AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **95(1)** (2026), Pages 83–90

Existence of multi-latin squares satisfying a persistent Sudoku condition

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This paper is dedicated to the memory of the late Dean Hoffman.

Abstract

Suppose that n, k, and d are positive integers and k divides n; say, $n = mk$. A *multi-latin square* is an $n \times n$ array on m symbols such that each symbol appears exactly k times in each row and in each column. Such a square is in the subcollection $\mathrm{ML}_d(n, k)$ if in each $d \times d$ block of the array—such a block is the intersection of d consecutive rows of the array with d consecutive columns—none of the m symbols appears more than once. We show that for $d > 1$, $\mathrm{ML}_d(n, k)$ multilatin squares exist if and only if $m \geq d^2 + 1$, and give a construction for an $\mathrm{ML}_d(n, k)$ array when this inequality holds.

1 Definitions and background

A multi-latin square is, for some positive integers m, n, and k, an $n \times n$ array of symbols from an alphabet of m letters such that each letter appears exactly k times in each row and each column. Clearly such an array exists only if $n = mk$. Given such m, n, and k, and a fixed alphabet of m letters, the set of such multilatin squares will be denoted $\mathrm{ML}(n, k)$. Note that an ordinary latin square of order n is an element of $\mathrm{ML}(n, 1)$.

Suppose $d < n$ is a positive integer. A $d \times d$ *block* in an $n \times n$ array is the intersection of d *consecutive rows* with d *consecutive columns*. An element A of $\mathrm{ML}(n, k)$ is in $\mathrm{ML}_d(n, k)$ if, and only if, in each $d \times d$ block B of A, none of the m letters appearing in A appears more than once in B. For an example of an element of $\mathrm{ML}_2(10, 2)$, a 10×10 array on the $10/2 = 5$ letters $0, 1, 2, 3, 4$, with each letter appearing twice in each row and each column and at most once in each 2×2 block, see Figure 2.

ISSN: 2202-3518

The title of this paper mentions a "persistent Sudoku condition". This mention of the Sudoku puzzles may engender in the mind of the reader a misunderstanding that will make it difficult for the reader to follow the arguments. In Sudoku—the usual type, in which the challenge is to complete a 9×9 partial latin square so that *certain* 3×3 blocks have each of the $m = 9$ symbols appears exactly once in each of *certain* (but not all) 3×3 blocks—the block requirement is imposed *only* on the 9 blocks of a tiling of the 9×9 square with 3×3 blocks. In the definition of $\mathrm{ML}_d(n, k)$, the block requirement–that none of the $m = n/k$ symbols appear more than once in the block–is imposed on *all* $(n - d + 1)^2$ $d \times d$ blocks of the $n \times n$ array. In fact, the reader can see in our main result, Theorem 2.4, which, with Theorem 2.2, gives a necessary and sufficient condition for $\mathrm{ML}_d(n, k) \neq \emptyset$, that it is not even required that $d|n$ for the existence of $A \in ML_d(n, k)$. That is, there may not even be a tiling of the $n \times n$ array with $d \times d$ blocks for multi-latin squares in $\mathrm{ML}_d(n, k)$ to exist! One of the referees reports that they used our construction to produce a square in $\mathrm{ML}_3(20, 2)$.

It will be convenient to have the $m = \frac{n}{k}$ symbols in an $\mathrm{ML}(n, k)$ array be $0, \ldots, m - 1$ and to think of these as the elements of $\mathbb{Z}_m$, the ring of congruence classes modulo m in $\mathbb{Z}$, the ring of integers. Similarly, the row and column indices for an $n \times n$ array will be $0, \ldots, n - 1$. It is easy to convert a latin square of order $n = mk$, with symbols $0, \ldots, n - 1$, into a multilatin square in $\mathrm{ML}(n, k)$: simply replace each appearance of $s \in \{0, \ldots, n - 1\}$ with the symbol from $\{0, \ldots, m - 1\}$ which is congruent to s mod m. We have not looked into the question of how far this obvious map of $\mathrm{ML}(n, 1)$ into $\mathrm{ML}(n, k)$ is from being surjective.

As most readers will know, the study of latin squares has been a great unifying phenomenon in combinatorics for almost 300 years. It may have started with the Euler officer problem, which in modern times has become: given n, how large a collection of pairwise "orthogonal" latin squares of order n can there be? The answer is known for many values of n, but not for all; not for $n = 10$, for instance [3].

