

Semisimple algebras related to immaculate tableaux

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Abstract

Given a direct sum A of full matrix algebras, if there is a combinatorial interpretation associated with both the dimension of A and the dimensions of the irreducible A -modules, then this can be thought of as providing an analogue of the famous Frobenius–Young identity $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$ derived from the semisimple structure of the symmetric group algebra $\mathbb{C}S_n$, letting f^λ denote the number of standard Young tableaux of partition shape $\lambda \vdash n$. By letting g^α denote the number of standard immaculate tableaux of composition shape $\alpha \vDash n$, we construct an algebra $\mathbb{C}\mathcal{I}_n$ with a semisimple structure such that $\dim \mathbb{C}\mathcal{I}_n = \sum_{\alpha \vDash n} (g^\alpha)^2$ and such that $\mathbb{C}\mathcal{I}_n$ contains an isomorphic copy of $\mathbb{C}S_n$. We bijectively prove a recurrence for $\dim \mathbb{C}\mathcal{I}_n$ so as to construct a basis of $\mathbb{C}\mathcal{I}_n$ indexed by permutation-like objects that we refer to as immacutations. We form a basis \mathcal{B}_n of $\mathbb{C}\mathcal{I}_n$ such that $\mathbb{C}\mathcal{B}_n$ has the structure of a monoid algebra in such a way so that \mathcal{B}_n is closed under the multiplicative operation of $\mathbb{C}\mathcal{I}_n$, yielding a monoid structure on the set of order- n immacutations.

1 Introduction

The representation theory of the symmetric group S_n is of basic importance in many areas of mathematics and may be seen as providing a foundation for a rich interplay among combinatorial and algebraic objects. Young’s construction of matrix units for the irreducible modules of the group algebra $\mathbb{C}S_n$ gives rise to one of the most famous formulas in the entire discipline of combinatorics, namely, the *Frobenius–Young identity* such that

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2 \quad (1)$$

for a positive integer n and for integer partitions λ of n and for the number f^λ of standard Young tableaux of shape λ , referring to Section 2 for preliminaries and for background material. The importance of the identity in (1) within both combinatorics and representation theory leads us to consider extensions and variants of (1)

from both combinatorial and representation-theoretic perspectives. This has led us to formulate and prove an analogue of the Frobenius–Young identity using a generalization of standard Young tableaux arising from the Pieri rule for the *immaculate basis* introduced in a seminal paper by Berg et al. [4].

By the semisimplicity of the group algebra $\mathbb{C}S_n$, and by Frobenius’ Fundamental Theorem of Representation Theory [11, §1] or an equivalent version of Maschke’s Theorem, there exists a basis of $\mathbb{C}S_n$ satisfying the matrix unit multiplication rules (reviewed in Section 2 below). The irreducible $\mathbb{C}S_n$ -modules are indexed by partitions $\lambda \vdash n$, and, for such a partition λ , every irreducible $\mathbb{C}S_n$ -module is of dimension f^λ and multiplicity f^λ . So, by taking all of the $(f^\lambda)^2$ matrix units among the irreducible $\mathbb{C}S_n$ -modules corresponding to a given partition λ of n , and by then taking the matrix units corresponding to the irreducible modules that remain, we obtain a basis of $\mathbb{C}S_n$ of size $\sum_{\lambda \vdash n} (f^\lambda)^2$, or, equivalently, of size $n!$. In an equivalent way, by letting $\mathcal{M}_m(\mathbb{C})$ denote the algebra consisting of $m \times m$ matrices with complex entries, the identity in (1) may be obtained by taking the dimension of each side of the isomorphic equivalence

$$\mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} \mathcal{M}_{f^\lambda}(\mathbb{C}), \tag{2}$$

with the decomposition of $\mathbb{C}S_n$ in (2) being referred to as the *Wedderburn decomposition* of $\mathbb{C}S_n$. The rich history associated with the bijective proof of (1) via the RSK correspondence motivates the development of research topics related to extensions or variants of (1). In this direction, since noncommutative Schur-like bases give rise, via Pieri rules, to composition tableau generalizations of Young tableaux, this raises questions as to what would be appropriate as an analogue of (1) involving composition tableaux and how such an analogue, in turn, could extend the semisimple structure indicated in (2).

Given a family of combinatorial objects related to Schur functions, such as the tableaux arising in product rules associated with the Schur basis, constructing analogues of such combinatorial objects with the use of noncommutative versions of Schur functions is often motivated by applications related to Schur positivity and to representation-theoretic problems. Informally, in view of how the Hopf algebra \mathbf{NSym} projects onto \mathbf{Sym} , Schur-like bases of the former Hopf algebra can be thought of as “containing more information” in a way that could, ideally, be useful in terms of developing a deeper understanding as to the behavior of commutative Schur functions. In this regard, active research related to the *immaculate basis* of \mathbf{NSym} [4] has to do with its close relationship with the Schur basis of \mathbf{Sym} .

Since standard immaculate tableaux provide a natural analogue of standard Young tableaux from both algebraic and combinatorial perspectives, this leads us to introduce an analogue of (1) with the use of standard immaculate tableaux. Again referring to Section 2 for background and preliminaries, by letting α denote an integer composition, and by letting g^α denote the number of standard immaculate tableaux of shape α , a main object of study in this work is the *immaculate algebra*

$$\mathbb{C}\mathcal{I}_n := \bigoplus_{\alpha \vDash n} \mathcal{M}_{g^\alpha}(\mathbb{C}) \tag{3}$$

that we introduce. Since $\mathbb{C}\mathcal{F}_n$ is defined by analogy with the direct sum decomposition in (2), this raises questions as to how the combinatorics associated with immaculate tableaux could give rise to a product rule for $\mathbb{C}\mathcal{F}_n$ that emulates or reflects the operation given by the composition of permutations. Such questions are inspired by past research concerning immaculate tableaux [10, 25], immaculate functions [5, 15, 16], and dual immaculate functions [1, 3, 6, 8, 18, 20, 21, 22, 23].

Experimentally, using the On-Line Encyclopedia of Integer Sequences [24], we have discovered a recursion for the sequence

$$(\dim \mathbb{C}\mathcal{F}_n : n \in \mathbb{N}) = (1, 2, 7, 35, 236, 2037, 21695, 277966, 4198635, 73558135, \dots), \quad (4)$$

and we introduce a bijective proof of this recurrence. Our bijective approach gives rise to a combinatorial interpretation for the dimension of $\mathbb{C}\mathcal{F}_n$ in terms of tuples of integer sets that may be seen as analogues of permutations and that we refer to *immaculations* of order n . This and the definition in (3) give rise to our analogue

$$\# \text{ of immaculations of order } n = \sum_{\alpha \vdash n} (g^\alpha)^2 \quad (5)$$

of the Frobenius–Young identity.

2 Background and preliminaries

Since the Frobenius–Young identity and how it can be generalized in relation to noncommutative symmetric functions are central to our work, we begin with a review of both Young tableaux and the algebra Sym , as below.

