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# Minimum embedding of a $\lambda$-fold $P_3$-design into a $\lambda$-fold kite system for any index $\lambda$

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**Abstract**

Let $G'$ be a subgraph of a simple finite graph $G$ and $U$ be a subset of a set $V$. We say that a $\lambda$-fold $G'$-design $(U, \mathcal{C})$ of order $u$ is *embedded* into a $\mu$-fold $G$-design $(V, \mathcal{B})$ of order $u+w$, $\mu \geq \lambda$, if there is an injective function $f : \mathcal{C} \to \mathcal{B}$ such that $C$ is a subgraph of $f(C)$ for every $C \in \mathcal{C}$. The mapping $f$ is called the *embedding* of $(U, \mathcal{C})$ into $(V, \mathcal{B})$. If $w$ attains the minimum possible value, then $f$ is a *minimum* embedding. In this paper a complete solution is given to the problem of determining a minimum embedding of a $\lambda$-fold $P_3$-design into a $\lambda$-fold kite system ($\lambda = \mu$) for any index $\lambda$.

## 1 Introduction

If $G$ is a graph, then let $V(G)$ and $E(G)$ denote the vertex set and edge set of $G$, respectively. The complete graph on the vertices $x_1, x_2, \dots, x_v$ is denoted by $K_v(x_1, x_2, \dots, x_v)$ or, simply, $K_v$. A complete graph of order 3 is called a *triangle*. The path on $k$ vertices is denoted by $P_k$. The complete bipartite graph with bipartition $(A, B)$, where $|A| = m$ and $|B| = n$, is denoted by $K_{m,n}(A, B)$ or, simply, $K_{m,n}$. The graph $K_v \setminus K_h$ has vertex set $V(K_v)$ containing a distinguished subset $H$ of size

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$h$ and edge set $E(K_v)$ with the $\binom{h}{2}$ edges between the $h$ distinguished vertices of $H$ removed (this graph is sometimes referred to as a complete graph of order $v$ with a *hole* of size $h$). The graph $\lambda G$ is obtained by replacing each edge of a $G$ by $\lambda$ parallel edges.

Let $\Gamma$ be a finite graph and $G$ be a subgraph of $\Gamma$. A *$G$-decomposition* (or *$G$-design*) of $\Gamma$ is a pair $(X, \mathcal{B})$ where $X = V(\Gamma)$ and $\mathcal{B}$ is a collection of isomorphic copies of $G$ (*$G$-blocks* or, simply, *blocks*), whose edges partition $E(\Gamma)$. If $\Gamma = \lambda K_v$, then we refer to such a design as a *$\lambda$-fold $G$-design of order $v$* (usually, the integer $\lambda$ is said to be the *index* of the design and when $\lambda = 1$ the term "1-fold" is dropped). The set of all $v \in N$ such that a $\lambda$-fold $G$-design of order $v$ exists is called the *spectrum* for $\lambda$-fold $G$-designs and is denoted by $\Sigma_\lambda(G)$.

Let $G'$ be a subgraph of $G$ and $U$ be a subset of a set $V$. We say that a $\lambda$-fold $G'$-design $(U, \mathcal{C})$ of order $u$ is *embedded* into a $\mu$-fold $G$-design $(V, \mathcal{B})$ of order $u + w$, $\mu \geq \lambda$, if there is an injective function $f : \mathcal{C} \to \mathcal{B}$ such that $C$ is a subgraph of $f(C)$ for every $C \in \mathcal{C}$. The mapping $f$ is called the *embedding* of $(U, \mathcal{C})$ into $(V, \mathcal{B})$. If $w$ attains the minimum possible value, then $f$ is a *minimum* embedding. If $\Gamma'$ is a subgraph of a graph $\Gamma$, by extending the above notion of embedding, we can also speak of a $G'$-design of $\Gamma'$ embedded into a $G$-design of $\Gamma$ with obvious meaning of the terms. Note that a special case occurs when $G = G'$ and the related embedding problem is better known as the Doyen-Wilson problem (see [9,10,14–16]). Embedding problems have been investigated for several pairs of graphs $G'$, $G$ (see, for instance, [6–8, 11, 17, 18, 20, 21]). Their study is motivated by cost optimization in dynamic SONET/WDM networks. In this context, "traffic grooming" aims to minimize the number of Add-Drop Multiplexers (ADMs) by efficiently packing low-rate signals. Unlike static grooming, dynamic grooming accounts for traffic fluctuations (e.g., day-to-night variations). As shown in [8], a "two-times network" can be modeled by embedding one graph decomposition into another. Specifically, the embedding of a $P_3$-design of order $u$ into a kite-design of order $v$ provides a theoretical solution for networks with parameters $(v, u, 4, 2)$, ensuring that the hardware infrastructure can support multiple traffic configurations.

Let $K$ be a *kite*, i.e., the graph consisting of a triangle with one pendant edge (see Figure 1). A $K$-design is usually referred to as a *kite system*. The existence of a $P_3$-design of order $u$ which is embedded into a kite-design of order $v$ realizes a two-times network. The minimum embedding problem of a $P_3$-design into a kite system has been studied and completely solved in [5, 19]. Here we completely solved the above embedding problem for a $\lambda$-fold $P_3$-design of order $u$ into a $\lambda$-fold kite system of order $v$, with $\lambda \geq 2$. Note that the non-trivial paths contained in a kite as subgraphs are $P_3$ and $P_4$. However, the embedding of a $\lambda$-fold $P_4$-design into a $\lambda$-fold kite system is impossible; this is because completing a $P_4$ to a kite requires adding an edge that joins two existing vertices of the $P_4$, which would necessarily increase the index. Minimum embeddings of $\lambda$-fold $P_k$-designs (for $k = 3, 4$) into $\mu$-fold kite systems with $\mu > \lambda$ are investigated in [22]. The main result of this paper, which we will prove in Section 5, is the following:

**Theorem 5.1** For any pair $\lambda \geq 1$ and $u \in \Sigma_\lambda(P_3)$, except for $\lambda \equiv 0 \pmod 4$ and $u = 4, 5, 7$, the minimum order of a $\mathrm{KS}(v, \lambda)$ which embeds a $\mathrm{P}_3(u, \lambda)$ is the minimum integer $v \in \Sigma_\lambda(K)$ such that:

- $v > (3u - 1)/2$ if $\lambda$ is odd;
- $v \geq (3u - 1)/2$ if $\lambda$ is even.

For $\lambda \equiv 0 \pmod 4$ and $u = 4, 5, 7$, the minimum order is $v = \lceil \frac{3u-1}{2} \rceil + 1$.

## 2 Preliminaries

In what follows, we will denote:

- the path $P_3$ consisting of the edges $\{a, b\}$ and $\{b, c\}$ by $[a, b, c]$;
- the triangle $K_3(a, b, c)$ by $(a, b, c)$;
- the kite consisting of the triangle $(a, b, c)$ and the pendant edge $\{c, d\}$ by $(a, b, c) - d$ (see Figure 1).

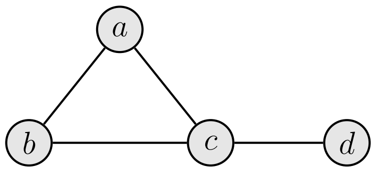


Figure 1: The kite consisting of the triangle $(a, b, c)$ and the tail edge $\{c, d\}$.