Over the past 80 years we have the results on completions and embeddings of latin squares ([1], [3], [4]) and of special classes of latin squares (symmetric or idempotent or both: [3]).

And then came the Sudoku puzzles. It did not take mathematicians long to go from 3 to N, and our first co-author, now deceased, went even further [2].

It was Hoffman who came up with the question we answer in this paper, and much of the answer. For those interested in how ideas beget new ideas, we offer the conjecture that if Sudoku had never been, Hoffman would not have thought of $\mathrm{ML}_d(n, k)$.

2 Main results

Throughout, n, m, k, and d are positive integers, and $n = mk$.

Lemma 2.1. *If $ML_d(n, k) \neq \emptyset$ then $m \geq d^2$.*

Proof. In any $A \in \mathrm{ML}_d(n,k)$, in any $d \times d$ block, the d^2 entries are distinct, and are elements of $\{0,\dots,m-1\}$. □

Definitions.

Let $\mathbb{N}$ denote the set of non-negative integers. For $x, y \in \mathbb{N}$, $x \le y$, let $[x,y] = \{z \in \mathbb{N} | x \le z \le y\}$.

For an $n \times n$ array A with rows and columns indexed by the integers $0,\dots,n-1$, and integers $0 \le a_1 \le a_2 \le n-1$ and $0 \le b_1 \le b_2 \le n-1$, let $D(a_1,a_2,b_1,b_2)$ denote the rectangular subarray which is the intersection of the rows indexed by the elements of $[a_1,a_2]$ with the columns indexed by the elements of $[b_1,b_2]$.

For an integer $d \in [1, n-1]$ and integers $a, b \in [0, n-d]$, let $B_d(a,b) = D(a, a+d-1, b, b+d-1)$ which is a $d \times d$ subsquare of A of the very type, if $n = mk$, with which we are concerned in our search for arrays in $\mathrm{ML}_d(n,k)$. That is, if $n = mk$, $A \in ML(n,k)$ is in $\mathrm{ML}_d(n,k)$ if and only if each of the m symbols appearing in A appears no more than once in each subsquare $B_d(a,b)$, $0 \le a, b \le n-d$.

Theorem 2.2. *If $m = d^2 > 1$ then $ML_d(n,k) = \emptyset$.*

Proof. Suppose $d > 1$ and $A \in \mathrm{ML}_d(kd^2, k)$. Since all $m = d^2$ symbols $0,\dots,d^2-1$ appear m times in each d-block, $B_d(a,b) = D(a, a+d-1, b, b+d-1)$, $0 \le a, b \le n-d$, it follows that if $b \le n-d-1$ then the set of entries of the first (left-most) column of $B_d(a,b)$ is the same as the set of entries of the last column of $B_d(a, b+1)$, namely, the complement of the set of entries of $D(a, a+d-1, b+1, b+d-1)$ in $\{0,\dots,d^2-1\}$.

Similarly, if one takes any d consecutive cells in any row r of A, the set of entries in those cells will be the same as the set of entries in row(s) $r \pm d$ of A, in the same columns.

Keeping these facts in mind, let f be the symbol in cell $(d-1, d-1)$ of A. Then f appears in the rightmost (last) column $D(0, d-1, xd-1, xd-1)$ of $B_d(0, (x-1)d)$, $x = 1,\dots,kd$. There are $kd > k$ of these appearances and f can appear only k times in row $d-1$ of A. Therefore there is a smallest integer $x > 1$ such that f appears in cell $(r, xd-1)$ of A for some $r \in \{0,\dots,d-2\}$. That is, f appears in cells $(r, xd-1)$ and $(d-1, (x-1)d-1)$. See Figure 1.

Similarly, for some $1 < y \le kd$ and $c \in \{0,\dots,d-2\}$, f appears in cells $(yd-1, c)$ and $((y-1)d-1, d-1)$.

Consider the d-blocks $S_1 = B_d((y-2))d, (x-2)d)$ and $S_2 = B_d((y-1)d, (x-1)d)$. Since S_1 is the horizontal translate of $B_d((y-2)d, 0)$, in which f appears in cell $((y-1)d-1, d-1)$, by $(x-2)d$ units, and also the vertical translate of $B_d(0, (x-2)d)$ by $(y-2)d$ units, f must appear in the intersection of S_1 with row $(y-1)d-1$ of A and in the intersection of S_1 with column $(x-1)d-1$ of A. Since f can appear at most once in S_1, it follows that f appears in cell $((y-1)d-1, (x-1)d-1)$.