2.1 Young tableaux and commutative symmetric functions

Much about our terminology and notation concerning symmetric functions is adapted from Macdonald’s classic monograph on symmetric functions [17]. In this direction, we set $\text{Sym}^{(n)} = \mathbb{k}[x_1, x_2, \dots, x_n]^{S_n}$ for indeterminates x_1, x_2, \dots, x_n and for a field \mathbb{k} that we set as \mathbb{Q} for convenience, and where S_n acts on the polynomial ring $\mathbb{Q}[x_1, x_2, \dots, x_n]$ by permuting the given variables (cf. [17, §1.2]). This leads us to a graded ring decomposition, writing

$$\text{Sym}^{(n)} = \bigoplus_{k \geq 0} \text{Sym}_k^{(n)},$$

letting $\text{Sym}_k^{(n)}$ consist of the zero polynomial and homogeneous symmetric polynomials of degree k . We then construct an inverse limit

$$\text{Sym}_k := \varprojlim_n \text{Sym}_k^{(n)}, \quad (6)$$

and we invite the interested reader to refer to Macdonald’s text for details [17, pp. 17–19]. The inverse limit in (6) leads us to set

$$\mathbf{Sym} := \bigoplus_{k \geq 0} \mathbf{Sym}_k.$$

Define the r^{th} elementary symmetric generator e_r so that $e_0 = 1$ and so that

$$e_r = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

We then define the complete homogeneous generator h_m for $m \in \mathbb{N}_0$ recursively so that $h_0 = 1$ and so that $\sum_{r=0}^n (-1)^r e_r h_{n-r} = 0$. This leads us to equivalently define \mathbf{Sym} so that

$$\mathbf{Sym} = \mathbb{Q}[h_1, h_2, \dots]. \tag{7}$$

By setting $\deg h_n = n$ for positive integers n , we thus have that \mathbf{Sym} is the free commutative \mathbb{Q} -algebra with one generator h_n in each degree n .

An integer partition λ is a finite tuple of positive integers in weakly decreasing order, and we write $\ell(\lambda)$ in place of the length of this tuple, and write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$. We let $|\lambda|$ denote the sum of the entries of a nonempty partition λ and adopt the convention whereby the empty tuple $()$ is such that $|\cdot| = 0$. If $|\lambda| = n$, then we may let this be denoted by writing $\lambda \vdash n$.

For a nonempty integer partition λ , we write $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$, and we write $h_{()} = h_0$. By letting \mathcal{P} denote the set of all integer partitions, we thus obtain that the family $\{h_\lambda\}_{\lambda \in \mathcal{P}}$ is a basis of \mathbf{Sym} , i.e., the complete homogeneous basis of \mathbf{Sym} .

An integer composition α is a finite tuple of positive integers. By analogy with the above notation/conventions for integer partitions, the expression $\ell(\alpha)$ denotes the number of entries of α , with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)})$, and with $|\alpha|$ denoting the sum of the entries of α if α is nonempty, and writing $\alpha \vDash n$ in place of $|\alpha| = n$. We may identify an integer composition (or partition) α with a tableau with $\ell(\alpha)$ rows consisting of α_i cells in its i^{th} row for $i \in \{1, 2, \dots, \ell(\alpha)\}$. We henceforth make use of French notation for denoting tableaux, so that the first row of a tableaux is depicted as being the bottommost row, referring to classic texts by Macdonald [17] and Fulton [9] for details and for further background material related to the tableaux considered below.

A semistandard Young tableau is a tableau of partition shape labeled with positive integers and with strictly increasing columns and with weakly increasing rows. A standard Young tableau is a tableau of partition shape such that the labels of this tableau are distinct and consecutive positive integers starting with 1 and such that the rows and columns are strictly increasing.

Example 2.1. The standard Young tableaux of size 3 are as below.

$$\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}
 \tag{8}$$

The tableaux displayed in (8) illustrate the special case of the Frobenius–Young identity whereby

$$3! = \left(f \begin{array}{|c|} \hline \square \\ \hline \end{array} \right)^2 + \left(f \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right)^2 + \left(f \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \right)^2.$$

Letting m denote the maximal label appearing in a semistandard tableau T , the *weight* or *content* of T is the integer composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ such that the number of labels of T equal to i is α_i for $i \in \{1, 2, \dots, m\}$. For integer partitions λ and μ (of equal order), the *Kostka coefficient* $K_{\mu, \lambda}$ is equal to the number of semistandard Young tableaux of weight λ and shape μ . The Schur basis $\{s_\lambda\}_{\lambda \in \mathcal{P}}$ of \mathbf{Sym} is typically regarded as *the* basis of \mathbf{Sym} and may be defined according to the classical rule

$$h_\lambda = \sum_{\mu} K_{\mu, \lambda} s_\mu. \tag{9}$$

2.2 Semisimple algebras and Young’s construction

Let A be a finite-dimensional \mathbb{k} -algebra for a field \mathbb{k} . Suppose there exists a family consisting of nonzero elements $e_{i,j}^\lambda \in A$, for λ in some set I_1 , where, for $\lambda \in I_1$, the expression $e_{i,j}^\lambda$ is defined for i and j in some index set $I_2(\lambda) = I_2$, and where such expressions (whenever defined) satisfy

$$e_{i_1, i_2}^\lambda e_{i_3, i_4}^\mu = \begin{cases} e_{i_1, i_4}^\lambda & \text{if } i_2 = i_3 \text{ and } \lambda = \mu, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

We refer to the multiplication rules in (10) as the *matrix unit multiplication rules*, since, if we let $E_{i,j}^{(n)}$ for $i, j \in \{1, 2, \dots, n\}$ denote the $n \times n$ matrix with the value of 1 in its (i, j) -entry and with 0-values everywhere else, then we find that $E_{i_1, i_2}^{(n)} E_{i_3, i_4}^{(n)} = E_{i_1, i_4}^{(n)}$ if $i_2 = i_3$ and that $E_{i_1, i_2}^{(n)} E_{i_3, i_4}^{(n)}$ equals the $n \times n$ zero matrix, otherwise. Matrices of the form $E_{i,j}^{(n)}$ are referred to as *matrix units*. The orthogonality relations in (10) for the case whereby the superscripts are unequal may be thought of as being given by the case whereby matrix units are taken from different components in a direct sum of matrix algebras.

Given a basis of A or for an A -module under the action of left- or right-multiplication by elements of A satisfying the matrix unit multiplication rules, we refer to the elements in such a basis as *matrix units*, and this is consistent with Young’s terminology, with regard what is referred to as *Young’s construction* [11, §1]. We refer to any basis of A consisting of elements $e_{i,j}^\lambda$ satisfying (10) as a *matrix unit basis*.

Again with reference to the work of Garsia and Egecioglu [11, §1], if A has a matrix unit basis, then, according to our notation associated with (10), we obtain the direct sum decomposition

$$A \cong \bigoplus_{\lambda \in I_1} \mathcal{M}_{|I_2(\lambda)|}(\mathbb{k}). \tag{11}$$

Algebras that are isomorphic to (finite) direct sums of full matrix algebras are said to be *semisimple*. The equivalence in (11) gives us that

$$\dim A = \sum_{\lambda \in I_1} |I_2(\lambda)|^2, \tag{12}$$

and Young’s construction of matrix unit bases for symmetric group algebras gives us, from the identity in (12), a proof of the Frobenius–Young identity.