A $\lambda$-fold $G$-design of order $v$ with $G = P_3$ or $G = K$ will be denoted by $\mathrm{P}_3(v, \lambda)$ or $\mathrm{KS}(v, \lambda)$, respectively. It is well-known that a $\mathrm{P}_3(v, \lambda)$ exists if and only if $\lambda v(v-1) \equiv 0 \pmod 4$ (see [4]) and a $\mathrm{KS}(v, \lambda)$ exists if and only if $\lambda v(v-1) \equiv 0 \pmod 8$ (see [3]). The above necessary and sufficient conditions for $P_3$-designs and kite systems are detailed in Tables 1 and 2, where the spectra are partitioned into suitable congruence classes of $\lambda$.

$\Sigma_\lambda(P_3)$:

| $\lambda \pmod 2$ | $v \geq 3$ |
|---|---|
| 1 | $v \equiv 0, 1 \pmod 4$ |
| 0 | any |

Table 1: Spectra for $P_3$

$\Sigma_\lambda(K)$:

| $\lambda \pmod 4$ | $v \geq 4$ |
|---|---|
| 1, 3 | $v \equiv 0, 1 \pmod 8$ |
| 2 | $v \equiv 0, 1 \pmod 4$ |
| 0 | any |

Table 2: Spectra for $K$

The following lemma provides a necessary condition for embedding a $\mathrm{P}_3(u, \lambda)$ into a $\mathrm{KS}(v, \lambda)$ for any $\lambda \geq 1$.

**Lemma 2.1** *If a* $\mathrm{P}_3(u,\lambda)$ *is embedded into a* $\mathrm{KS}(v,\lambda)$*, then:*

- $v > (3u-1)/2$ *if* $\lambda$ *is odd;*
- $v \geq (3u-1)/2$ *if* $\lambda$ *is even.*

*Proof* If there exists a $\mathrm{P}_3(u,\lambda)$ $(U,\mathcal{C})$ embedded into a $\mathrm{KS}(v,\lambda)$ $(V,\mathcal{B})$, then $2|\mathcal{C}| \leq \lambda|U|(v-u)$, which implies $v \geq (3u-1)/2$. For odd $\lambda$, the case $v=(3u-1)/2$ requires $v \equiv 1 \pmod{24}$ and $u \equiv 1 \pmod{16}$ and embedding a $P_3(16k+1,\lambda)$ $(U,\mathcal{C})$ into a $KS(24k+1,\lambda)$ $(U \cup W,\mathcal{B})$ is impossible. Indeed, every path in $\mathcal{C}$ must be a subgraph of a block in $\mathcal{B}$, and each vertex in $W$ can be attached to at most $\left\lfloor \frac{\lambda(16k+1)}{2} \right\rfloor$ paths of $\mathcal{C}$. This leads to a contradiction, as $|\mathcal{C}| > |W| \left\lfloor \frac{\lambda(16k+1)}{2} \right\rfloor$. □

Here, for every index $\lambda$ and every order $u$ which is admissible for the existence of a $\mathrm{P}_3(u,\lambda)$, we determine the minimum integer $v$ such that there exists a $\mathrm{KS}(v,\lambda)$ $(U \cup W,\mathcal{B})$ which embeds a $\mathrm{P}_3(u,\lambda)$ $(U,\mathcal{C})$, by proving our main result in Theorem 5.1. For convenience in describing embeddings, we write $a \triangleleft P$ to denote the kite $(a,b,c)-d$ where $P=[b,c,d]$, separating the vertex $a \in W$ from the path $P$ (which lies in $U$). Thus, in order to define the collections $\mathcal{C}$ and $\mathcal{B}$ and simultaneously the embedding $f:\mathcal{C} \to \mathcal{B}$, we always provide a partition $\mathcal{B} = \mathcal{B}_e \cup \mathcal{B}_c$, where the subcollection $\mathcal{B}_e$, whose kites are given in the above form $a \triangleleft P$, embeds $\mathcal{C} = \{P : a \triangleleft P \in \mathcal{B}_e\}$, which defines a $\mathrm{P}_3(u,2)$ on $U$. The embedding is the mapping $f:\mathcal{C} \to \mathcal{B}$, defined by $f(P) = a \triangleleft P$, for every $P \in \mathcal{C}$, where $a$ is the vertex in $W$ such that $a \triangleleft P \in \mathcal{B}_e$ (clearly, $f(\mathcal{C}) = \mathcal{B}_e$).

If $(V,\mathcal{B}_i)$, $i=1,2$, is a $\mathrm{KS}(v,\lambda_i)$ which embeds a $\mathrm{P}_3(u,\lambda_i)$ $(U,\mathcal{C}_i)$, then $(V,\mathcal{B}_1 \cup \mathcal{B}_2)$ is trivially a $\mathrm{KS}(v,\lambda_1+\lambda_2)$ which embeds a $\mathrm{P}_3(u,\lambda_1+\lambda_2)$ (this technique follows the standard approach described in [13] and is widely employed in constructing designs with index $\lambda > 1$). As a consequence of the above, the union of $\rho$ copies of a $\mathrm{KS}(v,\lambda)$ embedding a $\mathrm{P}_3(u,\lambda)$ gives a $\mathrm{KS}(v,\rho\lambda)$ which embeds a $\mathrm{P}_3(u,\rho\lambda)$. As shown in Tables 1 and 2, the spectra $\Sigma_\lambda(P_3)$ and $\Sigma_\lambda(K)$ coincide with $\Sigma_1(P_3)$ and $\Sigma_1(K)$, respectively, for any odd $\lambda$. Consequently, since the necessary conditions established in Lemma 2.1 for any odd $\lambda$ coincide with those for the case $\lambda=1$, a solution to our problem can be directly obtained by taking the union of $\lambda$ copies of a $\mathrm{KS}(v,1)$ which embeds a $\mathrm{P}_3(u,1)$, the existence of which is guaranteed by the following theorem.

**Theorem 2.1** *( [5]) Let* $v \equiv 0,1 \pmod 8$*. There exists a* $\mathrm{P}_3(u,1)$ *embedded into a* $\mathrm{KS}(v,1)$ *for every* $4 \leq u \leq (2v-\beta(v))/3$*, where* $\beta(v)=0$ *if* $v \equiv 0 \pmod{24}$*,* $\beta(v)=2$ *if* $v \equiv 1 \pmod{24}$*,* $\beta(v)=1$ *if* $v \equiv 8 \pmod{24}$*,* $\beta(v)=3$ *if* $v \equiv 9 \pmod{24}$*,* $\beta(v)=5$ *if* $v \equiv 16 \pmod{24}$*,* $\beta(v)=7$ *if* $v \equiv 17 \pmod{24}$*.*

**Corollary 2.1** *For any* $\lambda \equiv 1 \pmod 2$ *and* $u \equiv 0,1 \pmod 4$*, there exists a* $\mathrm{P}_3(u,\lambda)$ *embedded into a* $\mathrm{KS}(v,\lambda)$ *where* $v$ *is the minimum integer* $v \equiv 0,1 \pmod 8$ *such that* $v > \frac{3u-1}{2}$*.*

For $\lambda \equiv 0 \pmod 2$, it will be sufficient to settle the cases $\lambda = 2$ (Section 3) and $\lambda = 4$ (Section 4) and paste copies of suitable kite systems of smaller indeces to settle any even index $\lambda$. In order to solve the cases $\lambda = 2, 4$, we will make a massive use of the *difference method*, both pure and mixed differences (see [1, 2]).

