}(x) = \sum_{i \geqslant 0} \gamma_{n,i}(k) x^i (1 + x)^{n-2i},$$

as desired. This completes the proof. □

For $\sigma \in \mathfrak{S}_n^B$, let $\operatorname{neg}(\sigma)$ be the number of negative entries of σ. For example, $\operatorname{neg}(\overline{3}5124\overline{6}) = 2$. The *q-Eulerian polynomials of type B* are defined by

$$B_n(x, q) = \sum_{\sigma \in \mathfrak{S}_n^B} x^{\operatorname{des}_B(\sigma)} q^{\operatorname{neg}(\sigma)}.$$

In [2, Theorem 3.4], Brenti obtained that

$$B_n(x, q) = (1 + ((1 + q)n - 1)x)B_{n-1}(x, q) + (1 + q)x(1 - x)\frac{\partial}{\partial x} B_{n-1}(x, q), \qquad (9)$$

with $B_0(x, q) = 1$. In particular, $B_1(x, q) = 1 + qx$ and $B_2(x, q) = 1 + (1 + 4q + q^2)x + q^2x^2$. A unified extension of (1) and (3) is given as follows.

Theorem 2.6. *For any $n \geqslant 1$, one has*

$$\left((1 + q)x\frac{\mathrm{d}}{\mathrm{d}x} + 1\right)^n \frac{1}{1 - x} = \frac{B_n(x, q)}{(1 - x)^{n+1}}.$$

Proof. Note that

$$\left((1 + q)x\frac{\mathrm{d}}{\mathrm{d}x} + 1\right) \frac{1}{1 - x} = \frac{1 + qx}{(1 - x)^2}.$$

Assume that the result holds for $n = m$, where $m \geqslant 1$. Then by (9), we get

$$\begin{aligned}
&\left((1 + q)x\frac{\mathrm{d}}{\mathrm{d}x} + 1\right)^{m+1} \frac{1}{1 - x} \\
&= \left((1 + q)x\frac{\mathrm{d}}{\mathrm{d}x} + 1\right) \frac{B_m(x, q)}{(1 - x)^{m+1}} \\
&= (1 + q)x\frac{(1 - x)\frac{\mathrm{d}}{\mathrm{d}x}B_m(x, q) + (m + 1)B_m(x, q)}{(1 - x)^{m+2}} + \frac{B_m(x, q)}{(1 - x)^{m+1}} \\
&= \frac{B_{m+1}(x, q)}{(1 - x)^{m+2}},
\end{aligned}$$

as desired. This completes the proof. □

3 Variants of the Euler operator

3.1 On the second-order Eulerian polynomials of types A and B

Following Carlitz [4], the *second-order Eulerian polynomials* $C_n(x)$ can be defined by

$$\left(\frac{x}{1-x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{1-x} = \frac{C_n(x)}{(1-x)^{2n+1}}. \tag{10}$$

It is well known that

$$C_{n+1}(x) = (1+2n)xC_n(x) + x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}C_n(x),\ C_0(x) = 1. \tag{11}$$

In particular, $C_1(x) = x, C_2(x) = x + 2x^2$ and $C_3(x) = x + 8x^2 + 6x^3$, see [7, 12] for instance.

Let $[n]_2 = \{1, 1, 2, 2, \ldots, n, n\}$ be a multiset, where each i appears two times. We say that the multipermutation σ of $[n]_2$ is a *Stirling permutation* if whenever $\sigma_i = \sigma_j$, then $\sigma_k > \sigma_i$ for all $i < k < j$. Let $\mathcal{Q}_n$ be the set of Stirling permutations of $[n]_2$. A descent of $\sigma \in \mathcal{Q}_n$ is an element σ_i such that $\sigma_i > \sigma_{i+1}$, where $i \in [2n]$ and $\sigma_{2n+1} = 0$. Gessel and Stanley [7] discovered that

$$C_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)},$$

which has been extensively studied in recent years, see [12, 13] for instance.

