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Perfect 1-factorisations of $K_{11,11}$

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Abstract

A perfect 1-factorisation of a graph is a decomposition of that graph into 1-factors such that the union of any two 1-factors is a Hamiltonian cycle. A Latin square of order n is row-Hamiltonian if for every pair (r, s) of distinct rows, the permutation mapping r to s has a single cycle of length n. We report the results of a computer enumeration of the perfect 1-factorisations of the complete bipartite graph $K_{11,11}$. This also allows us to find all row-Hamiltonian Latin squares of order 11. Finally, we plug a gap in the literature regarding how many row-Hamiltonian Latin squares are associated with the classical families of perfect 1-factorisations of complete graphs.

1 Introduction

A 1-*factor*, or *perfect matching*, of a graph G is a set of edges of G with the property that every vertex of G is in exactly one of the edges. A 1-*factorisation* of G is a partition of its edge set into 1-factors. Let $\mathcal{F}$ be a 1-factorisation of G and let f and f' be distinct 1-factors in $\mathcal{F}$. The edges in f and f' together form a subgraph of G which is a union of cycles of even length. If $f \cup f'$ induces a Hamiltonian cycle in G, regardless of the choice of f and f', then $\mathcal{F}$ is a *perfect* 1-*factorisation*. Two 1-factorisations $\mathcal{F}$ and $\mathcal{E}$ of G are *isomorphic* if there exists a permutation ϕ of the vertices of G which maps the set of 1-factors in $\mathcal{F}$ onto the set of 1-factors in $\mathcal{E}$. In this case, ϕ is an *isomorphism* from $\mathcal{F}$ to $\mathcal{E}$. An *automorphism* of $\mathcal{F}$ is an isomorphism from $\mathcal{F}$ to itself. The *automorphism group* of $\mathcal{F}$ is the set of all automorphisms of $\mathcal{F}$ under composition.

The main purpose of this paper is to report the results of a computer enumeration of the perfect 1-factorisations of the complete bipartite graph $K_{11,11}$. It is known that

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a perfect 1-factorisation of $K_{n,n}$ can only exist if $n = 2$ or n is odd (see, e.g., [18]). It is conjectured that a perfect 1-factorisation of $K_{n,n}$ does exist in these cases. However, this conjecture is a long way from being resolved. There are few known infinite families of perfect 1-factorisations of complete bipartite graphs [2, 5, 6], and these only cover graphs $K_{n,n}$ where $n \in \{p, 2p-1, p^2\}$ for some odd prime p. Up to isomorphism there are 1, 1, 1, 2 and 37 perfect 1-factorisations of $K_{2,2}$, $K_{3,3}$, $K_{5,5}$, $K_{7,7}$ and $K_{9,9}$, respectively [18].

Perfect 1-factorisations of complete bipartite graphs are related to perfect 1-factorisations of complete graphs (see [20] for full details of this relationship). In particular, a perfect 1-factorisation of K_{2n} can be used to build a perfect 1-factorisation of $K_{2n-1,2n-1}$ using a construction that we call Kotzig's construction, which is given explicitly in §5. So the existence of a perfect 1-factorisation of K_{2n} implies the existence of a perfect 1-factorisation of $K_{2n-1,2n-1}$, but it is not known whether the converse holds. In 1964, Kotzig [10] famously conjectured that a perfect 1-factorisation of K_{2n} exists for all positive integers n. This conjecture remains even further from resolution than the conjecture on the existence of perfect 1-factorisations of complete bipartite graphs. There are only three known infinite families of perfect 1-factorisations of complete graphs [5], and these only cover graphs K_{2n} where $2n \in \{p+1, 2p\}$ for an odd prime p. Up to isomorphism there are 1, 1, 1, 1, 1, 5, 23 and 3155 perfect 1-factorisations of K_2, K_4, K_6, K_8, K_{10}, K_{12}, K_{14} and K_{16}, respectively [7, 8, 9, 13, 15].

The main result of this paper is the following theorem.

Theorem 1.1. *There are* 687 121 *perfect* 1*-factorisations of* $K_{11,11}$ *up to isomorphism. Of these,* 2657 *have a non-trivial automorphism group.*

The structure of this paper is as follows. In §2 we discuss our enumeration algorithm for proving Theorem 1.1. There is an equivalence between 1-factorisations of complete bipartite graphs and Latin squares. As a result, the catalogue behind Theorem 1.1 allows us to enumerate several interesting classes of Latin squares of order 11, as discussed in §3. In §4, we discuss how useful various invariants are for distinguishing our enumerated objects. In §5, we prove a new property of a well known family of perfect 1-factorisations of complete graphs.

To reduce the risk of programming errors, all computations described in this paper were performed independently by each author, then crosschecked. The combined computation time was under two CPU years.

2 The algorithm

In this section we describe how we generated the perfect 1-factorisations of $K_{11,11}$. The algorithm we used is similar to the algorithm used in [9] to generate the perfect 1-factorisations of K_{16}.

A *partial* 1*-factorisation* of a graph G is a collection of pairwise disjoint 1-factors of G. Let $\mathcal{P}$ be a partial 1-factorisation of G and let f and f' be distinct 1-factors in $\mathcal{P}$. If $f \cup f'$ induces a Hamiltonian cycle in G then (f, f') is a *perfect pair.* If every pair of distinct 1-factors in $\mathcal{P}$ is perfect, then $\mathcal{P}$ is called perfect. An *ordered* partial 1-factorisation is a partial 1-factorisation with an order on its 1-factors. We use $\mathcal{F} = [f_1, f_2, \ldots, f_a]$ to denote an ordered partial 1-factorisation with 1-factors $f_1, \ldots, f_a$ and then $\mathcal{F} \| f_{a+1}$ to denote $[f_1, f_2, \ldots, f_a, f_{a+1}]$, the ordered partial 1-factorisation obtained by appending f_{a+1} to $\mathcal{F}$. Two ordered partial 1-factorisations $\mathcal{F} = [f_1, f_2, \ldots, f_a]$ and $\mathcal{E} = [e_1, e_2, \ldots, e_a]$ of G are *isomorphic* if there is a permutation ψ of $\{1, 2, \ldots, a\}$ and a permutation ϕ of the vertices of G which maps f_i onto $e_{\psi(i)}$ for every $i \in \{1, 2, \ldots, a\}$. In this section, we will primarily be discussing ordered 1-factorisations of $K_{n,n}$. However, for most of our purposes the order will be inconsequential, so we will often just refer to such objects as 1-factorisations.

Throughout this section, we will assume that the vertices of $K_{n,n}$ are labelled by $u_1, u_2, \ldots, u_n$ and $v_1, v_2 \ldots, v_n$, where there is an edge between u_i and v_j for all $\{i, j\} \subseteq \{1, \ldots, n\}$. For brevity we will write the edge $\{u_i, v_j\}$ as $u_i v_j$, and similarly for other graphs. We will call an isomorphism *direct* if it preserves $\{u_1, u_2, \ldots, u_n\}$ and $\{v_1, v_2 \ldots, v_n\}$ setwise, and *indirect* if it exchanges these two sets.