In what follows, if $G$ is a graph whose vertices belong to $Z_n$, then we call the *orbit of $G$ under $Z_n$* the set of the *translates* of $G$, i.e., $Orb(G) = \{G + i : i \in Z_n\}$, where $G + i$ is the graph with $V(G + i) = \{x + i : x \in V(G)\}$ and $E(G + i) = \{\{x + i, y + i\} : \{x, y\} \in E(G)\}$. Instead of taking $Z_n$, we can consider vertices in the set $Z_n \times Z_t$, whose element $(x, j)$ is denoted by $x_j$ (in other words, we can take $t$ copies of each element of $Z_n$, distinguished by a subscript). In this case the differences arising from a pair $(x_j, y_k) \in Z_n \times Z_t$, denoted by $d_{jk}$ where $d = y - x$, are distinguished in *pure* or *mixed differences*, depending on $j = k$ or $j \neq k$, respectively. If $V(G) \subseteq Z_n \times Z_t$, then we speak of translates and the orbit of $G$ under $Z_n$ by means of $x_j + i = (x + i)_j$, with obvious meaning of the terms. If, further, $V(G) \subseteq Z_n \cup R$ (or $V(G) \subseteq (Z_n \times Z_t) \cup R$), where $R$ is disjoint from $Z_n$ (or $Z_n \times Z_t$), then we can again speak of translates and orbit of $G$ under $Z_n$ by means of $x + i = x$ for $x \in R$ (the elements of $R$ are sometimes referred to as *infinite* vertices). The difference method is an efficient tool to construct and describe graph decompositions, since their blocks can be defined by means of a set of orbit representatives (*base blocks*).

In order to obtain a $\lambda$-fold $G$-design of order $v$ on $Z_n$ (or $Z_n \times Z_t$), we need to construct base $G$-blocks which cover all differences in $Z_n$ (or all pure and mixed differences in $Z_n \times Z_t$). When some differences are not covered, the base blocks define a partial $\lambda$-fold $G$-design of order $v$. More generally, we speak of a *partial* $G$-design of a graph $\Gamma$ when the edges of the blocks of $\mathcal{B}$ partition a proper spanning subgraph of $\Gamma$. The triple $(X, \mathcal{B}, L)$, where $(X, \mathcal{B})$ is a partial $G$-design of $\Gamma$ and $L$ is the collection of edges of $\Gamma$ not belonging to any of the blocks of $\mathcal{B}$, is said to be a *$G$-packing* of $\Gamma$ with *leave* $L$.

We now present a simple motivating example to further clarify the problem, while simultaneously illustrating the difference based technique.

**Example 2.1** ($\lambda = 2$, $u = 9$) We want to embed a $P_3(9, 2)$ $(U, \mathcal{C})$ into a KS$(v, 2)$ $(U \cup W, \mathcal{B})$, where $v$ is the minimum possible; thus, by Lemma 2.1, it must be $v = 13$. Let $U = Z_9$ and $W = \{\infty_1, \infty_2, \infty_3, \infty_4\}$. As collection $\mathcal{B}_e$, take the union of the orbits under $Z_9$ of the following base blocks:

$$\infty_1 \triangleleft [1, 0, 8], \quad \infty_2 \triangleleft [2, 0, 7], \quad \infty_3 \triangleleft [3, 0, 6], \quad \infty_4 \triangleleft [4, 0, 5],$$

which can be written in a more compact form (highlighting the embedded paths) as $\infty_i \triangleleft P_i$ with $P_i = [i, 0, -i]$, for $i = 1, 2, 3, 4$. It is straightforward to verify that each infinite vertex is joined twice to every vertex in $U$, and that the paths $P_i$, for $i = 1, 2, 3, 4$, cover all differences in $Z_9$ exactly twice. Thus, $(U \cup W, \mathcal{B}_e)$ is a kite-decomposition of $2(K_{9+4} \setminus K_4)$ which embeds the $P_3(9, 2)$ $(U, \mathcal{C})$, where $\mathcal{C} = \cup_{i=1}^{4} Orb(P_i)$. Finally, consider a KS$(4, 2)$ $(W, \mathcal{B}_c)$, which is known to exist (see Table 2). The pair $(U \cup W, \mathcal{B}_e \cup \mathcal{B}_c)$ is a KS$(13, 2)$ which embeds $(U, \mathcal{C})$.

## 3 The case $\lambda = 2$

Taking into account that a $P_3(u, 2)$ exists for any $u \geq 3$ and a $KS(v, 2)$ exists if and only if $v \equiv 0, 1 \pmod 4$, here for every $u \geq 3$ we construct a $P_3(u, 2)$ $(U, \mathcal{C})$ which is embedded into a $KS(v, 2)$ $(U \cup W, \mathcal{B})$, where $v$ is the minimum integer such that $v \equiv 0, 1 \pmod 4$ and $v \geq \frac{3u-1}{2}$.

We distinguish the following eight cases.

1. $u = 8k$ and $v = 12k$, $k > 0$;
2. $u = 8k + 1$ and $v = 12k + 1$, $k > 0$;
3. $u = 8k + 2$ and $v = 12k + 4$, $k > 0$;
4. $u = 8k + 3$ and $v = 12k + 4$, $k \geq 0$;
5. $u = 8k + 4$ and $v = 12k + 8$, $k \geq 0$;
6. $u = 8k + 5$ and $v = 12k + 8$, $k \geq 0$;
7. $u = 8k + 6$ and $v = 12k + 9$, $k \geq 0$;
8. $u = 8k + 7$ and $v = 12k + 12$, $k \geq 0$.

**Proposition 3.1** *For every $u = 8k + r$, $r = 0, 1, 3, 6$ and $u \geq 3$, there exists a $P_3(u, 2)$ embedded into a $KS(v, 2)$ where $v = \lceil \frac{3u-1}{2} \rceil$.*

*Proof* Distinguish two cases depending on the parity of $u$. Let $u = 8k + r$, $r = 1, 3$, $u \geq 3$. Take $U = Z_u$ and $W = \{\infty_1, \infty_2, \dots, \infty_w\}$, where $w = (u - 1)/2$, and consider the collection $\mathcal{B}_e$ of kites defined by means of the base blocks $\infty_i \lhd P_i$ with $P_i = [i, 0, -i]$, for $i = 1, 2, \dots w$. It is easy to check that the pair $(U \cup W, \mathcal{B}_e)$ is a kite-decomposition of $2(K_{u+w} \setminus K_w)$, which embeds the $P_3(u, 2)$ $(U, \mathcal{C})$ where $\mathcal{C}$ is the collections of paths defined by means of the base blocks $P_i = [i, 0, -i]$, for $i = 1, 2, \dots w$. Now, consider a $KS(w, 2)$ $(W, \mathcal{B}_c)$, which exists because $w \equiv 0, 1 \pmod 4$ (if $u = 3$, then $\mathcal{B}_c = \emptyset$). The pair $(U \cup W, \mathcal{B}_e \cup \mathcal{B}_c)$ is a $KS(v, 2)$, $v = (3u-1)/2$, which embeds $(U, \mathcal{C})$.
Let $u = 8k + r$, $r = 0, 6$. Take $U = Z_{u-1} \cup \{a\}$ and $W = \{\infty_1, \infty_2, \dots, \infty_w\}$, where $w = u/2$, and define the collections $\mathcal{B}_e$ and $\mathcal{C}$ as in the odd case by means of the base paths $P_i = [i, 0, -i]$, for $i = 1, 2, \dots w - 2$, $P_{w-1} = [u/2 - 1, 0, a]$ and $P_w = [u/2 - 1, 0, a]$ (note that $P_{w-1}$ and $P_w$ are considered distinct for embedding purposes; since the paths form a multiset, each occurrence is mapped independently, allowing for distinct images under $f$).