Set $\bar{i} = -i$. Following Bala [17, A214406], a *signed Stirling permutation* of order n is a signed permutation τ of $\{\overline{n}, \overline{n}, \overline{n-1}, \overline{n-1}, \ldots, \overline{1}, \overline{1}, 1, 1, 2, 2, \ldots, n, n\}$ satisfying the conditions: (i) either the two copies of i or $\bar{i}$ appear in τ, (ii) if we ignore the signs of elements, then τ reduces to an ordinary Stirling permutation in $\mathcal{Q}_n$. Let $\mathcal{Q}_n^B$ be the set of signed Stirling permutations of order n. Clearly, we have $\mathcal{Q}_1^B = \{11, \overline{1}\,\overline{1}\}$ and

$$\mathcal{Q}_2^B = \{1122, 11\overline{2}\,\overline{2}, 1221, 1\overline{2}\,\overline{2}1, 2211, \overline{2}\,\overline{2}11, \overline{1}\,\overline{1}22, \overline{1}\,\overline{1}\,\overline{2}\,\overline{2}, \overline{1}22\overline{1}, \overline{1}\,\overline{2}\,\overline{2}\,\overline{1}, 22\overline{1}\,\overline{1}, \overline{2}\,\overline{2}\,\overline{1}\,\overline{1}\}.$$

Let $\mathrm{neg}\,(\tau)$ be the number of different negative entries of σ. For example, $\mathrm{neg}\,(\overline{1}2332\overline{1}\ \overline{4}\ \overline{4}) = 2$. The number of descents of τ is given by $\mathrm{des}\,(\tau) = \#\{i \in [2n] : \tau_i > \tau_{i+1}\}$, where $\tau_{2n+1} = 0$. We define the *second-order q-Eulerian polynomials of type B* as follows:

$$C_n^B(x,q) = \sum_{\tau \in \mathcal{Q}_n^B} x^{\mathrm{des}\,(\tau)} q^{\mathrm{neg}\,(\tau)}.$$

In particular, $C_1^B(x,q) = q + x$ and $C_2^B(x,q) = q^2 + (1 + 5q + 2q^2)x + (2+q)x^2$. It is clear that $C_n^B(x,1) = \sum_{\tau \in \mathcal{Q}_n^B} x^{\mathrm{asc}\,(\tau)}$, where $\mathrm{asc}\,(\tau) = \#\{i \in \{0, 1, 2, \ldots, 2n-1\} : \tau_i < \tau_{i+1}\}$, where $\tau_0 = 0$.

The $q = 1$ case of the following result can be found in [17, A214406].

Lemma 3.1. *We have*

$$C_{n+1}^B(x,q) = (q + (1+2n+2nq)x)C_n^B(x,q) + (1+q)x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}C_n^B(x,q).$$

Proof. We define

$$C_{n,i,j}^B = \#\{\tau \in \mathcal{Q}_n^B : \mathrm{des}\,(\tau) = i,\ \mathrm{neg}\,(\tau) = j\}.$$

Let $\tau' \in \mathcal{Q}_{n+1}^B$ be obtained from $\tau \in \mathcal{Q}_n^B$ by inserting the two copies of $n+1$ or $\overline{n+1}$. Assume that $\mathrm{des}\,(\tau') = i$ and $\mathrm{neg}\,(\tau') = j$. We distinguish four cases:

(*i*) If $\mathrm{des}\,(\tau) = i$ and $\mathrm{neg}\,(\tau) = j$, then we can only insert the two copies of $n+1$ right after a descent. This case gives the term $iC_{n,i,j}^B$;

(*ii*) If $\mathrm{des}\,(\tau) = i$ and $\mathrm{neg}\,(\tau) = j-1$, then we can insert the two copies of $\overline{n+1}$ right after a descent, or we can also put them at the front of τ. This case gives the term $(1+i)C_{n,i,j-1}^B$;

(*iii*) If $\mathrm{des}\,(\tau) = i-1$ and $\mathrm{neg}\,(\tau) = j$, then we can insert the two copies of $n+1$ right after an ascent, i.e., $\tau_k < \tau_{k+1}$ and $k \geqslant 1$. Moreover, we can also put two copies of $n+1$ at the front of τ. There are $(2n+1-(i-1)) = 2n-i+2$ possibilities and this case gives the term $(2n-i+2)C_{n,i-1,j}^B$;

(*iv*) If $\mathrm{des}\,(\tau) = i-1$ and $\mathrm{neg}\,(\tau) = j-1$, then we can only insert the two copies of $\overline{n+1}$ right after an ascent, i.e., $\tau_k < \tau_{k+1}$ and $k \geqslant 1$. There are $(2n-(i-1)) = 2n-i+1$ possibilities and this case gives the term $(2n-i+1)C_{n,i-1,j-1}^B$.

