AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **95(1)** (2026), Pages 131–143

A note on the strength of a hypercube

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Abstract

As a generalization of super magic strength, the strength of a graph was introduced by Ichishima et al. in [*Australas. J. Combin.* 72 (2018), 492–508]. For a vertex ordering f of graph G, the strength of f is the maximum sum of the labels on any pair of adjacent vertices. The strength of G is defined as the minimum strength of f, taken over all vertex orderings of G. Bounds on the strength of the hypercube are known and the upper bound will be improved in this paper.

1 Introduction

Let G be a graph on n vertices and f be a bijective function $f : V(G) \to \{1, 2, \dots, n\}$. The *strength of* f, denoted $\mathrm{str}_f(G)$, is defined as

$$\mathrm{str}_f(G) = \max\{f(u) + f(v) : uv \in E(G)\}.$$

The *strength* of G, denoted by $\mathrm{str}(G)$, is defined as

$$\mathrm{str}(G) = \min\{\mathrm{str}_f(G) : f \text{ is a vertex ordering of } G\}.$$

The strength of a graph was first defined in [4] and was motivated by super edge-magic labelings and the super magic strength of a graph. See the survey [6] for

ISSN: 2202-3518

more on super edge-magic labelings and [1, 4] for more on super magic strength. Ichishima et al. [4] determined the strength of a number of graph classes, including paths, cycles, complete graphs, and complete bipartite graphs. The strength of some trees, including caterpillars and complete n-ary trees, was determined in [5]. The n-dimensional hypercube, denoted Q_n, is the graph whose vertices correspond to the $(0,1)$-binary strings of length n, two vertices being adjacent if and only if the corresponding binary strings differ in exactly one digit. The n-dimensional hypercube, Q_n, was considered in [4], where they proved the following.

Theorem 1.1 ([4]). *For the n-dimensional hypercube Q_n, $\mathrm{str}(Q_1) = 3$, $\mathrm{str}(Q_2) = 6$, and for every integer $n \geq 3$,*

$$2^n + n \leq \mathrm{str}(Q_n) \leq 2^n + 2^{n-2} + 1.$$

For completeness, we note that the lower bound was improved in [2, Theorems 16, 17, and Corollary 18] to the following.

Theorem 1.2 ([2]). *For the n-dimensional hypercube Q_n, $\mathrm{str}(Q_3) = 11$, $\mathrm{str}(Q_4) = 21$, $\mathrm{str}(Q_5) = 40$,*

$$\begin{aligned} &\mathrm{str}(Q_n) \geq 2^n + 4n - 12 \quad \textit{for } 5 \leq n \leq 9, \textit{ and} \\ &\mathrm{str}(Q_n) \geq 2^n + \lfloor \tfrac{n^2}{4} \rfloor + 4 \quad \textit{for } n \geq 10. \end{aligned}$$

In this note, we improve the upper bound for $\mathrm{str}(Q_n)$. We define a bijection f between the set of all n-bit strings and $\{1, 2, \ldots, 2^n\}$. We first show the bijection gives the exact strength of Q_3, Q_4, and Q_5. We then determine the recurrence

$$\mathrm{str}(Q_n) \leq \mathrm{str}_f(Q_n) \leq \mathrm{str}_f(Q_{n-2}) + 3 \cdot 2^{n-2} + \binom{n-3}{\lceil \frac{n-3}{2} \rceil} + \binom{n-2}{\lceil \frac{n-2}{2} \rceil}$$

in Corollary 2.9. The recurrence provides an improvement over the upper bound of Theorem 1.1 of [4] for $n \in \{5, 6, \ldots, 13\}$. Ultimately, in Corollary 2.10, we show that for all $n \geq 14$,

$$\mathrm{str}(Q_n) \leq 2^n + 2^{n-3} + 28,$$

which is an improvement of $2^{n-3} - 27$ over the upper bound of Theorem 1.1 of [4].

2 An improved upper bound for $\mathrm{str}(\mathbf{Q_n})$

We state some important definitions before presenting the bijection between the n-bit strings and $\{1, 2, \ldots, 2^n\}$ and proving some initial observations.

Let $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_n$ be distinct n-bit strings; that is, x_i, $y_i \in \{0, 1\}$ for every $i \in \{1, 2, \ldots, n\}$. The *Hamming weight*, or simply *weight* of a bit string is the number of non-zero bits in the string. The *Hamming distance* between two n-bit strings x and y is the number of positions in which x and y differ. Consequently, two vertices of Q_n are adjacent if and only if the Hamming distance between the corresponding n-bit strings is one.