Here, the pair $(U \cup W, \mathcal{B}_e)$ is a kite-decomposition of $2(K_{u+w} \setminus K_{w+1})$, which has $W \cup \{a\}$ as hole and embeds $(U, \mathcal{C})$. As in the odd case, filling the hole with a $KS(w + 1, 2)$ $(W \cup \{a\}, \mathcal{B}_c)$, which exists because $w + 1 \equiv 0, 1 \pmod 4$, gives a $KS(v, 2)$, $v = 3u/2$, embedding $(U, \mathcal{C})$. □

The following lemma gives a kite-packing of a complete graph with a hole, which will be the starting point to construct a minimum embedding of a $P_3(u, 2)$ into a $KS(v, 2)$ $(U \cup W, \mathcal{B})$ in the remaining four cases. In what follows, the orbit of $\{0_j, d_k\} \subseteq Z_n \times Z_t$ under $Z_n$ is denoted by $Orb(d_{jk})$.

**Lemma 3.1** *Let* $\Gamma = K_{6h+3} \setminus K_{2h+1}$, $h \geq 1$, *be the complete graph on* $X = Z_{2h+1} \times Z_3$ *with* $H = Z_{2h+1} \times \{2\}$ *as hole. There exists a kite-packing* $(X, \mathcal{B}, L)$ *of* $2\Gamma$, *which embeds a* $\mathrm{P}_3(4h+2, 2)$ *on* $Z_{2h+1} \times \{0,1\}$ *and whose leave is*

$$L = Orb(0_{12}) \cup Orb(1_{02}).$$

*Proof* Consider the collection $\mathcal{B}$ of kites defined by means of the base blocks

$B_i = (-i)_2 \lhd P_i$, with $P_i = [i_1, 0_0, i_0]$, for $i = 1, 2, \ldots, 2h$,

$B'_i = (-i)_2 \lhd P'_i$, with $P'_i = [(2h+i)_0, 0_1, i_1]$, for $i = 1, 2, \ldots, 2h$,

$B = 0_2 \lhd P$, with $P = [0_1, 0_0, 1_1]$.

It can be verified that the collection $\mathcal{B}$ covers twice every pure difference $d_{jj}$, $j = 0, 1$, and every mixed difference $d_{jk}$, $j, k \in Z_3$, except the mixed differences $0_{12}$ and $1_{02}$, which are covered once (a detailed verification of the difference coverage is provided in the Appendix). Therefore, the triple $(X, \mathcal{B}, L)$, where

$$L = Orb(0_{12}) \cup Orb(1_{02}),$$

is a kite-packing of $2\Gamma$, which embeds the $\mathrm{P}_3(4h+2, 2)$ defined on $Z_{2h+1} \times \{0,1\}$ by means of the base blocks $P_i, P'_i$, for $i = 1, 2, \ldots, 2h$, and $P$. □

**Proposition 3.2** *For any integer* $k \geq 1$, *there exists a* $\mathrm{P}_3(8k+2, 2)$ *embedded into a* $\mathrm{KS}(12k+4, 2)$.

*Proof* Start from the kite-packing $(X, \mathcal{B}, L)$ of Lemma 3.1, where $X = Z_{2h+1} \times Z_3$ with $h = 2k$. Add an infinite vertex $\infty$ to $Z_{4k+1} \times \{2\}$, so to take $U = Z_{4k+1} \times \{0,1\}$ and $W = (Z_{4k+1} \times \{2\}) \cup \{\infty\}$. Replacing the base kite $0_2 \lhd P$ with $\infty \lhd P$ gives a kite-packing $(U \cup W, \mathcal{B}_e, L')$ of $2(K_{12k+4} \setminus K_{4k+1})$ (with $H = Z_{4k+1} \times \{2\}$ as hole), which embeds a $\mathrm{P}_3(8k+2, 2)$ on $U$ and whose leave is $L' = L_1 \cup L_2$ where

$L_1 = L \cup Orb(0_{02}) \cup Orb(0_{12})$,

$L_2 = K_{1,8k+2}(\{\infty\}, Z_{4k+1} \times \{0,1\}) \cup 2K_{1,4k+1}(\{\infty\}, Z_{4k+1} \times \{2\})$.

A collection of kites $\mathcal{B}_c$ which completes the decomposition and gives the required design is obtained by taking the translates of $(\infty, 0_1, 0_2) - 0_0$ and $(\infty, 0_0, 1_2) - 1_1$ (which along with the kites of $\mathcal{B}_e$ provides a kite-design of $2(K_{12k+4} \setminus K_{4k+1})$) and the block set of a $\mathrm{KS}(4k+1, 2)$ defined on $H$. □

**Proposition 3.3** *For any integer* $k \geq 0$, *there exists a* $\mathrm{P}_3(8k+4, 2)$ *embedded into a* $\mathrm{KS}(12k+8, 2)$.

*Proof* For $k = 0$, on the set $V = Z_4 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$ take $\mathcal{B}_e = \{\infty_1 \lhd [2,3,0], \infty_1 \lhd [3,1,2], \infty_2 \lhd [2,1,3], \infty_2 \lhd [2,3,0], \infty_3 \lhd [1,0,2], \infty_3 \lhd [2,0,1]\}$ and $\mathcal{B}_c = \{(0, \infty_1, \infty_4) - 1, (0, \infty_2, \infty_4) - 1, (3, \infty_2, \infty_4) - 2, (3, \infty_3, \infty_4) - 2, (1, \infty_2, \infty_3) - 3, (2, \infty_3, \infty_1) - 1, (0, \infty_2, \infty_1) - \infty_4, (\infty_1, \infty_2, \infty_3) - \infty_4\}$. For $k \geq 1$, start from the kite-packing $(X, \mathcal{B}, L)$ of Lemma 3.1, where $X = Z_{2h+1} \times Z_3$ with $h = 2k$.

Add two extra vertices, say $a$ and $b$, to $Z_{4k+1} \times \{0,1\}$ and three ones, say $\infty_1$, $\infty_2$ and $\infty_3$, to $Z_{4k+1} \times \{2\}$, so to take $U = (Z_{4k+1} \times \{0,1\}) \cup \{a,b\}$ and $W = (Z_{4k+1} \times \{2\}) \cup \{\infty_1, \infty_2, \infty_3\}$. Replace the base blocks $(-i)_2 \triangleleft P_i$, $i = 1,2,3$, of $\mathcal{B}$ with $\infty_i \triangleleft P_i$, $i = 1,2,3$ and use the mixed differences removed $d_{12}$, $d = 4k-3, 4k-1$, and $d_{02}$, $d = 4k-1, 4k$, to form the further base blocks

$$4k_2 \triangleleft [1_1, a, 0_0],\ \ 4k_2 \triangleleft [3_1, a, 0_0],\ \ 4k_2 \triangleleft [1_0, b, 0_1],$$

along with the kites

$$i_2 \triangleleft [(1+i)_0, b, (1+i)_1],\ \ i = 0, 1, \ldots, 4k-1,$$

$$4k_2 \triangleleft [0_0, b, a],\ \ \infty_1 \triangleleft [a, b, 0_1].$$

The collection of kites defined so far provides a kite-packing $(U \cup W, \mathcal{B}_e, L')$ of $2(K_{12k+8} \setminus K_{4k+1})$ (with $H = Z_{4k+1} \times \{2\}$ as hole), which embeds a $\mathrm{P}_3(8k+4, 2)$ on $U$ and whose leave is $L' = L_1 \cup L_2$ where

$$L_1 = L \cup Orb((4k-2)_{02}) \cup Orb((4k-5)_{12}),$$

$$L_2 = K_{3,8k+4}(\{\infty_1, \infty_2, \infty_3\}, U) \cup K_{2,2}(\{\infty_2, \infty_3\}, \{a, b\})$$
$$\cup\, 2K_3(\infty_1, \infty_2, \infty_3) \cup\ 2K_{3,4k+1}(\{\infty_1, \infty_2, \infty_3\}, Z_{4k+1} \times \{2\}).$$