In conclusion, we get that

$$\begin{aligned} C_{n+1,i,j}^B = {} & iC_{n,i,j}^B + (1+i)C_{n,i,j-1}^B + \\ & (2n-i+2)C_{n,i-1,j}^B + (2n-i+1)C_{n,i-1,j-1}^B. \end{aligned}$$

Multiplying both sides of this recursion by $x^i q^j$ and summing over all i and j, one can immediately get the recursion of $C_n^B(x,q)$. This completes the proof. □

Based on empirical evidence, comparing (11) with Lemma 3.1, we discover the following result.

Theorem 3.2. *We have*

$$\left(\frac{(1+q)x}{2(1-x)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{q}{2(1-x)}\right)^n \frac{1}{1-x} = \frac{C_n^B(x,q)}{2^n(1-x)^{2n+1}}. \tag{12}$$

Proof. Note that

$$\left(\frac{(1+q)x}{2(1-x)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{q}{2(1-x)}\right) \frac{1}{1-x} = \frac{q+x}{2(1-x)^3}.$$

Assume that the result holds for $n = m$, where $m \geqslant 1$. By Lemma 3.1, it is easy to verify that

$$\left(\frac{(1+q)x}{2(1-x)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{q}{2(1-x)}\right) \frac{C_m^B(x,q)}{2^m(1-x)^{2m+1}} = \frac{C_{m+1}^B(x,q)}{2^{m+1}(1-x)^{2m+3}}.$$

This completes the proof. □

It is clear that the $q = 0$ case of (12) reduces to (10).

3.2 On the Narayana polynomials of types A and B

It is well known that the *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$ and the *central binomial coefficients* $\binom{2n}{n}$ have the following expressions (see [6]):

$$C_n = \sum_{k=0}^{n-1} \frac{1}{n}\binom{n}{k+1}\binom{n}{k}, \quad \binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2.$$

The *Narayana polynomials of types A and B* are defined by

$$N_n(x) = \sum_{k=0}^{n-1} \frac{1}{n}\binom{n}{k+1}\binom{n}{k}x^k, \quad N_n^B(x) = \sum_{k=0}^{n}\binom{n}{k}^2 x^k.$$

The Narayana polynomials of types A and B share several similar properties, such as total positivity [18], real-rootedness [9] and γ-positivity [10].

In [1, Eq. (2)], Agapito discovered a new identity:

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right)^n \frac{1}{1-x} = \frac{n!(n+1)!xN_n(x)}{(1-x)^{2n+1}} \text{ for } n \geqslant 1. \tag{13}$$

This similarity between types A and B Narayana polynomials suggests the existence of a type B analogue of (13), which is unavailable in the literature. Our goal of this subsection is to fill in the missing pieces in the type B case.

Theorem 3.3. *We have*

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{1-x} = \frac{n!n!N_n^B(x)}{(1-x)^{2n+1}}.$$

Proof. Note that

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\right) \frac{1}{1-x} = \frac{1+x}{(1-x)^3}.$$

So the result holds for $n = 1$. Assume that the result holds for $n = m$. Then we have

$$\begin{aligned}\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\right)^{m+1} \frac{1}{1-x} &= \left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\right)\frac{m!m!N_m^B(x)}{(1-x)^{2m+1}} \\ &= x\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{m!m!N_m^B(x)}{(1-x)^{2m+1}} + \frac{\mathrm{d}}{\mathrm{d}x}\frac{m!m!N_m^B(x)}{(1-x)^{2m+1}}.\end{aligned}$$

Using the explicit formula $N_m^B(x) = \sum_{k=0}^{m}\binom{m}{k}^2 x^k$, after simplifying, it is easy to verify that

$$\left(x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\right)^{m+1} \frac{1}{1-x} = \frac{(m+1)!(m+1)!N_{m+1}^B(x)}{(1-x)^{2m+3}},$$

as desired. This completes the proof. □

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