We now define a partial order $\preccurlyeq$ on the set of ordered partial 1-factorisations of $K_{n,n}$. Let $\mathcal{F} = [f_1, f_2, \ldots, f_a]$ and $\mathcal{E} = [e_1, e_2, \ldots, e_b]$ be two distinct such partial 1-factorisations. If $f_i = e_i$ for all $i \leqslant \min(a, b)$ then $\mathcal{F}$ and $\mathcal{E}$ are incomparable. Otherwise, let j be minimal such that $f_j \neq e_j$. If $j > 4$ then we deem $\mathcal{F}$ and $\mathcal{E}$ incomparable. So suppose that $j \leqslant 4$. The edges in f_j can be written as $u_1x_1, u_2x_2, \ldots, u_nx_n$ where $\{x_1, \ldots, x_n\} = \{v_1, \ldots, v_n\}$. Similarly the edges in e_j can be written as $u_1y_1, u_2y_2, \ldots, u_ny_n$ where $\{y_1, \ldots, y_n\} = \{v_1, \ldots, v_n\}$. Let ℓ be minimal such that $x_\ell \neq y_\ell$. We say that $\mathcal{F} \prec \mathcal{E}$ if $x_\ell < y_\ell$ (in the lexicographical ordering). If $x_\ell > y_\ell$, we say that $\mathcal{E} \prec \mathcal{F}$. Let $\preccurlyeq$ denote the reflexive closure of $\prec$.

Let $\mathcal{F} = [f_1, f_2, \ldots, f_a]$ be an ordered partial 1-factorisation of $K_{n,n}$ with $a \geqslant 4$. Denote by $\mathcal{F}^i$ the ordered partial 1-factorisation $[f_1, f_2, \ldots, f_i]$. Say that $\mathcal{F}$ is *minimal* if $\mathcal{F}^4 \preccurlyeq \mathcal{E}^4$ for every ordered partial 1-factorisation $\mathcal{E}$ of $K_{n,n}$ that is isomorphic to $\mathcal{F}$. Note that if $\mathcal{F} = [f_1, f_2, \ldots, f_a]$ is a minimal perfect partial 1-factorisation then

$$\begin{aligned} f_1 &= \{u_1v_1, u_2v_2, \ldots, u_{n-1}v_{n-1}, u_nv_n\} \text{ and,} \\ f_2 &= \{u_1v_2, u_2v_3, \ldots, u_{n-1}v_n, u_nv_1\}. \end{aligned} \tag{2.1}$$

A graph isomorphism between vertex coloured graphs is *colour preserving* if the colour of each vertex matches the colour of its image under the isomorphism. The software `Nauty` [12] is a practical algorithm for testing whether there is a colour preserving graph isomorphism between two vertex coloured graphs. Isomorphism testing for 1-factorisations of bipartite graphs can be converted into an isomorphism problem on vertex coloured graphs as follows. For a 1-factorisation $\mathcal{F} = [f_1, f_2, \ldots, f_a]$ of a graph $G \subseteq K_{n,n}$ we construct a coloured graph $C(\mathcal{F})$ containing

- green vertices $f_1, f_2, \ldots, f_a$ each joined to a blue vertex F,
- green vertices $u_1, \ldots, u_n$ each joined to a red vertex U,

- green vertices $v_1, \ldots, v_n$ each joined to a red vertex V,
- one black vertex for each edge in G which is joined to one green vertex in each of the previous three categories to indicate the end points of the edge and the 1-factor that contains the edge.

It is routine to check that two partial 1-factorisations $\mathcal{F}$ and $\mathcal{E}$ are isomorphic if and only if there is a colour preserving graph isomorphism from $C(\mathcal{F})$ to $C(\mathcal{E})$. Also, the automorphism group of $\mathcal{F}$ is (group) isomorphic to the group of colour preserving automorphisms of $C(\mathcal{F})$, which `Nauty` counts. As an aside, the whole construction can be varied in an obvious way to solve the isomorphism problem for 1-factorisations of non-bipartite graphs.

Our algorithm for generating the perfect 1-factorisations of $K_{n,n}$ is described in Procedure 2, and its subroutine `AddFactor` described in Procedure 1. Steps 2 and 7 of Procedure 2 can be handled in a straightforward manner using `Nauty` as discussed above, and represent a negligible fraction of the computation time. As mentioned at the beginning of this section, our algorithm is similar to the one used in [9] to generate the perfect 1-factorisations of K_{16}. Apart from obvious adaptations to the bipartite setting, the main change is a refinement on when minimality checks are performed.

Procedure 1: Recursively add 1-factors to a perfect partial 1-factorisation

input: An odd integer $n \geqslant 5$
A perfect partial 1-factorisation $\mathcal{P}$ of $K_{n,n}$
A set $\mathcal{T}$ of 1-factors t for which $\mathcal{P} \| t$ is a perfect partial 1-factorisation

Procedure `AddFactor`$(n, \mathcal{P}, \mathcal{T})$
    if $|\mathcal{P}| = n$ **then**
        Output $\mathcal{P}$
    else
        Let e be an edge of $K_{n,n} \setminus \bigcup \mathcal{P}$ that is in the fewest 1-factors in $\mathcal{T}$
        for $t \in \mathcal{T}$ *containing* e **do**
            Let $\mathcal{T}^*$ be the set of 1-factors $t^* \in \mathcal{T}$ such that (t, t^*) is a perfect pair
            `AddFactor`$(n,\ \mathcal{P}\|t,\ \mathcal{T}^*)$
        end
    end
end

Our algorithm starts by producing the set $\mathcal{S}$ of all minimal perfect partial 1-factorisations of $K_{n,n}$ that contain four 1-factors and that include the edges u_1v_1, u_1v_2, u_1v_3 and u_1v_4. Note that $\preccurlyeq$ is a total order on $\mathcal{S}$ and it follows that no two elements of $\mathcal{S}$ are isomorphic. For each element $P \in \mathcal{S}$ we then find the set $\mathcal{T}$ of 1-factors whose addition to P preserves minimality and perfection. These 1-factors are then used to recursively extend our partial 1-factorisation by one 1-factor at a time

Procedure 2: Generate perfect 1-factorisations of $K_{n,n}$

input: An odd integer $n \geqslant 5$

Procedure `GenP1Fs`(n)

  Generate the set $\mathcal{S}$ of minimal perfect partial 1-factorisations of $K_{n,n}$ containing four 1-factors and edges u_1v_1, u_1v_2, u_1v_3, u_1v_4

  for $P \in \mathcal{S}$ **do**

    Let $\mathcal{T} = \{$1-factors t such that $P \| t$ is a minimal perfect partial 1-factorisation$\}$

    `AddFactor`(n, P, $\mathcal{T}$)

  end

  Screen the 1-factorisations output by `AddFactor` for isomorphism

end

until it has been completed to a perfect 1-factorisation. Other choices for the initial number of 1-factors could have been possible. Our choice of four initial 1-factors was made so that the cardinalities of both the sets $\mathcal{S}$ and $\mathcal{T}$ were manageable. For $n = 11$ we found that $|\mathcal{S}| = 13\,727\,482$ and $|\mathcal{T}|$ ranged from 13 954 down to 0. As a result of the minimality requirements, $|\mathcal{T}|$ generally trended downwards as P increased in the $\preccurlyeq$ order, resulting in significant speed-up for later parts of the computation.