We define $x < y$ to be in *lexicographic order* if $x_i = y_i$ for $i < k$ and $x_k < y_k$ for some $k \leq n$. We define *reverse lexicographic order* as the reverse ordering of the bit strings. That is, if $x < y < z$ then the sequence (x, y, z) is in lexicographic order and the sequence (z, y, x) is in reverse lexicographic order. We will often refer to the lexicographic or reverse lexicographic ordering of the set of bit strings of a fixed length and weight, so we define S_n^i to be the sequence of n-bit strings of weight i, taken in lexicographic order; and R_n^i be the sequence of n-bit strings of weight i, taken in reverse lexicographic order. Finally, define S_n to be the sequence:

$$S_n = \left(R_n^1, R_n^3, R_n^5, \ldots, R_n^{n-1}, S_n^n, S_n^{n-2}, \ldots, S_n^4, S_n^2, S_n^0\right) \text{ for even } n;$$

$$S_n = \left(R_n^1, R_n^3, R_n^5, \ldots, R_n^{n-2}, R_n^n, S_n^{n-1}, \ldots, S_n^4, S_n^2, S_n^0\right) \text{ for odd } n.$$

Observe that $S_n^n = R_n^n$ since both contain only the string of weight n (i.e. the all 1s string). Since every n-bit string appears exactly once in S_n we can define a bijection f between the sequence of all n-bit strings (in S_n) and $\{1, 2, \ldots, 2^n\}$. Note that the first 2^{n-1} strings in S_n each have odd weight and the second 2^{n-1} strings each have even weight. As an example, the sequences S_n for $n = 3, 4, 5, 6$ are given in Tables 1a, 1b, 1c, 1d, respectively. Although it is already known that $\text{str}(Q_3) = 11$, $\text{str}(Q_4) = 21$ and $\text{str}(Q_5) = 40$, the reader can verify from Tables 1a, 1b, and 1c that these values are achieved using the bijection provided by S_n.

The *one's complement* of a binary string is the string obtained by flipping all the bits in the representation, for example, 1011 is the one's complement of 0100. We will represent the one's complement of a string x as $\overline{x}$. For odd n, suppose x is an n-bit string with weight i for some odd positive integer i. Then $x \in R_n^i$ and $\overline{x} \in S_n^{n-i}$ and by construction of S_n,

$$f(x) + 2^{n-1} = f(\overline{x}). \tag{1}$$

Finally, we use the notation $0x$ or $x0$ to indicate that we prefix 0 to x or append 0 to x, respectively.

2.1 Preliminary observations

Remark 2.1. For any integer $n \geq 1$, let x and y be two n-bit strings. The vertices of Q_n that correspond to x and y are adjacent in Q_n if and only if x and y have Hamming distance 1. Fixing a bijective function f from the n-bit strings to $\{1, 2, \ldots, 2^n\}$ necessarily implies that there exist n-bit strings x and y such that $\text{str}_f(Q_n) = f(x) + f(y)$.

For a pair x, y of n-bit strings for which $\text{str}_f(Q_n) = f(x) + f(y)$ and x has weight i, we make the following assumption throughout this note:

$$x \text{ has odd weight } i \text{ and } y \text{ has even weight } i-1.$$

Since x and y have Hamming distance 1, their weights will be of different parity; thus we may assume i is odd. Consequently, y must have even weight $i-1$ or $i+1$. Strings of weight $i-1$ fall later in sequence S_n (and therefore map to larger function values) than strings of weight $i+1$. Thus, if x has odd weight i then y has even weight $i-1$.

bit string x	$f(x)$	bit string x	$f(x)$
100	1	011	5
010	2	101	6
001	3	110	7
111	4	000	8

(a) Sequence S_3 with R_3^1, R_3^3 on the left, S_3^2, S_3^0 on the right.

bit string x	$f(x)$	bit string x	$f(x)$
1000	1	1111	9
0100	2	0011	10
0010	3	0101	11
0001	4	0110	12
1110	5	1001	13
1101	6	1010	14
1011	7	1100	15
0111	8	0000	16

(b) Sequence S_4 with R_4^1, R_4^3 on the left, S_4^4, S_4^2, S_4^0 on the right.

bit string x	$f(x)$	bit string x	$f(x)$
10000	1	01111	17
01000	2	10111	18
00100	3	11011	19
00010	4	11101	20
00001	5	11110	21
11100	6	00011	22
11010	7	00101	23
11001	8	00110	24
10110	9	01001	25
10101	10	01010	26
10011	11	01100	27
01110	12	10001	28
01101	13	10010	29
01011	14	10100	30
00111	15	11000	31
11111	16	00000	32

(c) Sequence S_5 with R_5^1, R_5^3, R_5^5 on the left, S_5^4, S_5^2, S_5^0 on the right.

bit string x	$f(x)$	bit string x	$f(x)$
100000	1	111111	33
010000	2	001111	34
001000	3	010111	35
000100	4	011011	36
000010	5	011101	37
000001	6	011110	38
111000	7	100111	39
110100	8	101011	40
110010	9	101101	41
110001	10	101110	42
101100	11	110011	43
101010	12	110101	44
101001	13	110110	45
100110	14	111001	46
100101	15	111010	47
100011	16	111100	48
011100	17	000011	49
011010	18	000101	50
011001	19	000110	51
010110	20	001001	52
010101	21	001010	53
010011	22	001100	54
001110	23	010001	55
001101	24	010010	56
001011	25	010100	57
000111	26	011000	58
111110	27	100001	59
111101	28	100010	60
111011	29	100100	61
110111	30	101000	62
101111	31	110000	63
011111	32	000000	64

(d) Sequence S_6 with R_6^1, R_6^3, R_6^5 on the left, S_6^6, S_6^4, S_6^2, S_6^0 on the right.