A collection of kites $\mathcal{B}_c$ which completes the decomposition and gives the required design is obtained as follows. Consider the translates of

$$(\infty_1, (4k-5)_2, 0_1) - \infty_3,\ \ (\infty_2, 0_1, 0_2) - \infty_3,$$

$$(\infty_2, (4k-2)_2, 0_0) - \infty_3,\ \ (\infty_1, 0_0, 1_2) - \infty_3,$$

along with the kites

$$(a, \infty_3, \infty_2) - \infty_1,\ \ (b, \infty_2, \infty_3) - \infty_1,\ \ (a, \infty_1, \infty_3) - b,\ \ (b, \infty_1, \infty_2) - a,$$

in order to get a kite-decomposition of $2(K_{12k+8} \setminus K_{4k+1})$, and fill the hole $H$ with a $\mathrm{KS}(4k+1, 2)$. □

**Proposition 3.4** *For any integer $k \geq 0$, there exists a $\mathrm{P}_3(8k+5, 2)$ embedded into a $\mathrm{KS}(12k+8, 2)$.*

*Proof* For $k = 0$, on the set $V = Z_5 \cup \{\infty_1, \infty_2, \infty_3\}$ take

$$\begin{aligned}\mathcal{B}_e \ = \ \{&\infty_1 \triangleleft [3,2,4], \infty_1 \triangleleft [0,4,1], \infty_1 \triangleleft [3,1,2], \infty_2 \triangleleft [2,1,3], \infty_2 \triangleleft [3,2,4],\\ &\infty_2 \triangleleft [4,3,0], \infty_2 \triangleleft [0,4,1], \infty_3 \triangleleft [1,0,2], \infty_3 \triangleleft [4,3,0], \infty_3 \triangleleft [2,0,1]\}\end{aligned}$$

and

$$\mathcal{B}_c = \{(\infty_1, \infty_2, \infty_3) - 4, (0, \infty_2, \infty_1) - 4, (1, \infty_2, \infty_3) - 3, (2, \infty_3, \infty_1) - 1\}.$$

For $k \geq 1$, start from the kite-packing $(X, \mathcal{B}, L)$ of Lemma 3.1, where $X = Z_{2h+1} \times Z_3$ with $h = 2k$. Add three extra vertices, say $a, b, c$, to $Z_{4k+1} \times \{0,1\}$ and two ones, say $\infty_1$ and $\infty_2$, to $Z_{4k+1} \times \{2\}$, so to take $U = (Z_{4k+1} \times \{0,1\}) \cup \{a,b,c\}$ and $W = (Z_{4k+1} \times \{2\}) \cup \{\infty_1, \infty_2\}$. Replace the base blocks $(-i)_2 \triangleleft P_i$, $i = 1,2,3$, of $\mathcal{B}$ with $\infty_1 \triangleleft P_1$, $\infty_1 \triangleleft P_2$ and $\infty_2 \triangleleft P_3$, and use the mixed differences removed $d_{12}$, $d = 4k-5, 4k-3, 4k-1$, and $d_{02}$, $d = 4k-2, 4k-1, 4k$, to form the further base blocks

$4k_2 \lhd [1_1, a, 0_0]$, $4k_2 \lhd [0_0, a, 0_1]$, $4k_2 \lhd [3_1, b, 0_0]$, $4k_2 \lhd [1_0, b, 0_1]$
$4k_2 \lhd [5_1, c, 0_0]$, $4k_2 \lhd [2_0, c, 0_1]$,

along with the three kites

$\infty_1 \lhd [a, b, c]$, $\infty_1 \lhd [c, a, b]$, $\infty_2 \lhd [b, c, a]$.

The collection of kites defined so far provides a kite-packing $(U \cup W, \mathcal{B}_e, L')$ of $2(K_{12k+8} \setminus K_{4k+1})$ (with $H = Z_{4k+1} \times \{2\}$ as hole), which embeds a $\mathrm{P}_3(8k+5, 2)$ on $U$ and whose leave is $L' = L \cup L_1$ where

$$L_1 = K_{1,8k+4}(\{\infty_2\}, (Z_{4k+1} \times \{0,1\}) \cup \{b, c\}) \cup 2K_{2,4k+1}(\{\infty_1, \infty_2\}, Z_{4k+1} \times \{2\}) \cup \{\{\infty_1, b\}, \{\infty_1, c\}\} \cup 2\{\{\infty_1, \infty_2\}, \{\infty_2, a\}\}.$$

A collection of kites $\mathcal{B}_c$ which completes the decomposition and gives the required design is obtained as follows. Consider the translates of

$(\infty_2, 0_1, 0_2) - \infty_1$, $(\infty_2, 0_0, 1_2) - \infty_1$,

along with the two kites

$(b, \infty_1, \infty_2) - a$, $(c, \infty_1, \infty_2) - a$,

in order to get a kite-decomposition of $2(K_{12k+8} \setminus K_{4k+1})$, and fill the hole $H$ with a $\mathrm{KS}(4k+1, 2)$. □

**Proposition 3.5** *For any integer $k \geq 0$, there exists a $\mathrm{P}_3(8k+7, 2)$ embedded into a $\mathrm{KS}(12k+12, 2)$.*

*Proof* Start from the kite-packing $(X, \mathcal{B}, L)$ of Lemma 3.1, where $X = Z_{2h+1} \times Z_3$ with $h = 2k+1$, $k \geq 0$. Add one extra vertex $a$ to $Z_{4k+3} \times \{0,1\}$ and two ones $\infty_1$ and $\infty_2$ to $Z_{4k+3} \times \{2\}$, in order to take $U = (Z_{4k+3} \times \{0,1\}) \cup \{a\}$ and $W = (Z_{4k+3} \times \{2\}) \cup \{\infty_1, \infty_2\}$. Remove the base block $B = 0_2 \lhd [0_1, 0_0, 1_1]$ and use the differences $0_{01}$, $1_{01}$ and $0_{12}$ to form the three further base blocks

$0_2 \lhd [0_1, a, 0_0]$, $\infty_1 \lhd [0_0, 0_1, a]$, $\infty_2 \lhd [1_1, 0_0, a]$,

to obtain a kite-packing $(U \cup W, \mathcal{B}_e, L')$ of $2(K_{12k+12} \setminus K_{4k+3})$ (with $H = Z_{4k+3} \times \{2\}$ as hole), which embeds a $\mathrm{P}_3(8k+7, 2)$ on $U$ and whose leave is $L' = L_1 \cup L_2$ where

$$L_1 = L \cup Orb(0_{02}),$$
$$L_2 = K_{2,8k+6}(\{\infty_1, \infty_2\}, Z_{4k+3} \times \{0,1\}) \cup 2K_{2,4k+3}(\{\infty_1, \infty_2\}, Z_{4k+3} \times \{2\}) \cup \cup K_{1,4k+3}(\{a\}, Z_{4k+3} \times \{2\}) \cup 2K_3(a, \infty_1, \infty_2).$$

A collection of kites $\mathcal{B}_c$ which completes the decomposition and gives the required design is obtained as follows, by distinguishing the cases $k = 0$ and $k \geq 1$.