It is worth remarking that there are perfect partial 1-factorisations of $K_{n,n}$ that consist of four 1-factors (including the edges u_1v_1, u_1v_2, u_1v_3 and u_1v_4) but are not isomorphic to any element of $\mathcal{S}$. For example, form $\mathcal{F}_*$ from the two factors in (2.1) together with

$$f_3 = \{u_1v_3, u_2v_1, u_3v_5, u_4v_7, u_5v_4, u_6v_{10}, u_7v_{11}, u_8v_6, u_9v_2, u_{10}v_9, u_{11}v_8\}, \text{ and}$$
$$f_4 = \{u_1v_4, u_2v_{10}, u_3v_1, u_4v_{11}, u_5v_7, u_6v_9, u_7v_6, u_8v_2, u_9v_8, u_{10}v_3, u_{11}v_5\}.$$

Note that $\mathcal{F}_*$ is perfect, and contains the edges u_1v_1, u_1v_2, u_1v_3 and u_1v_4. However, the minimal member of the isomorphism class of $\mathcal{F}_*$ is formed from the two factors in (2.1) together with

$$f_3 = \{u_1v_3, u_2v_1, u_3v_5, u_4v_2, u_5v_9, u_6v_4, u_7v_6, u_8v_{11}, u_9v_8, u_{10}v_7, u_{11}v_{10}\}, \text{ and}$$
$$f_4 = \{u_1v_{10}, u_2v_5, u_3v_7, u_4v_{11}, u_5v_4, u_6v_3, u_7v_9, u_8v_6, u_9v_1, u_{10}v_2, u_{11}v_8\},$$

and does not contain u_1v_4. So, the isomorphism class of $\mathcal{F}_*$ has no representative in $\mathcal{S}$. Notwithstanding this example, we next show that our algorithm performs the desired enumeration.

Lemma 2.1. *The set of* 1*-factorisations returned by* `GenP1Fs`(n) *contains at least one representative from each isomorphism class of perfect* 1*-factorisations of* $K_{n,n}$.

Proof. Let $\mathcal{M}$ be an isomorphism class of ordered perfect 1-factorisations of $K_{n,n}$. Since $\mathcal{M}$ is finite it must contain a minimal element $\mathcal{F} = [f_1, \ldots, f_n]$. Minimality of $\mathcal{F}$ forces f_i to contain the edge u_1v_i for $1 \leqslant i \leqslant 4$ (cf. (2.1)), and hence $\mathcal{F}^4 \in \mathcal{S}$.

Count	From K_{12}	Direct automorphisms	Automorphisms
684464	0	1	1
100	15	1	2
2531	0	2	2
6	0	5	5
5	3	5	10
7	0	10	10
3	3	10	20
1	0	22	22
2	0	55	55
1	1	55	110
1	1	1210	2420

Table 1: Symmetries of perfect 1-factorisations of $K_{11,11}$

Let $U = \{f_5, f_6, \ldots, f_n\}$. The minimality of $\mathcal{F}$ implies that $\mathcal{F}^4 \| f$ is minimal for $f \in U$. So $U \subseteq \mathcal{T}$ when `AddFactor` is called with input $\mathcal{P} = \mathcal{F}^4$. By induction on $k \in \{5, \ldots, n\}$, subsequent recursive calls to `AddFactor` will be made with input $\mathcal{P}$ that consists of $\mathcal{F}^4$ together with $k - 4$ of the 1-factors in U, whilst the 1-factors in $U \setminus \mathcal{P}$ are in $\mathcal{T}$. In each inductive step the 1-factor t that is added to $\mathcal{P}$ will be whichever 1-factor in U contains the edge e defined by Line 5 of `AddFactor`. There must be such a 1-factor available, because U contains a 1-factorisation of $K_{n,n} \setminus \mathcal{F}^4$, and $\mathcal{T}$ inherits a 1-factorisation of the graph induced by whichever edges have not yet been included in $\mathcal{P}$.

In the case $k = n$, we see that `AddFactor` will output an ordered factorisation that equals $\mathcal{F}$, up to the order of its 1-factors. The result follows. □

Both of our implementations of `GenP1Fs` were used to generate the perfect 1-factorisations of $K_{n,n}$ for $n \in \{5, 7, 9, 11\}$. Results of both programs agreed with each other, and for $n \leqslant 9$ agreed with previously computed values [18]. Table 1 shows the 687 121 perfect 1-factorisations of $K_{11,11}$ categorised by the size of their automorphism group. The third column of the table lists the number of direct automorphisms of each 1-factorisation, and the fourth column lists the total number of automorphisms. The first column gives the number of isomorphism classes that attain the attributes listed in the row in question and the second column gives the number of such isomorphism classes that can be obtained from perfect 1-factorisations of K_{12} via Kotzig's construction. The number of direct automorphisms of a 1-factorisation $\mathcal{F}$ can be counted by using `Nauty` as described above, simply by changing the colour of the vertex V to yellow so that it can no longer be interchanged with U in any colour preserving automorphism of $C(\mathcal{F})$.

3 Latin squares

We start this section by giving some basic definitions regarding Latin squares, and explain their relationship to 1-factorisations of complete bipartite graphs. We then apply this previously known theory to our new catalogue of perfect 1-factorisations of $K_{11,11}$, uncovering some interesting Latin squares of order 11.

Let n and m be positive integers with $m \leqslant n$. An $m \times n$ *Latin rectangle* is an $m \times n$ matrix of n symbols, each of which occurs exactly once in each row and at most once in each column. A *Latin square* of order n is an $n \times n$ Latin rectangle. In this paper we will always assume that the rows and columns of a Latin square are indexed by its symbol set. Let L be an $m \times n$ Latin rectangle. A *subrectangle* of L is a submatrix of L that is itself a Latin rectangle. A $k \times \ell$ subrectangle is *proper* if $1 < k \leqslant \ell < n$. A *subsquare* of L is a subrectangle of L that is itself a Latin square. A *row cycle of length* k in L is a $2 \times k$ subrectangle of L that has no proper subrectangles. A *row-Hamiltonian* Latin square is a Latin square that has no proper subrectangles. Equivalently, a Latin square of order n is row-Hamiltonian if all of its row cycles have length n. A related but strictly weaker property is named N_∞, which applies to Latin squares that have no proper subsquares. Such properties are very natural for mathematicians to consider, but turn out to be quite elusive. After more than 50 years of studying the existence question it has only very recently been established in [1] that N_∞ Latin squares exist for all orders $n \notin \{4, 6\}$. The analogous but harder question for row-Hamiltonian Latin squares is very far from being solved, and provides one of the motivations for compiling catalogues for small orders.

Let L and L' be Latin squares. If L can be obtained from L' by applying a permutation α to its rows, a permutation β to its columns and a permutation γ to its symbols, then L and L' are *isotopic*, and (α, β, γ) is an *isotopism* from L' to L. Isotopism is an equivalence relation and the equivalence classes are called *isotopism classes*. Latin squares in the same isotopism class have the same number of subrectangles of each dimension, so the row-Hamiltonian property is an isotopism class invariant. An *autotopism* of L is an isotopism from L to itself. The *autotopism group* of L is the set of all autotopisms of L under composition.