Table 1

Lemma 2.2. *For any integer $n \geq 2$, there exists an $(n-1)$-bit string w such that $x = w1$, $y = w0$ and $\mathrm{str}_f(Q_n) = f(x) + f(y)$.*

Proof. Let x and y be n-bit strings for which $\mathrm{str}_f(Q_n) = f(x) + f(y)$. Consequently, x and y differ only in one digit. Assume, by Remark 2.1, that x has odd weight i and y has weight $i-1$. We first prove the last digit in y will be 0, regardless of the last digit in x.

First, suppose the last digit in x is 0. For a contradiction, assume the last digit in y is 1. As x and y differ only in the last digit, y has a larger weight than x, which provides a contradiction. Consequently, if the last digit in x is 0 then the last digit in y is 0.

Second, suppose the last digit of x is 1, so $x = z1$ for some $(n-1)$-bit string z of weight $i-1$. Of the n-bit strings of weight $i-1$ that differ from x in exactly one digit, $z0$ has the largest binary representation. Since the subsequence S_n^{i-1} in S_n is in lexicographic order, $f(z0) \geq f(y)$. Since $f(x) + f(y)$ is maximum, $y = z0$. Consequently, if the last digit in x is 1 then the last digit in y is 0.

Therefore, if $\mathrm{str}_f(Q_n) = f(x) + f(y)$ and x has odd weight, then the last digit of y is 0. Thus, $y = w0$ for some $(n-1)$-bit string w of weight $i-1$. Of the n-bit strings of weight i that differ from y in exactly one digit, $w1$ has the smallest binary representation. Since the subsequence R_n^i in S_n is in reverse lexicographic order, $f(w1) \geq f(x)$. Since $f(x) + f(y)$ is maximum, $x = w1$. □

We state the next observation for n-bit strings of weights i and $i-1$, but apply it to strings of other lengths and weights as well. For an n-bit string x of weight i, we define $\mathrm{A}(x, R_n^i)$ to be the set of strings in R_n^i that come after x. For an n-bit string y of weight $i-1$, we define $\mathrm{B}(y, S_n^{i-1})$ to be the set of strings in S_n^{i-1} that come before y.

Throughout the rest of the paper we will use the following binomial identities to simplify expressions. For $n, k \in \mathbb{Z}$, $n > k > 0$,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \tag{2}$$

For $n, k \in \mathbb{Z}$, $n \geq k \geq 0$,

$$\binom{n}{k} = \binom{n}{n-k}.$$

Lemma 2.3. *For any integer $n \geq 2$, let w be an $(n-1)$-bit string of weight $i-1$ for some odd integer i. Then*

$$f(w1) + f(w0) = 2^n + \binom{n-1}{i} - |\,\mathrm{A}(w1, R_n^i)| + |\,\mathrm{B}(w0, S_n^{i-1})| + 1.$$

Proof. For any integer $n \geq 2$, let w be an $(n-1)$-bit string of weight $i-1$ for some odd integer i. Since $w1 \in R_n^i$ and $w0 \in S_n^{i-1}$, we find

$$f(w1) = |R_n^1| + |R_n^3| + \cdots + |R_n^i| - |\,\mathrm{A}(w1, R_n^i)| = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{i} - |\,\mathrm{A}(w1, R_n^i)|$$

and

$$\begin{aligned} f(w0) &= 2^n - |S_n^0| - |S_n^2| - \cdots - |S_n^{i-1}| + |\,\mathrm{B}(w0, S_n^{i-1})| + 1 \\ &= 2^n - \binom{n}{0} - \binom{n}{2} - \cdots - \binom{n}{i-1} + |\,\mathrm{B}(w0, S_n^{i-1})| + 1. \end{aligned}$$

To simplify the sum $f(w1) + f(w0)$, we use the identity[1]

$$\sum_{k=0}^{\ell}(-1)^k\binom{N}{k} = (-1)^\ell\binom{N-1}{\ell};$$

so,

$$\begin{aligned} f(w1) + f(w0) &= 2^n - \Big(\sum_{k=0}^{i}(-1)^k\tbinom{n}{k}\Big) - |\,\mathrm{A}(w1, R_n^i)| + |\,\mathrm{B}(w0, S_n^{i-1})| + 1 \\ &= 2^n + \tbinom{n-1}{i} - |\,\mathrm{A}(w1, R_n^i)| + |\,\mathrm{B}(w0, S_n^{i-1})| + 1. \end{aligned}$$

□

2.2 Main results

We now provide an improved upper bound on the strength of the hypercube. Since $\mathrm{str}(Q_n)$ is known for $n \in \{1, 2, 3, 4\}$, we consider $n \geq 5$ in this subsection.