Young introduced an explicit construction of matrix units for all symmetric group algebras, and we refer to Garsia and Egecioglu’s exposition on this matrix unit construction [11, §1]. Omitting details, for a tableau T labeled with consecutive integers starting with 1 (i.e., an *injective tableau*) of a given shape $\lambda \vdash n$, we define

$$P(T) = \sum_{\alpha \in R(T)} \alpha \quad \text{and} \quad N(T) = \sum_{\beta \in C(T)} \text{sign}(\beta)\beta$$

for the row and column groups $R(T)$ and $C(T)$ associated with T (consisting of permutations that leave the labels of the rows and columns, respectively, of T invariant up to reorderings). We also let σ_{T_1, T_2} denote the unique permutation such that $T_1 = \sigma_{T_1, T_2} T_2$ for injective tableaux T_1 and T_2 . By then letting $R_1^\lambda <_{YFLO} R_2^\lambda <_{YFLO} \dots <_{YFLO} R_{f^\lambda}^\lambda$ denote the standard Young tableaux of shape λ ordered according to Young’s First Letter Order (whereby one injective tableau precedes another if the first entry of disagreement is lower), and by setting $\gamma_i^\lambda = \frac{f^\lambda}{n!} N(R_i^\lambda) P(R_i^\lambda)$, and by writing $\sigma_{R_i^\lambda, R_j^\lambda} = \sigma_{i,j}^\lambda$, the set of all expressions of the form

$$e_{i,j}^\lambda = \sigma_{i,j}^\lambda \gamma_j^\lambda (1 - \gamma_{j+1}^\lambda) (1 - \gamma_{j+2}^\lambda) \cdots (1 - \gamma_{f^\lambda}^\lambda). \tag{13}$$

is a matrix unit basis for $\mathbb{C}S_n$.

2.3 Immaculate functions and immaculate tableaux

By analogy with (7), we write

$$\mathbf{NSym} := \mathbb{Q}\langle H_1, H_2, \dots \rangle \tag{14}$$

to define the \mathbb{Q} -algebra version of the *algebra of noncommutative symmetric functions*, as introduced in a seminal paper by Gelfand et al. [12]. By setting the degree of the generator H_n as n for positive integers n , the definition in (14) gives us that \mathbf{NSym} is the free \mathbb{Q} -algebra with one generator in each degree.

From the definition in (14), we find that the bases of \mathbf{NSym} are indexed by compositions in a natural way, writing $H_\alpha = H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_{\ell(\alpha)}}$ for a nonempty composition α , and writing $H_{()} = H_0 = 1$. This gives rise to the *complete homogeneous basis* $\{H_\alpha\}_{\alpha \in \mathcal{C}}$ of \mathbf{NSym} , for the set \mathcal{C} of all integer compositions. A main object of study in this paper is given by a family of combinatorial objects (as defined below) arising from an analogue for \mathbf{NSym} of commutative Schur functions.

Definition 2.1. (Berg et al.) For compositions α and β , an *immaculate tableau* of shape α and content β is a tableau of the specified composition shape labelled with positive integers so that the following axioms hold:

1. the number of cells labeled with i is β_i ;
2. each row is weakly increasing (when read from left to right); and
3. the first column is strictly increasing (when read from bottom to top).

Moreover, an immaculate tableau is *standard* if it is of content $1^{|\alpha|}$ [4].

By writing $\mathcal{K}_{\alpha,\beta}$ in place of the number of immaculate tableaux of shape α and content β , the immaculate function \mathfrak{S}_α may be defined via the expansion rule such that

$$H_\beta = \sum_{\alpha \geq \ell \beta} \mathcal{K}_{\alpha,\beta} \mathfrak{S}_\alpha, \tag{15}$$

by analogy with (9), letting the lexicographic ordering on words and compositions be denoted with \leq_ℓ . This allows us to define the *immaculate basis* $\{\mathfrak{S}_\alpha\}_{\alpha \in \mathcal{C}}$ of \mathbf{NSym} [4].

Example 2.2. Inputting

```
from sage.combinat.ncsf_qsym.combinatorics import number_of_fCT
number_of_fCT(Composition([1,1,1,1,1]), Composition([2,1,2]))
```

into SageMath, we find that the number $\mathcal{K}_{(2,1,2),(1,1,1,1,1)}$ of standard immaculate tableaux of shape $(2, 1, 2)$ is 4, and this is illustrated below.

3	5	3	4	4	5	4	5
2		2		2		3	
1	4	1	5	1	3	1	2

Observe that in the second displayed tableau, the labels in the second column are not increasing (from bottom to top).

3 Immaculate algebras

Let the integer sequence $(a(n) : n \in \mathbb{N}_0)$ be defined recursively so that $a(0) = 1$ and so that

$$a(n) = \sum_{k=0}^{n-1} \binom{n-1}{k}^2 a(k), \tag{16}$$

with

$$(a(n) : n \in \mathbb{N}_0) = (1, 1, 2, 7, 35, 236, 2037, 21695, 277966, 4198635, 73558135, \dots), \tag{17}$$

noting the apparent agreement with (4). The sequence in (17) is indexed in the the On-Line Encyclopedia of Integer Sequences as A101514, and, as noted in this OEIS entry, we have previously conjectured that

$$\sum_{\alpha \vDash n} (g^\alpha)^2 = a(n) \tag{18}$$

holds for all $n \in \mathbb{N}$. We prove (18) bijectively, as below.

Remark 3.1. Our strategy is to construct a bijection between the sets \mathcal{A}_n and \mathcal{B}_n defined as follows. We let \mathcal{A}_n consist of 4-tuples of the form (T_1, T_2, R_1, R_2) for standard immaculate tableaux T_1 and T_2 of the same shape $\beta \vDash k$ for some $k \in \{0, 1, \dots, n\}$ (adopting the convention whereby there is a unique “empty tableau” \square with 0 cells), and for subsets R_1 and R_2 of $\{1, 2, \dots, n\}$ of the same size $n - k$. We also let \mathcal{B}_n consist of ordered pairs (T_3, T_4) of standard immaculate tableaux of the same shape $\alpha \vDash n + 1$. We introduce a procedure that gives us a bijection mapping a given domain element $(T_1, T_2, R_1, R_2) \in \mathcal{A}_n$ to a corresponding codomain element $(T_3, T_4) \in \mathcal{B}_n$, in such a way so that T_3 depends entirely on the pair (T_1, R_1) and T_4 depends entirely on the pair (T_2, R_2) according to the same rule that determines T_3 from (T_1, R_1) . For completeness, it is useful, for our purposes, to give our bijection from \mathcal{A}_n to \mathcal{B}_n explicitly, but this can be thought of as being simplified in terms of how (T_i, R_i) determines T_{i+1} for $i \in \{1, 2\}$.

Theorem 3.1. *For all positive integers n , the relation in (18) holds.*

Proof. The base case for $n = 1$ holds in an immediate way, and we assume that $\sum_{\alpha \vDash m} (g^\alpha)^2 = a(m)$ holds for all positive integers m such that $m \leq n$, inductively. So, it remains to prove that this implies that

$$\sum_{\alpha \vDash n+1} (g^\alpha)^2 = a(n + 1).$$

From the recursive definition of the a -sequence in (16), it remains to prove that our inductive hypothesis implies that

$$\sum_{k=0}^n a(k) \binom{n}{n-k}^2 = \sum_{\alpha \vDash n+1} (g^\alpha)^2, \tag{19}$$

noting the use of the symmetry of binomial coefficients. We proceed to define a mapping

$$\varphi_n: \mathcal{A}_n \rightarrow \mathcal{B}_n, \tag{20}$$

for \mathcal{A}_n and \mathcal{B}_n as in Remark 3.1. Let $(T_1, T_2, R_1, R_2) \in \mathcal{A}_n$. For $i \in \{1, 2\}$, let the $n - k$ elements in R_i be ordered by writing

$$R_i = \left\{ s_1^{(i)} < s_2^{(i)} < \dots < s_{n-k}^{(i)} \right\}.$$

We also write $t_j^{(i)} = s_j^{(i)} + 1$ for all $j \in \{1, 2, \dots, n - k\}$, and we write

$$\overline{R}_i = \left\{ t_1^{(i)} < t_2^{(i)} < \dots < t_{n-k}^{(i)} \right\}.$$