For $k = 0$, take $\mathcal{B}_c = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\mathcal{A}_0 = \{(\infty_1, 0_1, 0_2) - 2_2, (\infty_1, 1_1, 1_2) - 0_2, (\infty_1, 2_1, 2_2) - 1_2, (0_2, 1_2, 2_2) - a\},$$
$$\mathcal{A}_1 = Orb((0_2, 0_0, \infty_2) - 0_1) \cup Orb((\infty_1, 0_0, 1_2) - \infty_2),$$
$$\mathcal{A}_2 = \{(\infty_1, \infty_2, a) - 0_2, (\infty_1, \infty_2, a) - 1_2\}.$$

For $k \geq 1$, consider the collections of kites $\mathcal{A}_1$ and $\mathcal{A}_2$ defined as before along with the further collections

$$\begin{aligned}\mathcal{A}_3 &= \{(\infty_1, 0_1, 0_2) - (4k+2)_2, (\infty_1, 1_1, 1_2) - (4k+2)_2\} \\ &\quad \cup \{(\infty_1, (2+i)_1, (2+i)_2) - a : i = 0, 1, \ldots, 4k\}, \\ \mathcal{A}_4 &= \{(2i_2, (2+2i)_2, (1+2i)_2) - (3+2i)_2 : i = 0, 1, \ldots, 2k+1\}.\end{aligned}$$

To complete the construction for $k \geq 1$, we first observe that the union $\bigcup_{i=1}^{4} \mathcal{A}_i$ covers all the edges of the leave $L'$ and the pure differences $1_{22}$ and $2_{22}$ exactly once. Consequently, we define $\mathcal{B}_c$ by adding to $\bigcup_{i=1}^{4} \mathcal{A}_i$ the translates of the following $k$ base blocks (retaining only the first one in the case $k = 1$)

$(2_2, (2k+1)_2, 0_2) - (2k+2)_2$,

$((k+1-j)_2, (k+2+j)_2, 0_2) - (4k-1-2j)_2$, $j = 0, 1, \ldots, k-2$,

These blocks cover all the remaining pure differences $d_{22}$. □

Combining Propositions 3.1–3.5 gives the following result.

**Theorem 3.1** *For any $u \geq 3$, there exists a $\mathrm{P}_3(u, 2)$ embedded into a $\mathrm{KS}(v, 2)$ where $v$ is the minimum integer such that $v \in \Sigma_2(K)$ and $v \geq \frac{3u-1}{2}$.*

## 4 The case $\lambda = 4$

Taking into account that a $\mathrm{P}_3(u, 2)$ exists for any $u \geq 3$ and a $\mathrm{KS}(v, 4)$ exists for any $v \geq 4$, in this section for every $u \geq 3$, $u \neq 4, 5, 7$, we will construct a $\mathrm{P}_3(u, 4)$ $(U, \mathcal{C})$ which is embedded into a $\mathrm{KS}(v, 4)$ $(U \cup W, \mathcal{B})$, where $v = \lceil \frac{3u-1}{2} \rceil$. Moreover, for $u = 4, 5, 7$ we will prove that no $\mathrm{P}_3(u, 4)$ can be embedded into a $\mathrm{KS}(v, 4)$ of order $v = \lceil \frac{3u-1}{2} \rceil$ and a minimum embedding is attained for $v = \lceil \frac{3u-1}{2} \rceil + 1$. Here we use the notation introduced in Section 3 to define the collections $\mathcal{C}$ and $\mathcal{B}$, along with the embedding $f : \mathcal{C} \to \mathcal{B}$.

**Lemma 4.1** *For any $k > 0$, there exists a kite-decomposition $(X, \mathcal{B})$ of $4(K_{3k+1} \setminus K_k)$, where $X = Z_{2k+1} \cup H$, $H = \{\infty_1, \infty_2, \ldots, \infty_k\}$, and $\mathcal{B}$ embeds a $\mathrm{P}_3(2k+1, 4)$ on $Z_{2k+1}$.*

*Proof* Consider the collection $\mathcal{B}$ of kites defined by means of the base blocks

$$B_i = \infty_i \triangleleft P_i, \text{ with } P_i = [i, 0, -i], \text{ for } i = 1, 2, \ldots, 2k,$$

where $\infty_{k+i} = \infty_i$, for $i = 1, 2, \ldots, k$. □

**Proposition 4.1** *For any $u \equiv 1 \pmod 2$, $u \geq 3$ and $u \neq 5, 7$, there exists a $\mathrm{P}_3(u, 4)$ embedded into a $\mathrm{KS}(v, 4)$ of order $v = \frac{3u-1}{2}$.*

*Proof* Let $u = 2k+1$, $k \geq 1$ and $k \neq 2, 3$. Consider the kite-decomposition $(Z_{2k+1} \cup H, \mathcal{B})$ of $4(K_{3k+1} \setminus K_k)$ from Lemma 4.1 and place a $\mathrm{KS}(k, 4)$, say $(H, \mathcal{B}')$, into the hole $H$ (for $k = 1$, there is no hole to fill). The resulting design $(Z_{2k+1} \cup H, \mathcal{B} \cup \mathcal{B}')$ is a $\mathrm{KS}(v, 4)$ embedding a $\mathrm{P}_3(u, 4)$. □

**Lemma 4.2** *The minimum order of a* KS$(v,4)$ *which embeds a* P$_3(u,4)$ *of order* $u=5,7$ *is* $v=\frac{3u-1}{2}+1$.

*Proof* To start with, assume that $(U,\mathcal{C})$ is a P$_3(u,4)$, $u=5,7$, embedded into a KS$(v,4)$ $(U\cup W,\mathcal{B})$ of order $v=\frac{3u-1}{2}$ and let $f:\mathcal{C}\to\mathcal{B}$ be the embedding. To embed $(U,\mathcal{C})$ into $(U\cup W,\mathcal{B})$, we require $2|\mathcal{C}|$ edges joining a vertex of $U$ to a vertex of $W$. In both cases, $2|\mathcal{C}|=4|U|\cdot|W|$, which implies that the subcollection $\mathcal{B}_e=f(\mathcal{C})$ covers all edges of $4K_{u,w}(U,W)$, $(u,w)=(5,2),(7,3)$. Consequently, any remaining kites in $\mathcal{B}_c=\mathcal{B}\setminus\mathcal{B}_e$ would have to be contained entirely within $W$. However, this is impossible since $|W|<4$, whereas a kite is a graph of order 4. Therefore, if a KS$(v,4)$ embeds a P$_3(u,4)$ of order $u=5,7$, then $v\geq\frac{3u-1}{2}+1$.