Let L be a Latin square of order n. We can consider L as a set of n^2 triples of the form (row, column, symbol), called *entries*. A *conjugate* of L is a Latin square obtained from L by choosing a permutation of $\{1, 2, 3\}$ and applying it to the coordinates of each entry in L. Every conjugate of L can thus be labelled by a 1-line permutation of $\{1, 2, 3\}$, which gives the order of the coordinates of the conjugate relative to the order of the coordinates of L. For example, the (213)-conjugate of L is its matrix transpose. The (132)-conjugate of L is its *row-inverse*. If L is isotopic to some conjugate of L' then L and L' are *paratopic*. A *paratopism* from L' to L is a pair $(\mathcal{C}, \mathcal{A})$ where $\mathcal{A}$ is a 1-line permutation of $\{1, 2, 3\}$ specifying a conjugate L'' of L', and $\mathcal{C}$ is an isotopism from L'' to L. Paratopism is an equivalence relation and the equivalence classes are called *species*. An *autoparatopism* of L is a paratopism from L to itself. The *autoparatopism group* of L is the set of all autoparatopisms of L under composition.

Let L be a Latin square. Let $\nu(L)$ be the number of conjugates of L that are row-Hamiltonian. We will also say that L has $\nu = \nu(L)$. Since the row-Hamiltonian property is an isotopism class invariant, it follows that ν is a species invariant.

So if $\nu(L) = c$ then we will say that the species of Latin squares containing L has $\nu = c$. Latin squares with $\nu = 6$ are called *atomic*. It is known [18] that $\nu(L) \in \{0, 2, 4, 6\}$, since a Latin square is row-Hamiltonian if and only if its row-inverse is row-Hamiltonian.

There is a natural equivalence between Latin squares of order n and ordered 1-factorisations of $K_{n,n}$. This equivalence is studied in [18, 20], for example, where the following observations are spelt out in detail. Let L be a Latin square of order n. Label the vertices in one part of $K_{n,n}$ by $\{c_1, c_2, \dots, c_n\}$, corresponding to the columns of L, and the vertices in the other part by $\{s_1, s_2, \dots, s_n\}$, corresponding to the symbols of L. For row i of L, we define a 1-factor f_i of $K_{n,n}$ by adding the edge $c_j s_k$ to f_i whenever $L_{i,j} = k$. Then $\mathcal{F} = [f_1, f_2, \dots, f_n]$ is an ordered 1-factorisation of $K_{n,n}$, where the order on the 1-factors comes from the order of the rows of L. It is easy to see that this construction is also reversible, giving a map $\mathcal{F} \mapsto \mathcal{L}(\mathcal{F})$ from ordered 1-factorisations of $K_{n,n}$ to Latin squares of order n. The subgraph of $K_{n,n}$ induced by the 1-factors f_i and f_j is a union of cycles of even length, and it contains a cycle of length $2k$ if and only if there is a row cycle in $\mathcal{L}(\mathcal{F})$ of length k hitting rows i and j. Thus $\mathcal{F}$ is perfect if and only if $\mathcal{L}(\mathcal{F})$ is row-Hamiltonian.

Let $\mathcal{F}$ and $\mathcal{E}$ be ordered 1-factorisations of $K_{n,n}$. From the definition of $\mathcal{L}(\mathcal{F})$ and $\mathcal{L}(\mathcal{E})$, it is not hard to see that $\mathcal{F}$ is isomorphic to $\mathcal{E}$ if and only if $\mathcal{L}(\mathcal{F})$ is isotopic to $\mathcal{L}(\mathcal{E})$ or the row-inverse of $\mathcal{L}(\mathcal{E})$ (see [20] for details).

Lemma 3.1. *Suppose that L is any Latin square of order n that is isotopic to its transpose. For $X \in \{123, 132, 213, 231, 312, 321\}$ let $\mathcal{F}_X$ denote the ordered 1-factorisation of $K_{n,n}$ for which $\mathcal{L}(\mathcal{F}_X)$ is the (X)-conjugate of L. Then $\mathcal{F}_{123}$, $\mathcal{F}_{132}$, $\mathcal{F}_{213}$ and $\mathcal{F}_{231}$ are all isomorphic. Hence, if L is row-Hamiltonian then $\nu(L) \in \{4, 6\}$ and if the (321)-conjugate of L is row-Hamiltonian then $\nu(L) \in \{2, 6\}$.*

Proof. The proof is similar to that of [18, Lem. 5]. Any Latin square L would have an indirect isomorphism from $\mathcal{F}_{123}$ to $\mathcal{F}_{132}$ and also from $\mathcal{F}_{213}$ to $\mathcal{F}_{231}$. The fact that L is isotopic to its transpose means that $\mathcal{F}_{123}$ is isomorphic to $\mathcal{F}_{213}$. Hence, $\mathcal{F}_{123}, \mathcal{F}_{132}, \mathcal{F}_{213}$ and $\mathcal{F}_{231}$ are isomorphic to each other. Thus the following four statements are equivalent:

- L is row-Hamiltonian.
- The row-inverse of L is row-Hamiltonian.
- The transpose of L is row-Hamiltonian.
- The (231)-conjugate of L is row-Hamiltonian.

The lemma follows. □

Table 2 gives the number of species and isotopism classes containing row-Hamiltonian Latin squares, as well as the number of species containing atomic Latin squares of small orders. The data for orders up to 9 was determined by Wanless [18], and the number of species containing atomic Latin squares of order 11 was determined by Maenhaut and Wanless [11]. The number of species containing row-Hamiltonian

order	row-Hamiltonian species	row-Hamiltonian isotopism classes	atomic species
2	1	1	1
3	1	1	1
5	1	1	1
7	2	2	1
9	37	64	0
11	687 115	1 374 132	7

Table 2: Row-Hamiltonian and atomic Latin squares of small order

Latin squares of order n exactly matches the numbers of perfect 1-factorisations up to isomorphism of $K_{n,n}$ for all $n \in \{2, 3, 5, 7, 9\}$. However, this trend does not continue for order 11 as is shown concretely by (3.4) below. It was observed in [18] that there are no Latin squares of order n with $\nu = 4$ for $n \leqslant 9$, and that this trend also does not continue for $n = 11$.

From the set of representatives of isomorphism classes of perfect 1-factorisations of $K_{11,11}$, it is a simple task to obtain representatives of each species of row-Hamiltonian Latin squares of order 11. This can be achieved by using `Nauty` as described in §2, except that we recolour the vertex F red. Each autoparatopism group is also automatically calculated by `Nauty`, which allows us to deduce the following data.

Theorem 3.2.