Theorem 2.4. *For any integer $n \geq 5$, let x and y be n-bit strings for which $\mathrm{str}_f(Q_n) = f(x) + f(y)$. If x and y both begin with 0 then*

$$\mathrm{str}_f(Q_n) \leq \mathrm{str}_f(Q_{n-2}) + 3 \cdot 2^{n-2} + \binom{n-3}{\lceil\frac{n-3}{2}\rceil} + \binom{n-2}{\lceil\frac{n-2}{2}\rceil}.$$

Proof. Let x, y be n-bit strings for which $\mathrm{str}_f(Q_n) = f(x) + f(y)$, and assume x and y both begin with 0. By Remark 2.1, x has odd weight i and y has even weight $i-1$. By Lemma 2.2, x and y differ only in that the last bit in x is 1 and the last bit in y is 0. Therefore, $x = 0w1$ and $y = 0w0$ for some $(n-2)$-bit string w of weight $i-1$. We apply Lemma 2.3 to both $f(x) + f(y)$ and $f(w1) + f(w0)$:

$$\begin{aligned} &\Big(f(x) + f(y)\Big) - \Big(f(w1) + f(w0)\Big) \\ &= 2^n - 2^{n-1} + \tbinom{n-1}{i} - \tbinom{n-2}{i} - |\,\mathrm{A}(x, R_n^i)| + |\,\mathrm{B}(y, S_n^{i-1})| + |\,\mathrm{A}(w1, R_{n-1}^i)| - |\,\mathrm{B}(w0, S_{n-1}^{i-1})| \\ &= 2^{n-1} + \tbinom{n-2}{i-1} - |\,\mathrm{A}(0w1, R_n^i)| + |\,\mathrm{B}(0w0, S_n^{i-1})| + |\,\mathrm{A}(w1, R_{n-1}^i)| - |\,\mathrm{B}(w0, S_{n-1}^{i-1})|. \end{aligned} \tag{3}$$

Observe that R_n^i consists of subsequence R_{n-1}^{i-1} with 1 prefixed to each string; followed by subsequence R_{n-1}^i, with 0 prefixed to each string. Consequently, the strings that succeed $x = 0w1$ in R_n^i and the strings that succeed $w1$ in R_{n-1}^i are the same except that those in R_n^i that have a 0 prefixed. Therefore,

$$|\,\mathrm{A}(0w1, R_n^i)| = |\,\mathrm{A}(w1, R_{n-1}^i)|. \tag{4}$$

[1]This can be proved by induction on ℓ and using the identity (2); as it is an undergraduate exercise, we omit the proof here.

Similarly, S_n^{i-1} consists of subsequence S_{n-1}^{i-1} with 0 prefixed to each string; followed by subsequence S_{n-1}^{i-2} with 1 prefixed to each string. Thus,

$$|\,\mathrm{B}(0w0, S_n^{i-1})| = |\,\mathrm{B}(w0, S_{n-1}^{i-1})|. \tag{5}$$

Using Equations (4)-(5), the expression from (3) simplifies to

$$\Big(f(x)+f(y)\Big) - \Big(f(w1)+f(w0)\Big) = 2^{n-1} + \tbinom{n-2}{i-1} \leq 2^{n-1} + \tbinom{n-2}{\lceil (n-2)/2\rceil}.$$

Finally,

$$\begin{aligned} \mathrm{str}_f(Q_n) &= f(x)+f(y) \\ &\leq f(w1)+f(w0)+2^{n-1}+\tbinom{n-2}{\lceil (n-2)/2\rceil} \\ &\leq \mathrm{str}_f(Q_{n-1})+2^{n-1}+\tbinom{n-2}{\lceil (n-2)/2\rceil} \\ &\leq \Big[\mathrm{str}_f(Q_{n-2})+2^{n-2}+\tbinom{n-3}{\lceil (n-3)/2\rceil}\Big]+2^{n-1}+\tbinom{n-2}{\lceil (n-2)/2\rceil} \\ &= \mathrm{str}_f(Q_{n-2})+3\cdot 2^{n-2}+\tbinom{n-3}{\lceil (n-3)/2\rceil}+\tbinom{n-2}{\lceil (n-2)/2\rceil}. \end{aligned} \tag{6}$$

□

Note that the theorem statement is weaker than the result of Inequality (6). We do this to ensure that upper bounds for all cases in this section match.

Theorem 2.5. *For any integer $n \geq 5$, let x and y be n-bit strings for which $\mathrm{str}_f(Q_n) = f(x)+f(y)$. If x, y both begin with 1 and n is odd, then*

$$\mathrm{str}_f(Q_n) \leq \mathrm{str}_f(Q_{n-2}) + 3\cdot 2^{n-2} + \binom{n-3}{\lceil \frac{n-3}{2}\rceil} + \binom{n-2}{\lceil \frac{n-2}{2}\rceil}.$$

Proof. Let x and y be n-bit strings for which $str_f(Q_n) = f(x)+f(y)$, and assume n is odd and x and y both begin with 1. By Remark 2.1, x has odd weight i and y has even weight $i-1$. By Lemma 2.2, x and y differ only in that the last bit in x is 1 and the last bit in y is 0. Therefore, $x = 1w1$ and $y = 1w0$ for some $(n-2)$-bit string w of weight $i-2$.