A KS$(8,4)$ which embeds a P$_3(5,4)$ is $(Z_4\cup\{a\}\cup\{\infty_1,\infty_2,\infty_3\},\mathcal{B}_e\cup\mathcal{B}_c)$, where $\mathcal{B}_e$ contains the translates of the four base blocks $\infty_1\lhd[1,0,a]$, repeated twice, $\infty_2\lhd[1,0,a]$ and $\infty_3\lhd[2,0,a]$, along with the kites $\infty_2\lhd[2,0,1]$, $\infty_2\lhd[3,1,2]$, $\infty_3\lhd[0,2,3]$, $\infty_3\lhd[1,3,0]$, and $\mathcal{B}_c=\{(0,\infty_3,\infty_2)-a,(1,\infty_3,\infty_2)-a,(2,\infty_2,\infty_3)-a,(3,\infty_2,\infty_3)-a\}\cup 2\{(a,\infty_3,\infty_1)-\infty_2,(a,\infty_2,\infty_1)-\infty_3\}$.

A KS$(11,4)$ which embeds a P$_3(7,4)$ is $(Z_6\cup\{a\}\cup\{\infty_1,\infty_2,\infty_3,\infty_4\},\mathcal{B}_e\cup\mathcal{B}_c)$ where $\mathcal{B}_e$ contains the translates of the six base blocks $\infty_1\lhd[1,0,a]$, repeated twice, $\infty_2\lhd[1,0,a]$, repeated twice, $\infty_3\lhd[2,0,4]$ and $\infty_4\lhd[3,0,2]$, along with the kites $\infty_3\lhd[3,0,2]$, $\infty_3\lhd[4,1,3]$, $\infty_3\lhd[5,2,4]$, $\infty_3\lhd[0,3,5]$, $\infty_3\lhd[1,4,0]$, $\infty_3\lhd[2,5,1]$, and $\mathcal{B}_c=\{(0,\infty_3,\infty_4)-4,(1,\infty_4,\infty_3)-4,(2,\infty_3,\infty_4)-5,(3,\infty_4,\infty_3)-5,(\infty_3,\infty_1,\infty_0)-\infty_2,(\infty_3,\infty_2,\infty_1)-\infty_0,(\infty_3,\infty_0,\infty_2)-\infty_1\}\cup 2\{(\infty_4,\infty_1,\infty_0)-\infty_3,(\infty_4,\infty_2,\infty_1)-\infty_3,(\infty_4,\infty_0,\infty_2)-\infty_3\}$. □

**Proposition 4.2** *For any* $u\equiv 0 \pmod 2$, $u\geq 6$, *there exists a* P$_3(u,4)$ *embedded into a* KS$(v,4)$ *of order* $v=\frac{3u}{2}$.

*Proof* Let $u=2k+2$, $k\geq 2$. Start from the kite-decomposition $(X,\mathcal{B})$ of Lemma 4.1 and add an extra vertex $a$ to $Z_{2k+1}$ and an extra vertex $\infty$ to $H$, so to take $U=Z_{2k+1}\cup\{a\}$ and $W=H\cup\{\infty\}$. Replacing the base kite $B_1,B_2\in\mathcal{B}$ with the four base blocks

$$\infty_1\lhd[1,0,a],\ \ \infty_2\lhd[2,0,a],\ \ \infty\lhd[1,0,a],\ \ \infty\lhd[2,0,a],$$

gives a kite-decomposition of $4(K_{3k+3}\setminus K_{k+2})$, wich embeds a P$_3(2k+2,4)$ on $U$ and admits $W\cup\{a\}$ as hole. Filling the hole with a KS$(k+2,4)$ gives the required design. □

**Lemma 4.3** *The minimum order of a* KS$(v,4)$ *which embeds a* P$_3(4,4)$ *is* $v=7$.

*Proof* To start with, assume that there exists an embedding $f:\mathcal{C}\to\mathcal{B}$ is of a P$_3(4,4)$ into a KS$(6,4)$ $(U\cup W,\mathcal{B}_e\cup\mathcal{B}_c)$, where $\mathcal{B}_e=f(\mathcal{C})$. It is easy to see that each of the three blocks of $\mathcal{B}_c$ contains exactly three edges of $4K_{4,2}(U,W)$ and so $\mathcal{B}_c$ covers 9 edges of $4K_{4,2}(U,W)$, which is impossible because $\mathcal{B}_e$ covers 24 edges of $4K_{4,2}(U,W)$ and $|E(4K_{4,2})|=32$. Therefore, if a KS$(v,4)$ embeds a P$_3(4,4)$, then

$v \geq 7$. A KS(7, 4) which embeds a $P_3(4, 4)$ is $(Z_4 \cup \{\infty_1, \infty_2, \infty_3\}, \mathcal{B}_e \cup \mathcal{B}_c)$ where $\mathcal{B}_e = 2\{\infty_3 \lhd [1, 0, 2], \infty_3 \lhd [2, 0, 1], \infty_1 \lhd [2, 3, 0], \infty_1 \lhd [3, 1, 2], \infty_2 \lhd [2, 1, 3], \infty_2 \lhd [2, 3, 0]\}$ and $\mathcal{B}_c = \{(0, \infty_2, \infty_1) - 1, (2, \infty_3, \infty_1) - 0, (0, \infty_2, \infty_1) - 1, (3, \infty_3, \infty_2) - 0, (1, \infty_2, \infty_3) - 3, (2, \infty_1, \infty_3) - 3, (0, \infty_2, \infty_1) - \infty_3, (3, \infty_3, \infty_2) - \infty_1, (1, \infty_2, \infty_3) - \infty_1\}$. □

Combining Lemmas 4.2, 4.3 and Propositions 4.1, 4.2 gives the following result.

**Theorem 4.1** *For any $u \geq 3$, $u \neq 4, 5, 7$, there exists a $P_3(u, 4)$ embedded into a $KS(v, 4)$ of order $v = \lceil \frac{3u-1}{2} \rceil$. The minimum order of a $KS(v, 4)$ which embeds a $P_3(u, 4)$ of order $u = 4, 5, 7$ is $v = \lceil \frac{3u-1}{2} \rceil + 1$.*

# 5 Main result

By similar arguments as in the proof of Lemmas 4.2 and 4.3, it is easy to prove what follows.

**Lemma 5.1** *For any $\lambda \equiv 0 \pmod 4$, no $P_3(u, \lambda)$ of order $u = 4, 5, 7$ can be embedded into a $KS(v, \lambda)$ of order $v = \lceil \frac{3u-1}{2} \rceil$.*

Thanks to the above lemma and the results obtained in the previous sections we are now able to prove our main result.

**Theorem 5.1** *For any pair $\lambda \geq 1$ and $u \in \Sigma_\lambda(P_3)$, except for $\lambda \equiv 0 \pmod 4$ and $u = 4, 5, 7$, the minimum order of a $KS(v, \lambda)$ which embeds a $P_3(u, \lambda)$ is the minimum integer $v \in \Sigma_\lambda(K)$ such that:*

- *$v > (3u - 1)/2$ if $\lambda$ is odd;*
- *$v \geq (3u - 1)/2$ if $\lambda$ is even.*

*For $\lambda \equiv 0 \pmod 4$ and $u = 4, 5, 7$, the minimum order is $v = \lceil \frac{3u-1}{2} \rceil + 1$.*

*Proof* For any $\lambda \equiv 1 \pmod 2$, the existence of a $P_3(u, \lambda)$ embedded into a $KS(v, \lambda)$, where $v$ is the minimum integer in $\Sigma_\lambda(K)$ such that $v > \frac{3u-1}{2}$, follows by Corollary 2.1. For any $\lambda = 4l + 2$, paste $2l + 1$ copies of a KS$(v, 2)$ from Theorem 3.1. For any $\lambda = 4l$ and any $u \geq 3$, it is sufficient to paste $l$ copies of a KS$(v, 4)$ from Theorem 4.1 (see Lemma 5.1). □