- *There are* 687 115 *species containing row-Hamiltonian Latin squares of order* 11*. Of these,* 2660 *have a non-trivial autoparatopism group,* 687 096 *have* $\nu = 2$*,* 12 *have* $\nu = 4$ *and* 7 *have* $\nu = 6$*.*

- *There are* 1 374 132 *isotopism classes containing row-Hamiltonian Latin squares of order* 11*. Of these,* 5104 *have a non-trivial autotopism group.*

Theorem 3.2 allows us to fill in two previously unknown entries in the last row of Table 2. Table 3 shows the 687 115 species of row-Hamiltonian Latin squares of order 11 classified according to how much symmetry they have. In that table the second column lists how many species contain a symmetric Latin square whose (321)-conjugate is row-Hamiltonian, indicating that it can be obtained from one of the perfect 1-factorisations of K_{12} via Kotzig's construction. The third and fourth columns give the orders of the autotopism group and the autoparatopism group, respectively. The last column gives the value of ν and the first column reports how many species attain the attributes listed in the row in question.

Wanless [18] observed that 11 is the smallest order for which a Latin square with $\nu = 4$ exists. Theorem 3.2 tells us that there are 12 species of Latin squares of order 11 with $\nu = 4$, which we now catalogue. For $m \geqslant 1$ and $b \geqslant 0$ define $\mathbb{Z}_{m,b} = \mathbb{Z}_m \cup \{\infty_1, \infty_2, \ldots, \infty_b\}$ and

$$z^+ = \begin{cases} z + 1 & \text{if } z \in \mathbb{Z}_m, \\ z & \text{otherwise.} \end{cases}$$

Count	From K_{12}	Autotopisms	Autoparatopisms	ν
684455	0	1	1	2
99	14	1	2	2
8	0	1	2	4
1	1	1	2	6
2531	0	2	2	2
5	0	5	5	2
4	3	5	10	2
1	0	5	10	6
1	0	10	10	2
1	0	10	10	4
2	0	10	20	4
2	2	10	20	6
1	0	22	22	2
1	1	10	60	6
1	0	55	110	4
1	1	55	110	6
1	1	1210	7260	6

Table 3: Symmetries of species of row-Hamiltonian Latin squares of order 11

A *bordered diagonally cyclic Latin square* (BDCLS) of order $m+b$ is a Latin square L of order $m+b$ which satisfies the rule that if (i,j,k) is an entry of L then so is (i^+,j^+,k^+). Here we are using $\mathbb{Z}_{m,b}$ as the set of row indices, column indices and symbols. If $b=0$ then L is a *diagonally cyclic Latin square* (DCLS). For $b\in\{0,1\}$, a BDCLS is uniquely determined by its first row [17]. There are four species with $\nu=4$ that contain a BDCLS of order 11. The first row of a BDCLS representative for each such species is given below.

$$(0,10,4,8,7,6,1,3,5,2,9), \tag{3.1}$$
$$(0,2,6,8,7,\infty_1,3,5,4,1,9), \tag{3.2}$$
$$(0,3,7,9,8,\infty_1,4,6,5,2,1), \text{ and} \tag{3.3}$$
$$(\infty_1,1,9,7,5,3,8,6,4,2,0). \tag{3.4}$$

The DCLS whose first row is (3.1) comes from the only known infinite family of Latin squares with $\nu=4$ constructed in [2]. The BDCLS in (3.1), (3.2) and (3.3) are each symmetric so, by Lemma 3.1, each species gives rise to a single isomorphism class of perfect 1-factorisations. In contrast, the species represented by (3.4) gives rise to two isomorphism classes of perfect 1-factorisations. The remaining eight species all contain symmetric Latin squares. Figure 1 provides a symmetric representative of each of these species. As a consequence of their symmetry and Lemma 3.1, they also each give rise to a single isomorphism class of perfect 1-factorisations.

The seven species containing atomic Latin squares of order 11 were catalogued in [11]. From that study it can be inferred that they give rise to 12 isomorphism classes of perfect 1-factorisations. Of course, any species with $\nu=2$ can give rise to only a single isomorphism class of perfect 1-factorisations. This accounts for the $687\,121=687\,096+13+12$ perfect 1-factorisations of $K_{11,11}$ up to isomorphism. One

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0\\
2 & 3 & 9 & 8 & 0 & 4 & 5 & 10 & 7 & 1 & 6\\
3 & 4 & 8 & 5 & 9 & 7 & 2 & 6 & 10 & 0 & 1\\
4 & 5 & 0 & 9 & 7 & 8 & 10 & 1 & 3 & 6 & 2\\
5 & 6 & 4 & 7 & 8 & 10 & 0 & 2 & 1 & 3 & 9\\
6 & 7 & 5 & 2 & 10 & 0 & 1 & 9 & 4 & 8 & 3\\
7 & 8 & 10 & 6 & 1 & 2 & 9 & 3 & 0 & 5 & 4\\
8 & 9 & 7 & 10 & 3 & 1 & 4 & 0 & 6 & 2 & 5\\
9 & 10 & 1 & 0 & 6 & 3 & 8 & 5 & 2 & 4 & 7\\
10 & 0 & 6 & 1 & 2 & 9 & 3 & 4 & 5 & 7 & 8
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0\\
2 & 3 & 4 & 5 & 8 & 7 & 9 & 10 & 1 & 0 & 6\\
3 & 4 & 5 & 7 & 9 & 0 & 10 & 6 & 2 & 8 & 1\\
4 & 5 & 8 & 9 & 10 & 2 & 0 & 1 & 6 & 7 & 3\\
5 & 6 & 7 & 0 & 2 & 8 & 4 & 3 & 10 & 1 & 9\\
6 & 7 & 9 & 10 & 0 & 4 & 1 & 2 & 5 & 3 & 8\\
7 & 8 & 10 & 6 & 1 & 3 & 2 & 9 & 0 & 5 & 4\\
8 & 9 & 1 & 2 & 6 & 10 & 5 & 0 & 3 & 4 & 7\\
9 & 10 & 0 & 8 & 7 & 1 & 3 & 5 & 4 & 6 & 2\\
10 & 0 & 6 & 1 & 3 & 9 & 8 & 4 & 7 & 2 & 5
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0\\
2 & 3 & 5 & 8 & 7 & 1 & 10 & 0 & 6 & 4 & 9\\
3 & 4 & 8 & 9 & 0 & 7 & 2 & 10 & 5 & 6 & 1\\
4 & 5 & 7 & 0 & 8 & 10 & 9 & 6 & 1 & 3 & 2\\
5 & 6 & 1 & 7 & 10 & 4 & 0 & 9 & 2 & 8 & 3\\
6 & 7 & 10 & 2 & 9 & 0 & 3 & 5 & 4 & 1 & 8\\
7 & 8 & 0 & 10 & 6 & 9 & 5 & 1 & 3 & 2 & 4\\
8 & 9 & 6 & 5 & 1 & 2 & 4 & 3 & 10 & 0 & 7\\
9 & 10 & 4 & 6 & 3 & 8 & 1 & 2 & 0 & 7 & 5\\
10 & 0 & 9 & 1 & 2 & 3 & 8 & 4 & 7 & 5 & 6
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0\\
2 & 3 & 7 & 8 & 6 & 0 & 5 & 9 & 10 & 1 & 4\\
3 & 4 & 8 & 5 & 7 & 1 & 9 & 10 & 0 & 2 & 6\\
4 & 5 & 6 & 7 & 10 & 9 & 0 & 1 & 2 & 8 & 3\\
5 & 6 & 0 & 1 & 9 & 8 & 10 & 4 & 7 & 3 & 2\\
6 & 7 & 5 & 9 & 0 & 10 & 1 & 2 & 3 & 4 & 8\\
7 & 8 & 9 & 10 & 1 & 4 & 2 & 3 & 6 & 0 & 5\\
8 & 9 & 10 & 0 & 2 & 7 & 3 & 6 & 4 & 5 & 1\\
9 & 10 & 1 & 2 & 8 & 3 & 4 & 0 & 5 & 6 & 7\\
10 & 0 & 4 & 6 & 3 & 2 & 8 & 5 & 1 & 7 & 9
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0\\
2 & 3 & 5 & 8 & 7 & 10 & 0 & 6 & 4 & 1 & 9\\
3 & 4 & 8 & 9 & 0 & 7 & 1 & 2 & 10 & 5 & 6\\
4 & 5 & 7 & 0 & 1 & 2 & 10 & 9 & 6 & 8 & 3\\
5 & 6 & 10 & 7 & 2 & 3 & 9 & 1 & 0 & 4 & 8\\
6 & 7 & 0 & 1 & 10 & 9 & 8 & 4 & 5 & 3 & 2\\
7 & 8 & 6 & 2 & 9 & 1 & 4 & 10 & 3 & 0 & 5\\
8 & 9 & 4 & 10 & 6 & 0 & 5 & 3 & 7 & 2 & 1\\
9 & 10 & 1 & 5 & 8 & 4 & 3 & 0 & 2 & 6 & 7\\
10 & 0 & 9 & 6 & 3 & 8 & 2 & 5 & 1 & 7 & 4
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0\\
2 & 3 & 7 & 0 & 10 & 1 & 4 & 9 & 6 & 8 & 5\\
3 & 4 & 0 & 5 & 8 & 10 & 9 & 1 & 2 & 7 & 6\\
4 & 5 & 10 & 8 & 1 & 9 & 3 & 6 & 0 & 2 & 7\\
5 & 6 & 1 & 10 & 9 & 4 & 0 & 2 & 7 & 3 & 8\\
6 & 7 & 4 & 9 & 3 & 0 & 8 & 5 & 10 & 1 & 2\\
7 & 8 & 9 & 1 & 6 & 2 & 5 & 10 & 4 & 0 & 3\\
8 & 9 & 6 & 2 & 0 & 7 & 10 & 4 & 3 & 5 & 1\\
9 & 10 & 8 & 7 & 2 & 3 & 1 & 0 & 5 & 6 & 4\\
10 & 0 & 5 & 6 & 7 & 8 & 2 & 3 & 1 & 4 & 9
\end{pmatrix}$$