Then $x = 1w1 \in R_n^i$, and since n is odd, by Equation (1),

$$f(x)+2^{n-1} = f(1w1)+2^{n-1} = f(\overline{x}) \iff f(1w1)+2^{n-1} = f(0\overline{w}0).$$

Similarly, $y = 1w0 \in S_n^{i-1}$, and so $\overline{y} = 0\overline{w}1 \in R_n^{n-i+1}$. As $n-i+1$ and n are both odd,

$$f(0\overline{w}1)+2^{n-1} = f(1w0).$$

So

$$\mathrm{str}_f(Q_n) = f(x)+f(y) = f(1w1)+f(1w0) = f(0\overline{w}1)+f(0\overline{w}0).$$

Observe that $0\overline{w}1$ is an n-bit string of odd weight $n-i+1$ and $0\overline{w}0$ is an n-bit string of even weight $n-i$. Since $\mathrm{str}_f(Q_n) = f(0\overline{w}1)+f(0\overline{w}0)$, by Theorem 2.4,

$$\mathrm{str}_f(Q_n) \leq \mathrm{str}_f(Q_{n-2}) + 3\cdot 2^{n-2} + \tbinom{n-3}{\lceil (n-3)/2\rceil} + \tbinom{n-2}{\lceil (n-2)/2\rceil}.$$

□

Theorem 2.6. *For any integer $n \geq 5$, let x and y be n-bit strings for which* $\mathrm{str}_f(Q_n) = f(x) + f(y)$*. If x and y both begin with* 11 *and n is even, then*

$$\mathrm{str}_f(Q_n) \leq \mathrm{str}_f(Q_{n-2}) + 3 \cdot 2^{n-2} + \binom{n-3}{\lceil \frac{n-3}{2} \rceil} + \binom{n-2}{\lceil \frac{n-2}{2} \rceil}.$$

Proof. Let x, y be n-bit strings for which $\mathrm{str}_f(Q_n) = f(x) + f(y)$ and assume n is even, and x and y both begin with 11. By Remark 2.1, x has odd weight i and y has even weight $i-1$. By Lemma 2.2, x and y differ only in that the last bit in x is 1 and the last bit in y is 0. Therefore, $x = 11w1$ and $y = 11w0$ for some string w of length $n-3$ and weight $i-3$. We apply Lemma 2.3 to both $f(x)+f(y)$ and $f(w1) + f(w0)$:

$$f(x) + f(y) = 2^n + \tbinom{n-1}{i} - |\,\mathrm{A}(x, R_n^i)| + |\,\mathrm{B}(y, S_n^{i-1})| + 1,$$

$$f(w1) + f(w0) = 2^{n-2} + \tbinom{n-3}{i-2} - |\,\mathrm{A}(w1, R_{n-2}^{i-2})| + |\,\mathrm{B}(w0, S_{n-2}^{i-3})| + 1,$$

and so

$$\begin{aligned}\Big(f(x) + f(y)\Big) - \Big((f(w1) + f(w0)\Big) \;&=\; 3 \cdot 2^{n-2} + \tbinom{n-1}{i} - \tbinom{n-3}{i-2} \\ &- |\,\mathrm{A}(x, R_n^i)| + |\,\mathrm{A}(w1, R_{n-2}^{i-2})| + |\,\mathrm{B}(y, S_n^{i-1})| - |\,\mathrm{B}(w0, S_{n-2}^{i-3})|. \quad (7)\end{aligned}$$

To simplify Equation (7), we first relate $|\,\mathrm{A}(x, R_n^i)|$ to $|\,\mathrm{A}(w1, R_{n-2}^{i-2})|$. Observe that R_n^i consists of:

subsequence R_{n-2}^{i-2} with 11 prefixed to each string; followed by
subsequence R_{n-2}^{i-1} with 10 prefixed to each string, followed by
subsequence R_{n-2}^{i-1} with 01 prefixed to each string, followed by
subsequence R_{n-2}^{i} with 00 prefixed to each string.

Thus,

$$|\,\mathrm{A}(x, R_n^i)| = |\,\mathrm{A}(11w1, R_n^i)| = |\,\mathrm{A}(w1, R_{n-2}^{i-2})| + 2\tbinom{n-2}{i-1} + \tbinom{n-2}{i}. \quad (8)$$

We next relate $|\,\mathrm{B}(y, S_n^{i-1})|$ to $|\,\mathrm{B}(w0, S_{n-2}^{i-3})|$. Observe that S_n^{i-1} consists of:

subsequence S_{n-2}^{i-1} with 00 prefixed to each string; followed by
subsequence S_{n-2}^{i-2} with 01 prefixed to each string, followed by
subsequence S_{n-2}^{i-2} with 10 prefixed to each string, followed by
subsequence S_{n-2}^{i-3} with 11 prefixed to each string.

Thus,

$$|\,\mathrm{B}(y, S_n^{i-1})| = |\,\mathrm{B}(11w1, S_n^{i-1})| = \tbinom{n-2}{i-1} + 2\tbinom{n-2}{i-2} + |\,\mathrm{B}(w0, S_{n-2}^{i-3})|. \quad (9)$$

We now use Equations (8) and (9), along with Equation (2), to simplify Equation (7) to

$$\begin{aligned}\Big(f(x)+f(y)\Big) &- \Big(f(w1)+f(w0)\Big)\\ &= 3\cdot 2^{n-2}+\tbinom{n-1}{i}-\tbinom{n-3}{i-2}-\tbinom{n-2}{i-1}-\tbinom{n-2}{i}+2\tbinom{n-2}{i-2}\\ &= 3\cdot 2^{n-2}-\tbinom{n-3}{i-2}+2\tbinom{n-2}{i-2}\\ &= 3\cdot 2^{n-2}+\tbinom{n-3}{i-3}+\tbinom{n-2}{i-2}\\ &\le 3\cdot 2^{n-2}+\tbinom{n-3}{\lceil (n-3)/2\rceil}+\tbinom{n-2}{\lceil (n-2)/2\rceil}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{str}_f(Q_n)=f(x)+f(y) &\le f(w1)+f(w0)+3\cdot 2^{n-2}+\tbinom{n-3}{\lceil (n-3)/2\rceil}+\tbinom{n-2}{\lceil (n-2)/2\rceil}\\ &\le \mathrm{str}_f(Q_{n-2})+3\cdot 2^{n-2}+\tbinom{n-3}{\lceil (n-3)/2\rceil}+\tbinom{n-2}{\lceil (n-2)/2\rceil}.\end{aligned}$$