For each of $i \in \{1, 2\}$, we form the tableau

$$U_i = \begin{array}{|c|c|c|c|} \hline 1 & t_1^{(i)} & t_2^{(i)} & \cdots t_{n-k}^{(i)} \\ \hline \end{array}. \tag{21}$$

As above, we let T_1 and T_2 be of the same shape $\beta \vDash k$. For each of $i \in \{1, 2\}$, we form a tableau V_i by relabelling T_i by replacing the consecutive labels $1, 2, \dots, k$ with, in order, the entries in the tuple obtained by ordering $\{2, 3, \dots, n + 1\} \setminus \overline{R_i}$. We then, again letting $i \in \{1, 2\}$, form a tableau T_{i+2} by adjoining U_i and V_i , i.e., by placing V_i on top of U_i to form a new tableau with $n + 1$ labeled cells. We then let (20) be defined so that

$$\varphi_n(T_1, T_2, R_1, R_2) = (T_3, T_4). \tag{22}$$

We see that $\varphi_n(T_1, T_2, R_1, R_2)$ is a pair of standard immaculate tableaux of the same shape with $n + 1$ cells, since, by construction, for each of $i \in \{1, 2\}$, the first column of T_{i+2} begins with 1 and then strictly increases according to how the first column of T_i is relabelled with (certain) elements in $\{2, 3, \dots, n + 1\} \setminus \overline{R_i}$ in increasing order, and since there are no restrictions on the orderings of the labels in the remaining columns, and since the first row in (21) is strictly increasing, and since the shifted rows of V_i are strictly increasing, and since the number of cells in T_{i+2} is equal to the number $n - k + 1$ of cells in U_i plus the number k of cells in V_i (or in T_i). We claim that the mapping defined in (22) is bijective.

Let (T_1, T_2, R_1, R_2) and (T'_1, T'_2, R'_1, R'_2) be distinct elements in the domain in (20), and write

$$\varphi_n(T'_1, T'_2, R'_1, R'_2) = (T'_3, T'_4).$$

For each of $i \in \{1, 2\}$, if $R_i \neq R'_i$, then the first row of T_{i+2} will not be the same as the first row of T'_{i+2} , according to (21). Now, suppose that $T_i \neq T'_i$, again letting $i \in \{1, 2\}$. If T_i and T'_i are not of the same shape, then T_{i+2} and T'_{i+2} will not be of the same shape, according to how T_{i+2} is formed by adjoining U_i and V_i , and similarly for T'_{i+2} . If T_i and T'_i are of the same shape but $R_i \neq R'_i$, then T_{i+2} would not have the same initial row as T'_{i+2} , as shown previously. So, it remains to consider the case whereby T_i and T'_i are of the same shape (but with labels arranged in different ways by assumption that $T_i \neq T'_i$) and $R_i = R'_i$. As above, let T_i consist of k cells. We thus have that there is a non-identity permutation σ of $\{1, 2, \dots, k\}$ such that $T_i = \sigma T'_i$. Since $R_i = R'_i$, we find that $U_i = U'_i$, so that V_i is obtained by replacing the consecutive labels $1, 2, \dots, k$ of T_i with the consecutive entries of the tuple \mathcal{T} obtained by ordering $\{2, 3, \dots, n + 1\} \setminus \overline{R_i}$, and V'_i is obtained by replacing the consecutive labels $1, 2, \dots, k$ of T'_i with the consecutive entries of the same tuple \mathcal{T} , noting that the same bijective relabelling function $f: \{1, 2, \dots, k\} \rightarrow \{2, 3, \dots, n + 1\} \setminus \overline{R_i}$ is used in both cases, with $j_1 < j_2$ implying that $f(j_1) < f(j_2)$. This gives us that $V_i = \rho V'_i$ for a non-identity permutation ρ of $\{2, 3, \dots, n + 1\} \setminus \overline{R_i}$ such that $\rho = f \circ \sigma$. Since ρ is not the identity permutation, we have that T_i and T'_i are not the same, giving us the injectivity of φ_n .

Let (T_3, T_4) denote a pair of standard immaculate tableaux of the same shape with $n + 1$ cells. For $i \in \{1, 2\}$, we take the first row \mathcal{U}_i of T_{i+2} , remove the 1-labeled cell, and then take the set $\overline{\mathcal{S}}_i$ of the remaining labels, and then shift each

label downwards by 1, yielding a subset \mathcal{S}_i of $\{1, 2, \dots, n\}$ of size $n - k$ for some $k \in \{0, 1, \dots, n\}$. Again for $i \in \{1, 2\}$, we let \mathcal{V}_i denote the truncated version of T_{i+2} obtained by removing the first row of T_{i+2} . This truncated tableau \mathcal{V}_i is labeled with the elements in $\{2, 3, \dots, n + 1\} \setminus \overline{\mathcal{S}_i}$, and we relabel \mathcal{V}_i by replacing the j^{th} largest label with j , yielding a standard immaculate tableau T_i . By applying the mapping φ_n to (T_1, T_2, R_1, R_2) , our construction gives us that that $\mathcal{S}_i = R_i$ and $\overline{R}_i = R_i$ and $\mathcal{U}_i = U_i$ and $\mathcal{V}_i = V_i$, so that φ_n maps (T_1, T_2, R_1, R_2) to (T_3, T_4) . \square

Example 3.1. To illustrate the bijection associated with the case of (19) whereby

$$a(4) = a(0) \binom{3}{3}^2 + a(1) \binom{3}{2}^2 + a(2) \binom{3}{1}^2 + a(3) \binom{3}{0}^2, \tag{23}$$

we can think of the initial term $a(0) \binom{3}{3}^2 = 1$ as corresponding to the ordered 4-tuple shown in the correspondence whereby

$$(\Xi, \Xi, \{1, 2, 3\}, \{1, 2, 3\}) \mapsto \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right), \tag{24}$$

where the highlighted cells among the tableaux in the tuple shown on the right of (24) are meant to illustrate the definition of (20), in the sense that the members of R_i for $i \in \{1, 2\}$ can be thought of as being ordered and then placed to the right of the lowest left cell in the tableau T_i and then having their labels shifted upwards by 1. The term $a(1) \binom{3}{2}^2 = 9$ in the expansion in (23) is in correspondence with the 9 pairs (T_3, T_4) of order-4 standard immaculate tableaux of the same shape with 2 cells in the bottom row apart from the 1-labeled cell, as illustrated with

$$\left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \{2, 3\}, \{1, 2\} \right) \mapsto \left(\begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right).$$

The full bijection for the case of (19) in (23) is given in the Appendix below.