$$\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0\\
2 & 3 & 6 & 0 & 7 & 4 & 10 & 9 & 1 & 8 & 5\\
3 & 4 & 0 & 5 & 6 & 10 & 8 & 2 & 7 & 1 & 9\\
4 & 5 & 7 & 6 & 10 & 9 & 1 & 3 & 2 & 0 & 8\\
5 & 6 & 4 & 10 & 9 & 8 & 2 & 0 & 3 & 7 & 1\\
6 & 7 & 10 & 8 & 1 & 2 & 9 & 5 & 0 & 4 & 3\\
7 & 8 & 9 & 2 & 3 & 0 & 5 & 1 & 10 & 6 & 4\\
8 & 9 & 1 & 7 & 2 & 3 & 0 & 10 & 4 & 5 & 6\\
9 & 10 & 8 & 1 & 0 & 7 & 4 & 6 & 5 & 3 & 2\\
10 & 0 & 5 & 9 & 8 & 1 & 3 & 4 & 6 & 2 & 7
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 0\\
2 & 3 & 6 & 5 & 0 & 8 & 1 & 4 & 10 & 7 & 9\\
3 & 4 & 5 & 1 & 6 & 7 & 10 & 9 & 2 & 0 & 8\\
4 & 5 & 0 & 6 & 7 & 10 & 9 & 3 & 1 & 8 & 2\\
5 & 6 & 8 & 7 & 10 & 9 & 0 & 2 & 4 & 3 & 1\\
6 & 7 & 1 & 10 & 9 & 0 & 8 & 5 & 3 & 2 & 4\\
7 & 8 & 4 & 9 & 3 & 2 & 5 & 10 & 0 & 1 & 6\\
8 & 9 & 10 & 2 & 1 & 4 & 3 & 0 & 5 & 6 & 7\\
9 & 10 & 7 & 0 & 8 & 3 & 2 & 1 & 6 & 4 & 5\\
10 & 0 & 9 & 8 & 2 & 1 & 4 & 6 & 7 & 5 & 3
\end{pmatrix}$$

Figure 1: Eight symmetric row-Hamiltonian Latin squares

representative from each species containing row-Hamiltonian Latin squares of order 11 can be found at [19].

Up to paratopism, there are nine row-Hamiltonian Latin squares of order 11 that have trivial autotopism group but non-trivial autoparatopism group, and which give rise to perfect 1-factorisations with trivial automorphism group. They are the eight squares given in Figure 1 and a symmetric atomic Latin square in the class $\mathcal{A}_{11}^5$ from [11]. There are two isomorphism classes of perfect 1-factorisations which arise from $\mathcal{A}_{11}^5$. One of these has trivial automorphism group and the other has automorphism group of cardinality 2.

We have already given details for the Latin squares reported in Table 3 with $\nu > 2$. The most symmetric species with $\nu = 2$ is represented by the DCLS with first row $(0, 2, 8, 5, 7, 1, 10, 4, 6, 3, 9)$. It has an autotopism that applies the permutation $(0, 10)(1, 9)(2, 8)(3, 7)(4, 6)$ to the rows, columns and symbols. Together with the diagonally cyclic symmetry, this generates an autotopism group of order 22. The next most symmetric Latin square from Table 3 with $\nu = 2$ is

$$\begin{pmatrix} 5 & 7 & 0 & 4 & 9 & 6 & 10 & 8 & 2 & 1 & 3 \\ 10 & 5 & 8 & 1 & 0 & 7 & 4 & 6 & 9 & 3 & 2 \\ 1 & 6 & 5 & 9 & 2 & 8 & 3 & 0 & 7 & 10 & 4 \\ 3 & 2 & 7 & 5 & 10 & 9 & 0 & 4 & 1 & 8 & 6 \\ 6 & 4 & 3 & 8 & 5 & 10 & 7 & 1 & 0 & 2 & 9 \\ 4 & 0 & 1 & 2 & 3 & 5 & 8 & 9 & 10 & 6 & 7 \\ 2 & 9 & 6 & 10 & 4 & 1 & 5 & 3 & 8 & 7 & 0 \\ 0 & 3 & 10 & 7 & 6 & 2 & 1 & 5 & 4 & 9 & 8 \\ 7 & 1 & 4 & 6 & 8 & 3 & 9 & 2 & 5 & 0 & 10 \\ 9 & 8 & 2 & 0 & 7 & 4 & 6 & 10 & 3 & 5 & 1 \\ 8 & 10 & 9 & 3 & 1 & 0 & 2 & 7 & 6 & 4 & 5 \end{pmatrix}.$$

Its autotopism group is isomorphic to the dihedral group of order 10.