□

What remains to consider is the case where x and y both begin with 10 and n is even; and the following lemma will be used. The lemma relates subsequence S^j_{n-2} to subsequence S^{n-j-2}_{n-2} (and also to subsequences R^j_{n-2} and R^{n-j-2}_{n-2}). This will be helpful in relating the number of $(n-2)$-bit strings that precede (and succeed) a particular $(n-2)$-bit string to the number of $(n-2)$-bit strings that precede (and succeed) its one's complement.

Lemma 2.7. *Let $n-2$ be an even positive integer and $j : 0 \le j \le n-2$, with $j \ne \frac{n-2}{2}$ and $S^j_{n-2} = (v_1, v_2, \dots, v_{k-1}, v_k)$, where $k = \binom{n-2}{j}$. Then*

$S^{n-j-2}_{n-2} = (\overline{v_k}, \overline{v_{k-1}}, \dots, \overline{v_2}, \overline{v_1})$;

$R^j_{n-2} = (v_k, v_{k-1}, \dots, v_2, v_1)$; *and*

$R^{n-j-2}_{n-2} = (\overline{v_1}, \overline{v_2}, \dots, \overline{v_{k-1}}, \overline{v_k})$.

Proof. Let $v_a, v_b \in S^j_{n-2}$ where $v_a < v_b$. Suppose v_a and v_b agree in the first $i-1$ digits, for some integer $i : 0 \le i-1 \le n-3$ and disagree in the ith digit. Since $v_a < v_b$, the ith digits of v_a and v_b are 0 and 1, respectively. Thus, $\overline{v_b} < \overline{v_a}$. Applying the above argument to every pair v_a, v_b for which $v_a < v_b$, the conclusion that $S^{n-j-2}_{n-2} = (\overline{v_k}, \overline{v_{k-1}}, \dots, \overline{v_2}, \overline{v_1})$ follows.

If $S^j_{n-2} = (v_1, v_2, \dots, v_{k-1}, v_k)$ then $R^j_{n-2} = (v_k, v_{k-1}, \dots, v_2, v_1)$. A similar argument shows $R^{n-j-2}_{n-2} = (\overline{v_1}, \overline{v_2}, \dots, \overline{v_{k-1}}, \overline{v_k})$. □

Theorem 2.8. *For any integer $n \ge 5$, let x and y be n-bit strings for which $\mathrm{str}_f(Q_n) = f(x) + f(y)$. If x and y both begin with 10 and n is even, then*

$$\mathrm{str}_f(Q_n) \le \mathrm{str}_f(Q_{n-2}) + 3\cdot 2^{n-2} + \binom{n-3}{\lceil \frac{n-3}{2}\rceil} + \binom{n-2}{\lceil \frac{n-2}{2}\rceil}.$$

Proof. Let x and y be n-bit strings for which $\mathrm{str}_f(Q_n) = f(x) + f(y)$, and assume n is even, and x and y both begin with 10. By Remark 2.1, x has weight i and y has weight $i-1$. By Lemma 2.2, x and y differ only in that the last bit in x is 1 and the last bit in y is 0. Thus, $x = 10w1$ and $y = 10w0$ for some string w of length $n-3$ and weight $i-2$. We will bound the difference $(f(x)+f(y)) - (f(\overline{w}1)+f(\overline{w}0))$ because we can exploit the relationships between $f(x)$ and $f(\overline{w}0)$ and between $f(y)$ and $f(\overline{w}1)$.

By Lemma 2.3,

$$f(x) + f(y) = 2^n + \tbinom{n-1}{i} - |\,\mathrm{A}(x, R_n^i)| + |\,\mathrm{B}(y, S_n^{i-1})| + 1$$

and since $\overline{w}$ is an $(n-3)$-bit string of weight $n-i-1$, by the same lemma:

$$f(\overline{w}1) + f(\overline{w}0) = 2^{n-2} + \tbinom{n-3}{n-i} - |\,\mathrm{A}(\overline{w}1, R_{n-2}^{n-i})| + |\,\mathrm{B}(\overline{w}0, S_{n-2}^{n-i-1})| + 1.$$

Thus,

$$\begin{aligned}&\Big(f(x) + f(y)\Big) - \Big(f(\overline{w}1) + f(\overline{w}0)\Big)\\ &= 3\cdot 2^{n-2} + \tbinom{n-1}{i} - \tbinom{n-3}{n-i} - |\,\mathrm{A}(x, R_n^i)| + |\,\mathrm{B}(y, S_n^{i-1})| + |\,\mathrm{A}(\overline{w}1, R_{n-2}^{n-i})| - |\,\mathrm{B}(\overline{w}0, S_{n-2}^{n-i-1})|\\ &= 3\cdot 2^{n-2} + \tbinom{n-1}{i} - \tbinom{n-3}{i-3} - |\,\mathrm{A}(x, R_n^i)| + |\,\mathrm{B}(y, S_n^{i-1})| + |\,\mathrm{A}(\overline{w}1, R_{n-2}^{n-i})| - |\,\mathrm{B}(\overline{w}0, S_{n-2}^{n-i-1})|.\end{aligned} \tag{10}$$