Remark 3.2. It was pointed out to us by an anonymous reviewer that relations as in (16) and (23) are reminiscent of the binomial sum formula for the size of the symmetric inverse semigroup/monoid, which is also referred to as the rook monoid R_n , and which consists of order- n square binary matrices with at most one nonzero entry in each row and each column, with

$$|R_n| = \sum_{k=0}^n k! \binom{n}{k}^2.$$

This motivates explorations based on how our methods and constructions could be applied in relation to the rook monoid algebra $\mathbb{C}R_n$, which is closely related to the partition algebra $\mathbb{C}A_n(x)$, which is considered in relation to the immaculate algebra $\mathbb{C}\mathcal{I}_n$ in the Conclusion in Section 4. The irreducible representations of $\mathbb{C}R_n$ have been expressed by Munn [19] in terms of the irreducible representations of $\mathbb{C}S_k$ for $k \in \{0, 1, \dots, n\}$, which raises questions as to how our extensions of symmetric group algebras could be further extended, i.e., to extend $\mathbb{C}R_n$. We leave this to a separate project.

To construct a family of combinatorial objects to enumerate the sequence a according to the recursion in (16), we require the preliminaries given as follows. For a finite tuple t , we let $\ell(t)$ denote its length or the number of its entries, and we write $t = (t_1, t_2, \dots, t_{\ell(t)})$. A *weak composition* is a finite tuple of nonnegative integers. Correspondingly, we let a *weak partition* refer to a finite tuple of weakly decreasing nonnegative integers.

Definition 3.1. Let n be a positive integer. Let λ be a strictly decreasing weak partition ending with 0 and with a largest part $\leq n - 1$. An *immaculation* of order n is a tuple t of length $2\ell(\lambda)$ such that t_1 and t_2 are subsets of $\{1, 2, \dots, n - 1\}$ of size $n - 1 - \lambda_1$, and such that t_{2i-1} and t_{2i} are subsets of $\{1, 2, \dots, \lambda_{i-1} - 1\}$ of size $\lambda_{i-1} - 1 - \lambda_i$, for $i \in \{2, 3, \dots, \ell(\lambda)\}$. We refer to λ as the *class* of t .

Example 3.2. Let $n = 4$ and let λ denote the weak partition $(3, 2, 0)$. Then

$$t = \left(\underbrace{\emptyset, \emptyset}_{\lambda_1=3}, \underbrace{\emptyset, \emptyset}_{\lambda_2=2}, \underbrace{\{1\}, \{1\}}_{\lambda_3=0} \right) \tag{25}$$

is an immaculation of order n , where the groupings of entries suggested in (25) are meant to illustrate the following. The initial entries $t_1 = \emptyset$ and $t_2 = \emptyset$ are subsets of $\{1, 2, \dots, n - 1\} = \{1, 2, 3\}$ of size $n - 1 - \lambda_1 = 0$, and, letting $i = 2$, the next two entries $t_{2i-1} = t_3 = \emptyset$ and $t_{2i} = t_4 = \emptyset$ are subsets of $\{1, 2, \dots, \lambda_{i-1} - 1\} = \{1, 2\}$ of size $\lambda_{i-1} - 1 - \lambda_i = 0$, and, letting $i = 3$, the next two entries $t_{2i-1} = t_5 = \{1\}$ and $t_{2i} = t_6 = \{1\}$ are subsets of $\{1, 2, \dots, \lambda_{i-1} - 1\} = \{1\}$ of size $\lambda_{i-1} - 1 - \lambda_i = 1$.

Theorem 3.2. For positive integers n , the number of immaculations of order n is equal to $a(n)$.

Proof. Inductively, we assume that the number of order- m immaculations is equal to $a(m)$ for $m \leq n$, with the base case holding in an immediate way. So, from the defining recurrence for the a -sequence in (16), together with our inductive hypothesis, it remains to prove that

$$\begin{aligned} \# \text{ of order-}(n + 1) \text{ immaculations} = \\ \sum_{k=0}^n \binom{n}{n - k}^2 (\# \text{ of order-}k \text{ immaculations}). \end{aligned}$$

Let t denote an immaculation of order k and class λ . By taking subsets R_1 and R_2 of $\{1, 2, \dots, n\}$, with each of R_1 and R_2 being of size $n - k$ for some fixed $k \in \{0, 1, \dots, n\}$, we then form a tuple u by concatenating the 2-tuple (R_1, R_2) and t . We then let μ denote the tuple obtained by concatenating the 1-tuple (k) and λ . Since the order- k immaculation t is of class λ , the initial entry of λ satisfies $\lambda_1 \leq k - 1$, and any subsequent entries would form a strictly decreasing sequence ending with 0, so we find that μ is a strictly decreasing weak partition ending with 0 and with largest part $\leq n$. So, we find that u is a tuple of length $2\ell(\mu)$ such that $u_1 = R_1$ and $u_2 = R_2$ are subsets of $\{1, 2, \dots, n\}$ of size $n - \mu_1 = n - k$. For $i = 2$, we have that

$u_{2i-1} = t_1$ and $u_{2i} = t_2$, and, since t is an immaculation of class λ and order k , we have that t_1 and t_2 are subsets of $\{1, 2, \dots, k - 1\}$ of size $k - 1 - \lambda_1$, so that u_{2i-1} and u_{2i} (again for $i = 2$) are subsets of $\{1, 2, \dots, \mu_{i-1} - 1\}$ of size $\mu_{i-1} - 1 - \mu_2$. The desired property for higher indices i then holds because t is an immaculation. \square

Example 3.3. The unique order-4 immaculation of class $(3, 2, 1, 0)$ is

$$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset),$$

and the order-4 immaculations of class $(3, 1, 0)$ are

$$\begin{aligned} &(\emptyset, \emptyset, \{1\}, \{1\}, \emptyset, \emptyset), \\ &(\emptyset, \emptyset, \{1\}, \{2\}, \emptyset, \emptyset), \\ &(\emptyset, \emptyset, \{2\}, \{1\}, \emptyset, \emptyset), \text{ and} \\ &(\emptyset, \emptyset, \{2\}, \{2\}, \emptyset, \emptyset), \end{aligned}$$

and we may confirm that there are $a(4) = 35$ order-4 immaculations in total.

Definition 3.2. Let \mathcal{B}_n denote the set consisting of pairs of standard immaculate tableaux of the same shape $\alpha \vDash n$, and let \mathcal{C}_n denote the set consisting of immaculations of order n . Define

$$f: \mathcal{B}_n \rightarrow \mathcal{C}_n \tag{26}$$

so that, for a pair $(T_1^{(1)}, T_2^{(1)})$ of standard immaculate tableaux of the same shape $\alpha \vDash n$, we take the (possibly empty) set $R_1^{(1)}$ (resp. $R_2^{(1)}$) of labels to the right of the 1-cell in the first row of $T_1^{(1)}$ (resp. $T_2^{(1)}$) and we then form a set $\overline{R_1^{(1)}}$ (resp. $\overline{R_2^{(1)}}$) by shifting each member of $R_1^{(1)}$ (resp. $R_2^{(1)}$) downwards by 1, and we record the sets $\overline{R_1^{(1)}}$ and $\overline{R_2^{(1)}}$ as consecutive entries to form an initial tuple $t^{(1)}$. We then recursively define $t^{(i)}$ and $T_1^{(i)}$ (resp. $T_2^{(i)}$) as follows. We form $T_1^{(i)}$ (resp. $T_2^{(i)}$) by taking $T_1^{(i-1)}$ (resp. $T_2^{(i-1)}$) and removing its first row and by then relabelling any remaining labels with consecutive integers starting with 1, according to the increasing order of any labels in $T_1^{(i-1)}$ (resp. $T_2^{(i-1)}$). By taking the (possibly empty) set $R_1^{(i)}$ (resp. $R_2^{(i)}$) of labels to the right of the 1-cell in the first row of $T_1^{(i)}$ (resp. $T_2^{(i)}$), we then form a set $\overline{R_1^{(i)}}$ (resp. $\overline{R_2^{(i)}}$) by shifting each member of $R_1^{(i)}$ (resp. $R_2^{(i)}$) downwards by 1, and we then set $t^{(i)}$ as the concatenation of $t^{(i-1)}$ and the tuple $(\overline{R_1^{(i)}}, \overline{R_2^{(i)}})$. We then let f map $(T_1^{(1)}, T_2^{(1)})$ to the immaculation $t^{(j)}$ obtained after the given procedure terminates (just before empty tableaux would be obtained from successive truncations).