4 Invariants

Let $\mathcal{B}$ be the set of row-Hamiltonian Latin squares of order 11, and let $\mathcal{R}(\mathcal{B})$ be the set of species representatives of $\mathcal{B}$ that we generated. Similarly, let $\mathcal{D}$ be the set of perfect 1-factorisations of $K_{11,11}$, and let $\mathcal{R}(\mathcal{D})$ be the set of isomorphism class representatives of $\mathcal{D}$ that we generated. In this section we discuss some old and new invariants, and examine how useful they are for distinguishing elements of $\mathcal{B}$ and elements of $\mathcal{D}$. A *complete species invariant on* $\mathcal{B}$ is a function $\mathcal{I}$ on $\mathcal{B}$ such that $\mathcal{I}(L_1) = \mathcal{I}(L_2)$ if and only if Latin squares L_1 and L_2 are paratopic. A *complete isomorphism class invariant* for 1-factorisations can be defined similarly.

Let L be a Latin square of order n. A *transversal* of L is a selection of n of its entries such that no two entries share a row, column or symbol. Let $N(L)$ denote the number of transversals of L. It is immediate that N is a species invariant.

Let L be a Latin square with symbol set S of cardinality n. Define $G = G(L)$ to be a digraph with vertex set S^3 such that each vertex has a unique outgoing arc. The arc from (a, b, c) goes to the triple (x, y, z) where (a, b, z), (a, y, c) and (x, b, c)

are entries of L. The graph G is called the *train* of L, and the isomorphism class of G is a species invariant [16]. Thus, the indegree sequence of G (a sorted list of the indegrees of the vertices) is also a species invariant. Denote this indegree sequence by $I(L)$.

Recall that a row cycle of a Latin square is a $2 \times k$ subrectangle that contains no proper subrectangles. We can analogously define *column cycles* and *symbol cycles*, and taking conjugates interchanges these objects. For a Latin square L let $C(L)$ be a sorted list of the lengths of its row, column and symbol cycles. Then C is a species invariant. Also define $S(L)$ to be a multiset consisting of three sorted lists, one giving the lengths of its row cycles, one giving the lengths of its column cycles and one giving the lengths of its symbol cycles. Then S is also a species invariant.

We determined how well the above invariants distinguish squares in $\mathcal{B}$ and obtained the following results. When applied to every square in $\mathcal{R}(\mathcal{B})$:

- N took 630 values,
- I took 283 518 values,
- C took 151 412 values,
- S took 675 110 values,
- (I, C) took 687 069 values,
- (N, I, C) took 687 115 values, thus is a complete invariant on $\mathcal{B}$,
- (I, S) took 687 115 values, thus is a complete invariant on $\mathcal{B}$.

Let $\mathcal{F}$ and $\mathcal{E}$ be non-isomorphic perfect 1-factorisations of $K_{11,11}$ such that $\mathcal{L}(\mathcal{F})$ is paratopic to $\mathcal{L}(\mathcal{E})$. Since each of N, I, C and S are well known species invariants, they cannot possibly distinguish between $\mathcal{F}$ and $\mathcal{E}$. So we now define a new invariant, which is useful for distinguishing such perfect 1-factorisations.

Let $\mathcal{F} = \{f_1, f_2, \dots, f_n\}$ be a perfect 1-factorisation of $K_{n,n}$. Let $\{i, j, k\} \subseteq \{1, 2, \dots, n\}$ with $i < j$ and $k \notin \{i, j\}$. Let $\mathcal{F}_{i,j}$ denote the subgraph of $K_{n,n}$ with edge set $f_i \cup f_j$. Since $\mathcal{F}$ is perfect, $\mathcal{F}_{i,j}$ forms a Hamiltonian cycle in $K_{n,n}$. For each edge $e \in f_k$, define $p_{i,j,k,e}$ to be the distance between the endpoints of e in $\mathcal{F}_{i,j}$. Define

$$P(\mathcal{F}) = \sum_{i<j} \sum_{k \neq i,j} \prod_{e \in f_k} p_{i,j,k,e}.$$

Then P is invariant on isomorphism classes of perfect 1-factorisations of $K_{n,n}$.

When applied to every element of $\mathcal{R}(\mathcal{D})$, P took 687 115 values. The six pairs of elements in $\mathcal{R}(\mathcal{D})$ on which P coincided can be found at [19]. For any invariant $\mathcal{I} \in \{N, C, I, S\}$, the pair $(P, \mathcal{I})$ took 687 121 values, thus formed a complete invariant on $\mathcal{D}$.

5 The classical families of perfect 1-factorisations

The main purpose of this section is to revisit the classical families of perfect 1-factorisations to tie up a loose end in the literature, which we do in Theorem 5.2 below. As mentioned in §1, given a perfect 1-factorisation of the complete graph K_{n+1}, there is a known method for constructing perfect 1-factorisations of $K_{n,n}$. We refer to this as Kotzig's construction. Let n be an odd integer, let V be the vertex set of K_{n+1}, and suppose that $\mathcal{F}$ is a perfect 1-factorisation of K_{n+1}. For distinct x and y in V let $h_{x,y}$ denote the unique 1-factor in $\mathcal{F}$ containing the edge xy. Fix a vertex $v \in V$ called the *root* vertex. We associate to the pair $(\mathcal{F}, v)$ a Latin square of order n, denoted by $\mathcal{L}(\mathcal{F}, v)$, whose row index set, column index set and symbol set is $V \setminus \{v\}$, and is defined by

$$\mathcal{L}(\mathcal{F}, v)_{i,j} = \begin{cases} i & \text{if } j = i, \\ k & \text{if } j \neq i, \text{ where } k \in V \setminus \{v\} \text{ is such that } kv \in h_{i,j}. \end{cases}$$

Then $\mathcal{L}(\mathcal{F}, v)$ is a symmetric Latin square whose (321)-conjugate is row-Hamiltonian and hence encodes a perfect 1-factorisation of $K_{n,n}$. Lemma 3.1 implies that $\nu(\mathcal{L}(\mathcal{F}, v)) \in \{2, 6\}$. Furthermore, if $\{u, v\} \subseteq V$ then $\mathcal{L}(\mathcal{F}, v)$ is paratopic to $\mathcal{L}(\mathcal{F}, u)$ if and only if there is an automorphism of $\mathcal{F}$ that maps v to u. See [20] for more details.

We now discuss the known infinite families of row-Hamiltonian Latin squares that come from the construction given above. For each prime $p \geqslant 11$ there are two known non-isomorphic perfect 1-factorisations of K_{p+1} which come from infinite families. One is due to Kotzig and is commonly denoted by GK_{p+1}. The other is due to Bryant, Maenhaut, and Wanless [5], which we will denote by GB_{p+1}. There are two species that contain Latin squares $\mathcal{L}(GK_{p+1}, v)$ for some root vertex v, and there are three other species that contain Latin squares $\mathcal{L}(GB_{p+1}, v)$ for some root vertex v. If 2 is primitive modulo p then all five of these species have $\nu = 6$. If 2 is not primitive modulo p then two of these species have $\nu = 6$ and the remaining three have $\nu = 2$, see [5, 14, 18]. There is a well known perfect 1-factorisation of K_{2p} for every odd prime p, commonly denoted by GA_{2p}. Kotzig [10] stated that GA_{2p} is perfect for every odd prime p, and a proof was provided by Anderson [3]. For each odd prime p, every Latin square of the form $\mathcal{L}(GA_{2p}, v)$ lies in the same species. Our goal for this section is to show that this species has $\nu = 2$ unless $p = 3$, in which case it has $\nu = 6$.