Our goal is to simplify Equation (10). We first observe that

$$|\,\mathrm{B}(\overline{w}0, S_{n-2}^{n-i-1})| + |\,\mathrm{A}(\overline{w}0, S_{n-2}^{n-i-1})| + 1 = \tbinom{n-2}{n-i-1} = \tbinom{n-2}{i-1}. \tag{11}$$

We next relate $|\,\mathrm{A}(x, R_n^i)|$ to $|\,\mathrm{B}(\overline{w}0, S_{n-2}^{n-i-1})|$. The sequence R_n^i consists of

> sequence R_{n-2}^{i-2} with 11 prefixed to each string in the sequence; followed by
> sequence R_{n-2}^{i-1} with 10 prefixed to each string in the sequence; followed by
> sequence R_{n-2}^{i-1} with 01 prefixed to each string in the sequence; followed by
> sequence R_{n-2}^{i} with 00 prefixed to each string in the sequence.

Then

$$\begin{aligned}|\,\mathrm{A}(x, R_n^i)| &= |\,\mathrm{A}(10w1, R_n^i)|\\ &= |\,\mathrm{A}(w1, R_{n-2}^{i-1})| + \tbinom{n-2}{i-1} + \tbinom{n-2}{i}\\ &= |\,\mathrm{A}(\overline{w}0, S_{n-2}^{n-i-1})| + \tbinom{n-2}{i-1} + \tbinom{n-2}{i} \quad \text{by Lemma 2.7}\\ &= 2\tbinom{n-2}{i-1} + \tbinom{n-2}{i} - 1 - |\,\mathrm{B}(\overline{w}0, S_{n-2}^{n-i-1})| \quad \text{by Equation (11).}\end{aligned} \tag{12}$$

Equation (12) establishes the relationship between $|\,\mathrm{A}(x, R_n^i)|$ and $|\,\mathrm{B}(\overline{w}0, S_{n-2}^{n-i-1})|$.

We now observe that

$$|\,\mathrm{A}(\overline{w}1, R_{n-2}^{n-i})| + |\,\mathrm{B}(\overline{w}1, R_{n-2}^{n-i})| + 1 = \tbinom{n-2}{n-i} = \tbinom{n-2}{i-2} \tag{13}$$

and we next relate $|\,\mathrm{B}(y, S_n^{i-1})|$ to $|\,\mathrm{A}(\overline{w}1, R_{n-2}^{n-i})|$. The sequence S_n^{i-1} consists of sequence S^{i-1}_{n-2} with 00 prefixed to each string in the sequence; followed by sequence S^{i-2}_{n-2} with 01 prefixed to each string in the sequence; followed by sequence S^{i-2}_{n-2} with 10 prefixed to each string in the sequence; followed by sequence S^{i-3}_{n-2} with 11 prefixed to each string in the sequence.

Then

$$\begin{aligned} |\,\mathrm{B}(y, S^{i-1}_n)| &= |\,\mathrm{B}(10w0, S^{i-1}_n)| \\ &= |\,\mathrm{B}(w0, S^{i-2}_{n-2})| + \tbinom{n-2}{i-2} + \tbinom{n-2}{i-1} \\ &= |\,\mathrm{B}(\overline{w}1, R^{n-i}_{n-2})| + \tbinom{n-2}{i-2} + \tbinom{n-2}{i-1} \quad \text{by Lemma 2.7} \\ &= \tbinom{n-2}{i-2} - |\,\mathrm{A}(\overline{w}1, R^{n-i}_{n-2})| - 1 + \tbinom{n-2}{i-2} + \tbinom{n-2}{i-1} \quad \text{by Equation (13)} \\ &= 2\tbinom{n-2}{i-2} + \tbinom{n-2}{i-1} - 1 - |\,\mathrm{A}(\overline{w}1, R^{n-i}_{n-2})|. \end{aligned} \tag{14}$$

We use Equations (12) and (14) to simplify the expression in Equation (10) to:

$$\begin{aligned} \Big(f(x) + f(y)\Big) \;-\; \Big(f(\overline{w}1) + f(\overline{w}0)\Big) &= 3 \cdot 2^{n-2} + \tbinom{n-1}{i} - \tbinom{n-3}{i-3} + 2\tbinom{n-2}{i-2} - \tbinom{n-2}{i-1} - \tbinom{n-2}{i} \\ &= 3 \cdot 2^{n-2} + 2\tbinom{n-2}{i-2} - \tbinom{n-3}{i-3} \\ &= 3 \cdot 2^{n-2} + \tbinom{n-2}{i-2} + \tbinom{n-3}{i-2} \\ &\leq 3 \cdot 2^{n-2} + \tbinom{n-2}{\lceil (n-2)/2 \rceil} + \tbinom{n-3}{\lceil (n-3)/2 \rceil}. \end{aligned}$$

So $\mathrm{str}_f(Q_n) \leq \mathrm{str}_f(Q_{n-2}) + 3 \cdot 2^{n-2} + \binom{n-2}{\lceil (n-2)/2 \rceil} + \binom{n-3}{\lceil (n-3)/2 \rceil}$. □

The next corollary follows directly from Theorems 2.4, 2.5, 2.6 and Theorem 2.8.