Theorem 3.3. *The mapping in (26) is a bijection from \mathcal{B}_n to \mathcal{C}_n .*

Proof. This can be shown using the bijection in the proof of Theorem 3.1 together with the bijective approach applied to prove Theorem 3.2, leaving the details to the reader. \square

Example 3.4. Starting with the pair

$$\begin{array}{|c|c|} \hline 4 & 5 \\ \hline 3 & \\ \hline 2 & 6 \\ \hline 1 & \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|} \hline 5 & 6 \\ \hline 3 & \\ \hline 2 & 4 \\ \hline 1 & \\ \hline \end{array}
 \tag{27}$$

of standard immaculate tableaux of the same shape $(1, 2, 1, 2) \vDash 6$, we have, in the notation of Definition 3.2, that

$$t^{(1)} = (\emptyset, \emptyset),$$

and the truncation-and-relabelling process in Definition 3.2 applied to the pair in (27) gives rise to the pair

$$\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 2 & \\ \hline 1 & 5 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array},
 \tag{28}$$

which, in turn, yields the tuple

$$t^{(2)} = (\emptyset, \emptyset, \{4\}, \{2\}).$$

An application of the truncation-and-relabelling procedure from Definition 3.2 to (28) yields the pair

$$\begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & \\ \hline \end{array},$$

giving rise to the tuple

$$t^{(3)} = (\emptyset, \emptyset, \{4\}, \{2\}, \emptyset, \emptyset).$$

Mimicking the above steps, we obtain the pair

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}
 \quad
 \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array},$$

and this gives us the tuple

$$t^{(4)} = (\emptyset, \emptyset, \{4\}, \{2\}, \emptyset, \emptyset, \{1\}, \{1\}). \tag{29}$$

Observe that the tuple in (29) is an immaculation of order $n = 6$ and of class $\lambda = (5, 3, 2, 0)$. In this direction, we find that t_1 and t_2 are subsets of $\{1, 2, \dots, n - 1\} = \{1, 2, 3, 4, 5\}$ of size $n - 1 - \lambda_1 = 0$, and, setting $i = 2$, the following entries $t_{2i-1} = \{4\}$ and $t_{2i} = \{2\}$ are subsets of $\{1, 2, \dots, \lambda_{i-1} - 1\} = \{1, 2, 3, 4\}$ of size $\lambda_{i-1} - 1 - \lambda_i = 1$, and so forth.

By Theorem 3.3, by taking the dimension of both sides of (3), we obtain a proof of the Frobenius–Young-type identity in (5). By letting $\{e_{T_1, T_2}^\alpha\}$ denote a matrix basis of $\mathbb{C}\mathcal{F}_n$ indexed by pairs (T_1, T_2) of standard immaculate tableaux of the same shape $\alpha \vDash n$, Theorem 3.3 provides us with an equivalent basis that we denote as $\{e_{f(T_1, T_2)}^\alpha\}$ and that is indexed by order- n immaculations.

3.1 The structure of immaculate algebras

Since we have focused on the combinatorics underlying the evaluation of $\dim \mathbb{C}\mathcal{F}_n$ and the dimensions of the components in the semisimple decomposition of $\mathbb{C}\mathcal{F}_n$, this leads us to turn toward considering the structure of $\mathbb{C}\mathcal{F}_n$, as below.

Theorem 3.4. *For all n , the immaculate algebra $\mathbb{C}\mathcal{F}_n$ contains an isomorphic copy of $\mathbb{C}S_n$.*

Proof. For an arbitrary integer partition $\lambda \vdash n$, we see that $f^\lambda \leq g^\lambda$, since every standard Young tableau of shape λ is a standard immaculate tableau of the same shape. So, since $\mathcal{M}_{f^\lambda}(\mathbb{C})$ is a subalgebra of $\mathcal{M}_{g^\lambda}(\mathbb{C})$, and similarly for direct sums of matrix algebras of the form $\mathcal{M}_{f^\lambda}(\mathbb{C})$, we have that $\bigoplus_{\lambda \vdash n} \mathcal{M}_{f^\lambda}(\mathbb{C})$ is a subalgebra of $\bigoplus_{\alpha \vdash n} \mathcal{M}_{g^\alpha}(\mathbb{C})$, so that the desired result then follows from the consequence of Young’s construction in (2). \square

The connection between immaculations and permutations can be made more explicit in the following sense. If we set $b(0) = 1$ and

$$b(n) = \sum_{k=0}^{n-1} \frac{(n-1)!}{k!} b(k) \tag{30}$$

by direct analogy with (16), then $b(n) = n!$ for all n .

Remark 3.3. An anonymous reviewer suggested how Theorem 3.4 raises questions as to how a given permutation could be expressed in terms of immaculations, i.e., since $\mathbb{C}S_n$ is contained in $\mathbb{C}\mathcal{F}_n$. The difficulty associated with such questions is outlined as follows. The Wedderburn decomposition in (2) can be thought of as a consequence of the non-singularity of the transition matrices between the usual permutation basis of $\mathbb{C}S_n$ and the family of Young’s matrix units defined in (13). By determining an explicit, combinatorial formula for expressing a given permutation as a linear combination of Young’s matrix units, we would then use the embedding of $\mathbb{C}S_n$ into $\mathbb{C}\mathcal{F}_n$, so that each of Young’s matrix units would be in correspondence with a pair of standard immaculate tableaux of the same shape, and we would then use the bijection in Definition 3.2 to associate a pair of the specified form with an immaculation. The main obstacle involved in this process is given by the determination of a cancellation-free, combinatorial formula for expressing a permutation in terms of Young’s matrix units, in view of the coefficients that arise upon the expansion of the right-hand side of (13). In this direction, there has been progress made by Armon and Halverson [2], who introduced and proved a formula for evaluating the entries of the transition matrices between Young’s natural and seminormal representations, but the associated bases of the symmetric group are not the same as Young’s matrix unit basis.

The relation in (30) together with Theorem 3.4 point toward why it would be desirable to construct a basis B of $\mathbb{C}\mathcal{F}_n$ indexed by order- n immaculations in such a way so that the product of two such basis elements is nonzero and a member of the same basis B (or a scalar multiple of an element in B), according to a combinatorial

rule by analogy with the composition of permutations. This leads us to make use of an analogue of the following result that is due to Hewitt and Zuckermann [14] and that is reproduced in Clifford and Preston’s text on the algebraic theory of semigroups [7, p. 167]. Recall that a *semigroup* is a set endowed with associative binary operation, and a *monoid* is a semigroup with an identity element.