There are some infinite families of row-Hamiltonian Latin squares that do not come from perfect 1-factorisations of complete graphs. For each prime $p \geqslant 11$, Bryant, Maenhaut and Wanless [6] constructed $(p-1)/2$ species of order p^2 with $\nu = 2$. Allsop and Wanless [2] constructed, for each prime $p \notin \{3, 19\}$ with $p \equiv 1 \bmod 8$ or $p \equiv 3 \bmod 8$, a Latin square of order p with $\nu = 4$. There are also some sporadic examples of row-Hamiltonian Latin squares [9, 16].

We now return to the family GA_{2p}. Let p be an odd prime and let the vertex set of K_{2p} be $\mathbb{Z}_p \times \{1,2\}$. For $i \in \mathbb{Z}_p$ define

$$f_i = \{(i+j,1)(i-j,1),(i+j,2)(i-j,2) : j \in \{1,2,\ldots,(p-1)/2\}\} \cup \{(i,1)(i,2)\}.$$

For $i \in \mathbb{Z}_p \setminus \{0\}$ define

$$g_i = \{(j,1)(i+j,2) : j \in \mathbb{Z}_p\}.$$

Then

$$GA_{2p} = \{f_i : i \in \mathbb{Z}_p\} \cup \{g_i : i \in \mathbb{Z}_p \setminus \{0\}\}$$

is a perfect 1-factorisation of K_{2p}.

Anderson [4] showed that the automorphism group of GA_{2p} acts transitively on the vertices of K_{2p}. Since we are only interested in the species of Latin square obtained from GA_{2p}, we may decide to work with the root vertex $v = (-1,2)$. Define $\mathscr{L}_p = \mathcal{L}(GA_{2p}, v)$. We can give a more explicit definition of $\mathscr{L}_p$.

Lemma 5.1. *The square $\mathscr{L} = \mathscr{L}_p$ is defined by*

$$\mathscr{L}_{(x,z),(y,w)} = \begin{cases} (x,z) & \text{if } (x,z)=(y,w), \\ (x+y+1,2) & \text{if } z=w \text{ and } x+y+2 \neq 0, \\ (-1,1) & \text{if } z=w \text{ and } x+y+2=0, \\ (2x+1,2) & \text{if } z \neq w, \text{ and } x=y, \\ (x-y-1,1) & \text{if } z=1, w=2, \text{ and } x \neq y, \\ (y-x-1,1) & \text{if } z=2, w=1 \text{ and } x \neq y. \end{cases}$$

Proof. Let $((x,z),(y,w)) \in (\mathbb{Z}_p \times \{1,2\})^2$ with $(x,z) \neq (y,w)$. First suppose that $z = 1 = w$. Let $i = 2^{-1}(x+y) \in \mathbb{Z}_p$ and note that $(x,z)(y,w) \in f_i$. If $x+y+2=0$ then $i=-1$ and so $(-1,1)(-1,2) \in f_i$. Hence $\mathscr{L}_{(x,z),(y,w)} = (-1,1)$. Now suppose that $x+y+2 \neq 0$. Let $j = i+1 \in \mathbb{Z}_p$ so that $i-j=-1$. Then $i+j = 2i+1 = x+y+1$ and thus $(x+y+1,2)(-1,2) \in f_i$. Hence $\mathscr{L}_{(x,z),(y,w)} = (x+y+1,2)$. Similar arguments can be used to prove that the claimed value of $\mathscr{L}_{(x,z),(y,w)}$ is correct when $z=2=w$.

Now assume that $z=1$ and $w=2$. We must distinguish two cases depending on whether or not $x=y$. First suppose that $x \neq y$. Let $i = y-x$ and note that $(x,z)(y,w) \in g_i$. Setting $i+j=-1$ we obtain $j=-i-1=x-y-1$. So $(x-y-1,1)(-1,2) \in g_i$ and thus $\mathscr{L}_{(x,z),(y,w)} = (x-y-1,1)$. Now suppose that $y=x$. Then $(x,z)(y,w) \in f_x$. Setting $x-j=-1$ yields $j=x+1$. Thus $(2x+1,2)(-1,2) \in f_x$ and so $\mathscr{L}_{(x,z),(y,w)} = (2x+1,2)$. Similar arguments can be used to prove that the claimed value of $\mathscr{L}_{(x,z),(y,w)}$ is correct when $z=2$ and $w=1$. □

We are now ready to determine $\nu(\mathscr{L}_p)$ for each odd prime p.

Theorem 5.2. *If $p=3$ then $\mathscr{L}_p$ is atomic and otherwise $\nu(\mathscr{L}_p)=2$.*

Proof. By Table 2, we know that any Latin square of order 5 that has $\nu > 0$ is atomic, so the theorem is true when $p = 3$. Now assume that $p \geqslant 5$. Using Lemma 5.1 it is easy to verify that the following ten triples are entries of $\mathscr{L}_p$:

$$\begin{array}{ll}
((0,1),(0,1),(0,1)), & ((0,2),(0,1),(1,2)),\\
((0,1),(0,2),(1,2)), & ((0,2),(0,2),(0,2)),\\
((0,1),(-1,1),(0,2)), & ((0,2),(-1,1),(-2,1)),\\
((0,1),(1,2),(-2,1)), & ((0,2),(1,2),(2,2)),\\
((0,1),(1,1),(2,2)), & ((0,2),(1,1),(0,1)).
\end{array}$$

These entries form a row cycle of length 5 in $\mathscr{L}_p$ and so $\mathscr{L}_p$ is not atomic. Since $\nu(\mathscr{L}_p) \in \{2, 6\}$ by Lemma 3.1, this proves the lemma. □

We end by discussing the result of applying Kotzig's construction to the perfect 1-factorisations of complete graphs of small order, where we have exhaustive catalogues. Suppose that $L = \mathcal{L}(\mathcal{F}, v)$ where $n \leqslant 15$ and $\mathcal{F}$ is any of the perfect 1-factorisations of K_{n+1}, with v being one of the vertices of K_{n+1}. By Lemma 3.1, we know that $\nu(L) = 2$ unless L is atomic. We can infer from [18] and [9] respectively that $\nu(L) = 2$ if $n \in \{9, 15\}$ since there are no symmetric atomic Latin squares of these orders. If $n = 13$ there are five species of atomic Latin squares that can be derived from perfect 1-factorisations of K_{14}, and these are described in [5]. The situation for $n = 11$ is covered in detail in [11]; there are six species of atomic Latin squares of order 11 that come from perfect 1-factorisations of K_{12}. For odd $n \leqslant 7$ it is known that there is a unique species of atomic Latin squares, which can be obtained from GK_{n+1}, the unique perfect 1-factorisation of K_{n+1} (although, for $n = 7$ it is important to choose the root vertex v to be the unique vertex that is fixed by all automorphisms of GK_8).

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