Tables 3 and 4 provide the minimum embedding order $v_{min}$ for each pair $(\lambda, u)$, distinguishing between the cases where $\lambda$ is odd or even, respectively. In Table 3, the values of $u$ are partitioned into congruence classes modulo 16, organized in two sets depending on the parity of $u$, with the corresponding $v_{min}$ reported below in the same column. Furthermore, the cases in which the lower bound established by Lemma 2.1 is not attained are explicitly highlighted (see Table 4 for those specific instances).

| $u \pmod{16}$ | 0 | 4 | 8 | 12 |
|---|---|---|---|---|
| $v_{min}$ | $\frac{3u+2}{2}$ | $\frac{3u+2}{2}+1$ | $\frac{3u+2}{2}+3$ | $\frac{3u+2}{2}+5$ |

| $u \pmod{16}$ | 1 | 5 | 9 | 13 |
|---|---|---|---|---|
| $v_{min}$ | $\frac{3u+1}{2}+6$ | $\frac{3u+1}{2}$ | $\frac{3u+1}{2}+2$ | $\frac{3u+1}{2}+4$ |

Table 3: Minimum embedding orders $v_{min}$ for odd $\lambda$.

| $\lambda$ | $u \pmod 8$ | $v_{min}$ | Exceptions $(u, v_{min})$ |
|---|---|---|---|
| $2 \pmod 4$ | $0,1,3,6$ | $\lceil \frac{3u-1}{2} \rceil$ | none |
| | $2,5$ | $\lceil \frac{3u-1}{2} \rceil + 1$ | none |
| | $4,7$ | $\lceil \frac{3u-1}{2} \rceil + 2$ | none |
| $0 \pmod 4$ | any | $\lceil \frac{3u-1}{2} \rceil$ | $(4,7)$, $(5,8)$, $(7,11)$ |

Table 4: Minimum embedding orders $v_{min}$ for even $\lambda$

## 6 Concluding remarks

A natural generalization for future research would be to investigate the embedding of a $P_k$-design, for a general $k$, into a $G$-design where $G$ is a minimal graph containing $P_k$ as a subgraph. This would extend the results obtained here for $P_3$ into kite systems to a broader class of graph designs, starting from the base case $\lambda = 1$. Interesting classes of minimal graphs (those obtained by adding only one or two edges to $P_k$) include cycles and dragon graphs (also known as tadpole graphs, consisting of a cycle with a path attached to one of its vertices [12]). A kite is, in fact, the smallest dragon graph. The difference method employed in this paper could be effectively adapted to solve these problems; indeed, the low degree of the vertices and the sparse nature of dragon graphs make the search for difference sets significantly more straightforward than for dense or complete graphs, allowing for a systematic and tractable construction of the required base blocks.

## Appendix

In this appendix, we provide a detailed verification of the difference set coverage for the proof of Lemma 3.1. This lemma is chosen as a representative case because of its importance for the subsequent results and because it contains the most significant and complex distribution of differences. This verification technique is particularly useful to verify all constructions where families of base blocks are defined by means of indices (e.g., $i = 1, \dots, 2h$). To avoid redundant repetitions and to enhance the readability of the manuscript, we limit our explicit analysis to this case.

Following the standard difference method (see, e.g., [2]), we recall that in $X = Z_n \times Z_t$ the set of all mixed differences consists of $d_{jk}$ for $d \in \{0, 1, \dots, n-1\}$ and for all $j, k \in Z_t$ with $j < k$. Regarding the pure differences, for every $j \in Z_t$ the set of all pure differences consists of $d_{jj}$ where $d \in \{1, 2, \dots, \frac{n-1}{2}\}$ if $n$ is odd and $d \in \{1, 2, \dots, \frac{n}{2}\}$ if $n$ is even. In our specific case where $X = Z_{2h+1} \times Z_3$, the mixed differences are $d_{jk}$, for $d \in \{0, 1, \dots, 2h\}$ and $j, k \in Z_3$ with $j < k$, and the pure differences are $d_{jj}$, for $d \in \{1, 2, \dots, h\}$ and $j \in Z_3$. Now, we must verify that the collection $\mathcal{B}$ consisting of the edges

$B_i = (-i)_2 \lhd P_i$, with $P_i = [i_1, 0_0, i_0]$, for $i = 1, 2, \dots, 2h$,

$B'_i = (-i)_2 \lhd P'_i$, with $P'_i = [(2h+i)_0, 0_1, i_1]$, for $i = 1, 2, \dots, 2h$,

$B = 0_2 \lhd P$, with $P = [0_1, 0_0, 1_1]$,

covers twice every pure difference $d_{jj}$, $j = 0, 1$, and every mixed difference $d_{jk}$, $j, k \in Z_3$, except the mixed differences $0_{12}$ and $1_{02}$, which are covered once.

| **Block** | **Edge 1** | **Edge 2** | **Edge 3** | **Tail** |
|---|---|---|---|---|
| $B_1$ | $(2h-1)_{12}$ | $(2h)_{02}$ | $1_{01}$ | $1_{00}$ |
| $B_2$ | $(2h-3)_{12}$ | $(2h-1)_{02}$ | $2_{01}$ | $2_{00}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $B_{2h-1}$ | $4_{12}$ | $2_{02}$ | $(2h-1)_{01}$ | $2_{00}$ |
| $B_{2h}$ | $2_{12}$ | $1_{02}$ | $(2h)_{01}$ | $1_{00}$ |
| $B'_1$ | $(2h)_{02}$ | $(2h)_{12}$ | $0_{01}$ | $1_{11}$ |
| $B'_2$ | $(2h-2)_{02}$ | $(2h-1)_{12}$ | $(2h)_{01}$ | $2_{11}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $B'_{2h-1}$ | $3_{02}$ | $2_{12}$ | $3_{01}$ | $1_{11}$ |
| $B'_{2h}$ | $1_{02}$ | $1_{12}$ | $2_{01}$ | $1_{11}$ |
| $B$ | $0_{12}$ | $0_{02}$ | $0_{01}$ | $1_{01}$ |

Table 5: Detailed differences for the base blocks in the proof of Lemma 3.1.

In Table 5 we list the differences covered by each base block. To facilitate the verification, for a kite denoted by $a \lhd [b, c, d]$, we order its edges and we call $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, and $\{c, d\}$, respectively, Edge 1, Edge 2, Edge 3, and Tail. Note that Edge 3 and the Tail are the edges of the path $[b, c, d]$. Consequently, a simultaneous verification of the differences covered by $P_i, P'_i$, for $i = 1, 2, \dots, 2h$, and $P$ is straightforward, confirming that these base blocks define a $\mathrm{P}_3(4h+2, 2)$ on $Z_{2h+1} \times \{0, 1\}$.

Following the convention that for a pair $\{x_j, y_k\}$ the mixed difference is $d_{jk} = (y-x)_{jk}$ with $j < k$, the table shows how the base blocks $B_i, B_i'$ and $B$ satisfy the required coverage. Specifically, the arithmetic progressions in the columns for Edge 1, 2, and 3, along with the difference in the last row and the Tail column, ensure that all mixed differences $d_{01}, d_{02}, d_{12}$ are covered exactly twice (with the exceptions noted in the proof), while the Tail column ensures the double coverage of pure differences $d_{00}$ and $d_{11}$.

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