Theorem 3.5. (Hewitt & Zuckerman [14]) *For a semisimple algebra A over a field F , if $A \cong FS$ for a finite semigroup S , then one of the simple components of A is of order 1 over F , and the converse holds if F is algebraically closed.*

Observe that $g^\alpha = 1$ in the cases whereby the composition α has a vertical or horizontal shape. So, by the Hewitt–Zuckerman Theorem, the semisimple algebra $\mathbb{C}\mathcal{J}_n$ has a semigroup algebra structure. To construct an analogue of the composition of permutations, we would want to construct a basis of $\mathbb{C}\mathcal{J}_n$ indexed by immaculations in such a way so as to give $\mathbb{C}\mathcal{J}_n$ the structure of a monoid algebra.

Theorem 3.6. *For all positive integers n , the immaculate algebra $\mathbb{C}\mathcal{J}_n$ has the structure of a monoid algebra.*

Proof. The 1-dimensional case for $n = 1$ holds in an immediate way, so we let $n > 2$. In this case, we have that there are at least two distinct simple components that are of $\mathbb{C}\mathcal{J}_n$ and that are of order 1 over \mathbb{C} , and these components correspond to the integer compositions $\alpha \vDash n$ of vertical and horizontal shapes.

From the direct sum decomposition in (3), we have that $\mathbb{C}\mathcal{J}_n$ is spanned by a family of matrix units, and these matrix units are in correspondence with pairs (T_1, T_2) of standard immaculate tableaux of the same shape α . We let a matrix unit of this form corresponding to the specified pair be denoted as e_{T_1, T_2}^α . We then define

$$h_{T_1, T_2}^\alpha = \begin{cases} e_{T_1, T_2}^{(1^n)} & \text{if } \alpha = (1^n), \\ \sum_{\beta \vDash n} e_{T, T}^\beta & \text{if } \alpha = (n), \\ e_{T_1, T_2}^{(1^n)} + e_{T_1, T_2}^\alpha & \text{otherwise.} \end{cases}$$

We find that the family $\{h_{T_1, T_2}^\alpha\}$ is a basis of $\mathbb{C}\mathcal{J}_n$, by considering the transition matrix given by expanding h -expressions into the e -basis, letting compositions be ordered so that $(1^n) < (n)$, yielding a (square) transition matrix with ones along the main diagonal, ones along the first column, certain 1-entries along the second row, and zeroes everywhere else, and this is necessarily of determinant 1. Since

$$h_{\begin{smallmatrix} \boxed{1} & \boxed{2} & \cdots & \boxed{n} \end{smallmatrix}, \begin{smallmatrix} \boxed{1} & \boxed{2} & \cdots & \boxed{n} \end{smallmatrix}}^{(n)} = \sum_{\alpha \vDash n} e_{T, T}^\alpha \tag{31}$$

is the sum of all idempotent matrix units in a matrix unit basis for $\mathbb{C}\mathcal{J}_n$, we write id in place of the basis element on the left-hand side of (31). We also write qid (in reference to a “quasi-identity” element) in place of the h -element

$$h_{\begin{smallmatrix} \boxed{n} & \boxed{n} \\ \vdots & \vdots \\ \boxed{2} & \boxed{2} \\ \boxed{1} & \boxed{1} \end{smallmatrix}, \begin{smallmatrix} \boxed{n} & \boxed{n} \\ \vdots & \vdots \\ \boxed{2} & \boxed{2} \\ \boxed{1} & \boxed{1} \end{smallmatrix}}^{(1^n)} = e_{\begin{smallmatrix} \boxed{n} & \boxed{n} \\ \vdots & \vdots \\ \boxed{2} & \boxed{2} \\ \boxed{1} & \boxed{1} \end{smallmatrix}, \begin{smallmatrix} \boxed{n} & \boxed{n} \\ \vdots & \vdots \\ \boxed{2} & \boxed{2} \\ \boxed{1} & \boxed{1} \end{smallmatrix}}^{(1^n)}.$$

We find that

$$h_{T_1, T_2}^\alpha h_{T_3, T_4}^\beta = h_{T_1, T_2}^\alpha$$

if $\beta = (n)$ and

$$h_{T_1, T_2}^\alpha h_{T_3, T_4}^\beta = h_{T_3, T_4}^\beta$$

if $\alpha = (n)$. If $\alpha \neq (n)$ and $\beta \neq (n)$, then

$$h_{T_1, T_2}^\alpha h_{T_3, T_4}^\beta = \begin{cases} \text{qid} & \text{if } T_2 \neq T_3 \text{ or } \alpha \neq \beta, \\ h_{T_1, T_4}^\alpha & \text{otherwise.} \end{cases} \tag{32}$$

This gives us that $\{h_{T_1, T_2}^\alpha\}$ is closed under the multiplicative operation on $\mathbb{C}\mathcal{F}_n$, giving $\{h_{T_1, T_2}^\alpha\}$ the structure of a monoid under the product operation on $\mathbb{C}\mathcal{F}_n$, with $\mathbb{C}\mathcal{F}_n$ equal to the monoid algebra $\mathbb{C}\{h_{T_1, T_2}^\alpha\}$. \square

Let integer compositions of n be linearly ordered so that $(1^n) \triangleleft (n)$ and so that $(n) \triangleleft \alpha$ for $\alpha \notin \{(1^n), (n)\}$ and so that $\alpha \triangleleft \beta$ for $\alpha, \beta \notin \{(1^n), (n)\}$ if and only if $\alpha <_\ell \beta$ for the lexicographic ordering $<_\ell$ on compositions. We then sort the set of order- n immaculations so that, for two such immaculations t_1 and t_2 , we have that $t_1 < t_2$ if and only if (a) the shape of either tableau in the pair $f^{-1}(t_1)$ is strictly less than (with respect to \triangleleft) the shape of either tableau in the pair $f^{-1}(t_2)$; or (b) these shapes are equal and t_1 is strictly less than t_2 lexicographically, by identifying \emptyset with 0 and by identifying a set of positive integers with the word obtained by sorting its entries. Disregarding superscripts for h -basis elements, we then define the product of t_1 and t_2 as the inverse image under f of the index of $h_{f^{-1}(t_1)} h_{f^{-1}(t_2)}$.

Example 3.5. There are 7 immaculations of order 3, which are ordered according to \triangleleft so that

$$\begin{aligned} t_1 &= (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset), \\ t_2 &= (\{1, 2\}, \{1, 2\}), \\ t_3 &= (\emptyset, \emptyset, \{1\}, \{1\}), \\ t_4 &= (\{1\}, \{1\}, \emptyset, \emptyset), \\ t_5 &= (\{1\}, \{2\}, \emptyset, \emptyset), \\ t_6 &= (\{2\}, \{1\}, \emptyset, \emptyset), \\ t_7 &= (\{2\}, \{2\}, \emptyset, \emptyset). \end{aligned}$$

We thus have that $\{t_1, t_2, \dots, t_7\}$ has the structure of a monoid under the operation of $\mathbb{C}\mathcal{F}_3$ in the manner described above. Explicitly, a composition table for the monoid $\{t_1, t_2, \dots, t_7\}$ is given below.

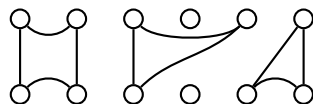
	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	t_1	t_1	t_1	t_1	t_1	t_1	t_1
t_2	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_3	t_1	t_3	t_3	t_1	t_1	t_1	t_1
t_4	t_1	t_4	t_1	t_4	t_5	t_1	t_1
t_5	t_1	t_5	t_1	t_1	t_1	t_4	t_5
t_6	t_1	t_6	t_1	t_6	t_7	t_1	t_1
t_7	t_1	t_7	t_1	t_1	t_1	t_6	t_7

4 Diagrammatics

An anonymous reviewer asked whether the monoid algebra structure on $\mathbb{C}\mathcal{F}_n$ together with the given definition of an immaculation could give rise to analogues of the partition diagrams indexing the bases of diagram algebras, and whether there is a visual way of making sense of the given product rule on immaculations. Although we leave it to a separate project to fully explore how our methods and results could be applied in relation to diagram algebras, we conclude, as below, by considering the use of partition diagram-like constructions to reformulate the given multiplication rule on $\mathbb{C}\mathcal{F}_n$.