Corollary 2.9. *For $n \geq 5$,*

$$\mathrm{str}(Q_n) \leq \mathrm{str}_f(Q_{n-2}) + 3 \cdot 2^{n-2} + \binom{n-3}{\lceil \frac{n-3}{2} \rceil} + \binom{n-2}{\lceil \frac{n-2}{2} \rceil}.$$

2.3 A comparison of upper bounds

Table 2 compares the upper bound given by Corollary 2.9 to the upper bound given by Theorem 1.1 [4] for $n \in \{5, 6, \ldots, 13\}$. For larger values of n, we compare the bounds by using the following inequality, due to [3], which holds for $k \geq 1$:

$$\tbinom{2k}{k} < \frac{4^k}{\sqrt{\pi k}}. \tag{15}$$

We further observe that if $k \geq 6$ then $\sqrt{\pi k} > 4$. From Inequality (15), it follows that for $k \geq 6$,

$$\tbinom{2k}{k} < 4^{k-1}. \tag{16}$$

Corollary 2.10. *For all $n \geq 14$, $\mathrm{str}(Q_n) \leq 2^n + 2^{n-3} + 28$.*

n	upper bound on $\mathrm{str}(Q_n)$ from Theorem 1.1	upper bound on $\mathrm{str}(Q_n)$ from Corollary 2.9
5	41	40
6	81	78
7	161	152
8	321	300
9	641	591
10	1281	1173
11	2561	2323
12	5121	4623
13	10241	9181

Table 2: Comparison of upper bounds for small values of n.

Proof. First, suppose $n-2$ is even and $n \geq 6$. We use the fact that $\binom{2k-1}{k} = \frac{1}{2}\binom{2k}{k}$ and $\mathrm{str}(Q_4) = 21$ to simplify the recurrence from Corollary 2.9 to that of Inequality (17).

$$\begin{aligned}
\mathrm{str}(Q_n) &\leq 21 + 3\Big(2^4 + 2^6 + \cdots + 2^{n-2}\Big) + \tfrac{3}{2}\Big(\tbinom{4}{2} + \tbinom{6}{3} + \cdots + \tbinom{(n-2)}{(n-2)/2}\Big) && (17)\\
&= 21 + 3\Big(\sum_{k=2}^{(n-2)/2} 4^k\Big) + \tfrac{3}{2}\Big(\sum_{k=2}^{(n-2)/2} \tbinom{2k}{k}\Big) && (18)\\
&= 5 + 2^n + \tfrac{3}{2}\Big(\sum_{k=2}^{(n-2)/2} \tbinom{2k}{k}\Big) && (19)\\
&= 2^n + \tfrac{3}{2}\Big(\sum_{k=6}^{(n-2)/2} \tbinom{2k}{k}\Big) + 527 && (20)\\
&< 2^n + \tfrac{3}{2}\Big(\sum_{k=6}^{(n-2)/2} 4^{k-1}\Big) + 527 \qquad \text{using (16)}\\
&= 2^n + 2^{n-3} + 15.
\end{aligned}$$

Note that (17)–(19) hold for $n \geq 6$. However, to get (20), we split up a summation and this requires the condition that $n \geq 14$ (since the summation then runs from $k = 6$ to $k = (n-2)/2$).

If n is odd, then since $\mathrm{str}(Q_5) = 40$, a similar calculation shows that for odd $n \geq 15$, $\mathrm{str}(Q_n) \leq 2^n + 2^{n-3} + 28$. □

Finally, we relate the bound in Corollary 2.10 to the bound of Theorem 1.1 [4].

From Theorem 1.1 [4], $\mathrm{str}(Q_n) \leq 2^n + 2^{n-2} + 1 = 2^n + 2^{n-3} + 28 + (2^{n-3} - 27)$. Consequently, for $n \geq 14$, the bound from Corollary 2.10 is an improvement of $(2^{n-3} - 27)$ over Theorem 1.1 [4]. Observe there remains room for improvement of the upper bound for $\mathrm{str}(Q_n)$: we used Inequality (16), which holds for $k \geq 6$. However, one could choose a larger value of k to gain a different inequality; for example, if $k \geq 21$ then $\binom{2k}{k} < \frac{4^k}{8} = \frac{1}{2} \cdot 4^{k-1}$ and if $k \geq 82$ then $\binom{2k}{k} < \frac{4^k}{16} = 4^{k-2}$. The tradeoff however, is that the constant term in the bound increases greatly.

Acknowledgements

Dylan Pearson acknowledges research support from the Atlantic Association for Research in the Mathematical Sciences (AARMS) and Mount Allison University. Melissa A. Huggan acknowledges research support from NSERC (grant application 2023-03395). M.E. Messinger acknowledges research support from NSERC (grant application 2018-04059).

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(Received 31 July 2025; revised 19 Mar 2026)