The immaculate algebra $\mathbb{C}\mathcal{F}_n$ may be considered in relation to the partition algebra $\mathbb{C}A_n(x)$, adapting notation from Halverson and Ram [13], since, for a suitable parameter x , we have that $\mathbb{C}A_n(x)$ and $\mathbb{C}\mathcal{F}_n$ are both semisimple and both contain $\mathbb{C}S_n$ and both contain two inequivalent 1-dimensional submodules that correspond in natural ways to the 1-dimensional Specht modules associated with horizontal and vertical shapes. This further motivates the exploration of how our constructions and results can be applied in relation to diagram algebras.

Omitting details, *diagram algebras* broadly refer to subalgebras of $\mathbb{C}A_n(x)$, and the bases of $\mathbb{C}A_n(x)$ are indexed by set-partitions of $\{1, 2, \dots, n, 1', 2', \dots, n'\}$, and a set-partition E of this form is denoted as a *partition diagram*, i.e., as a simple graph with vertices labeled with $1, 2, \dots, n$ arranged into a top row and vertices labeled with $1', 2', \dots, n'$ arranged into a bottom row, so that the components are the elements in E . For example, the diagram basis element of $\mathbb{C}A_6(x)$ associated with the set-partition $\{\{1, 2, 1', 2'\}, \{3, 5, 3'\}, \{4\}, \{6, 5', 6'\}, \{4'\}\}$ may be denoted as and identified with the partition diagram

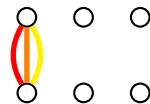


Omitting details, the multiplicative binary operation on $\mathbb{C}A_n(x)$ is defined in a combinatorial way using partition diagrams, by placing one such diagram on top of another, and then removing the central row so as to preserve topmost vertices being connected with bottommost vertices, and then, in certain cases, multiplying by a power of the parameter x . As suggested by an anonymous reviewer, Definition 3.1 together with the monoid algebra structure on $\mathbb{C}\mathcal{F}_n$ recall the combinatorics underlying the structure of diagram algebras, and this leads us to consider denoting immaculations in the following manner.

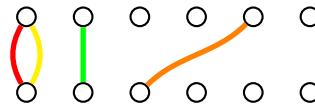
From Definition 3.1, an immaculation t of order n consists of pairs of (possibly empty) subsets of $\{1, 2, \dots, n - 1\}$. By analogy with the above notation for partition diagrams, we let vertices labeled with $0, 1, \dots, n - 1$ be arranged into a top row, and vertices labeled with $0, 1, \dots, n - 1$ be arranged into a bottom row. For a given pairing (t_i, t_{i+1}) of subsets of $\{1, 2, \dots, n - 1\}$ for an odd index i , if t_i and t_{i+1} are both nonempty (noting that these subsets are of the same size), we form a component in the graph under construction with the given vertices, so that the top (resp. bottom) vertices in this component are the elements in t_i (resp. t_{i+1}). If t_i and t_{i+1} are both

empty, we form a component connecting the upper and lower 0-labeled vertices. Moreover, since multiple edges may arise, to distinguish between equal pairings with distinct indices, we color the edges associated with the given components according to the ordering of pairings of the form (t_i, t_{i+1}) for odd indices i , by, say, letting the first such pairing be coloured with red, and this would be followed by orange, yellow, etc.

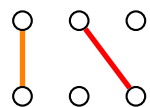
With the notational convention indicated above, the order-3 immaculation $t_1 = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$ given in Example 3.5 may be denoted as



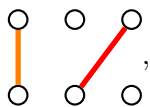
and is the quasi-identity immaculation in $\mathbb{C}\mathcal{I}_3$. Similarly, the order-6 immaculation $(\emptyset, \emptyset, \{4\}, \{2\}, \emptyset, \emptyset, \{1\}, \{1\})$ highlighted in (29) and in Example 3.4 may be denoted as



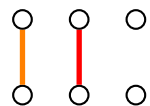
Denoting immaculations using the diagrammatic convention above gives light to both the combinatorics of and structural properties underlying immaculate algebras, especially in relation to both symmetric group algebras and partition algebras, and in terms of the specified product operation on $\mathbb{C}\mathcal{I}_n$ that gives it the monoid algebra structure indicated in Theorem 3.6. For example, returning to Example 3.5, by rewriting the order-3 immaculation $t_5 = (\{1\}, \{2\}, \emptyset, \emptyset)$ as



and by rewriting the order-3 immaculation $t_6 = (\{2\}, \{1\}, \emptyset, \emptyset)$ as



the expression of the product $t_5 t_6 = t_4 = (\{1\}, \{1\}, \emptyset, \emptyset)$, as



may be seen as a direct analogue of the partition diagram product evaluation

$$\left(\begin{array}{ccc} \circ & \circ & \circ \\ | & \diagdown & \\ \circ & \circ & \circ \end{array} \right) \left(\begin{array}{ccc} \circ & \circ & \circ \\ | & \diagup & \\ \circ & \circ & \circ \end{array} \right) = x \begin{array}{ccc} \circ & \circ & \circ \\ | & | & \\ \circ & \circ & \circ \end{array}$$

in the order-3 partition algebra $\mathbb{C}A_3(x)$. A full product rule for immaculations formulated strictly in terms of the diagrammatic notation for immaculations may be obtained using (32) and the inverse of the bijection f in (26), and we leave it to future projects to further study the combinatorics of and the structure of immaculate algebras via the partition diagram-like objects introduced above to denote immaculations.

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Appendix

The full bijection associated with the case of (19) indicated in (23) is given below.

$$\begin{aligned}
 (\square, \square, \{1, 2, 3\}, \{1, 2, 3\}) &\mapsto \left(\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline \end{array} \right) \\
 \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \{1, 2\}, \{1, 2\} \right) &\mapsto \left(\begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right) \\
 \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \{1, 2\}, \{1, 3\} \right) &\mapsto \left(\begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array} \right) \\
 \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \{1, 2\}, \{2, 3\} \right) &\mapsto \left(\begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array} \right) \\
 \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \{1, 3\}, \{1, 2\} \right) &\mapsto \left(\begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right) \\
 \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \{1, 3\}, \{1, 3\} \right) &\mapsto \left(\begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array} \right) \\
 \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \{1, 3\}, \{2, 3\} \right) &\mapsto \left(\begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array} \right) \\
 \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \{2, 3\}, \{1, 2\} \right) &\mapsto \left(\begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 4 & & \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right) \\
 \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \{2, 3\}, \{1, 3\} \right) &\mapsto \left(\begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 1 & 2 & 4 \\ \hline \end{array} \right) \\
 \left(\begin{array}{|c|} \hline 1 \\ \hline \end{array}, \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \{2, 3\}, \{2, 3\} \right) &\mapsto \left(\begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 3 & 4 \\ \hline \end{array} \right) \\
 \left(\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}, \{1\}, \{1\} \right) &\mapsto \left(\begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \right)
 \end{aligned}$$

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