Similarly, the single appearance of f in S_2 must be in cell $(y-1)d+r, (x-1)d+c)$. But then $(y-1)d+r-[(y-1)d-1] = r+1 \le d-1$ and $(x-1)d+c-[(x-1)d-1] =$

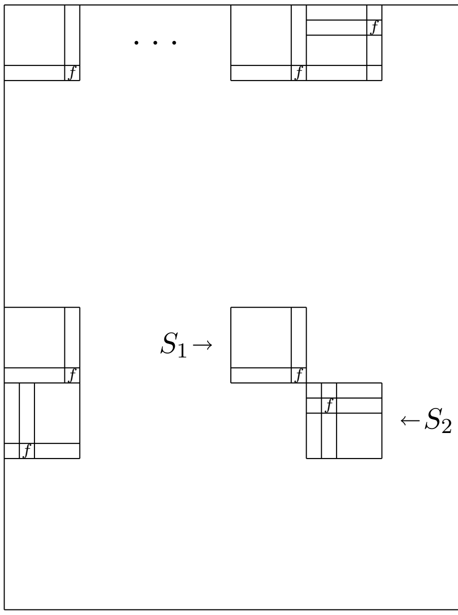

Figure 1. S_1 and S_2 are the two squares sharing one boundary point.

$c+1 \leq d-1$ implies that these two different appearances of f occur in the d-block $B_d((y-1)d-1,(x-1)d-1)$, contradicting the assumption that $A \in \mathrm{ML}_d(kd^2, d)$. $\square$

Corollary 2.3. *If $d > 1$, m and k are positive integers, and $ML_d(mk, k) \neq \emptyset$, then $m \geq d^2 + 1$.*

Our main result is the converse of Corollary 2.3 under the assumption that $d > 1$.

Theorem 2.4. *If $d > 1$, m and k are positive integers, and $m \geq d^2 + 1$, then $ML_d(mk, k) \neq \emptyset$.*

3 Proofs

We will prove Theorem 2.4 by giving a construction of an array in $\mathrm{ML}_d(mk, k)$ when d, m, and k satisfy the hypotheses of Theorem 2.4. Throughout this section we suppose that m, k, and d satisfy these hypotheses. (But see the Remark after Lemma 3.3.) We also have $n = mk$, $g = \gcd(n, d)$, $h = n/g$, $0 \leq p \leq h-1$, and $0 \leq q \leq g-1$.

Lemma 3.1. *With n, g, and h as above, as the pair (p, q) runs over $\{0, \ldots, h-1\} \times \{0, \ldots, g-1\}$, $r = p + qh$ runs over $\{0, \ldots, n-1\}$ exactly once—meaning, the map $(p, q) \rightarrow r = p + qh$ is one-to-one and onto.*

Proof. Since $hg = n = |\{0, \ldots, h-1\} \times \{0, \ldots, g-1\}|$, $(0,0) \rightarrow 0$, $(h-1, g-1) \rightarrow h-1+(g-1)h = n-1$, and $p + qh$ is strictly increasing in both p and q, it suffices to show that if $0 \leq p$, $p' \leq h-1$, $0 \leq q$, $q' \leq g-1$ and $p + qh = p' + q'h$, then $p = p'$ and $q = q'$.

$$\begin{aligned} p + qh = &\quad p' + q'h \\ \Rightarrow |p - p'| = &\quad |q' - q|h. \end{aligned}$$

Then $|p - p'| < h$ implies that $q = q'$ and also $p = p'$. □

Construction 3.2. *For a pair $(r, c) \in \{0, 1, \ldots, n-1\}^2$ find $(p, q) \in \{0, \ldots, h-1\} \times \{0, \ldots, g-1\}$ such that $r = p + qh$. Let $s' = c - (q + pd)$. Let $s = s(r, c) \in \{0, \ldots, m-1\}$ be such that $s' \equiv s$ mod m. Put s in cell (r, c).*

Example

row/column	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	0	1	2	3	4
1	3	4	0	1	2	3	4	0	1	2
2	1	2	3	4	0	1	2	3	4	0
3	4	0	1	2	3	4	0	1	2	3
4	2	3	4	0	1	2	3	4	0	1
5	4	0	1	2	3	4	0	1	2	3
6	2	3	4	0	1	2	3	4	0	1
7	0	1	2	3	4	0	1	2	3	4
8	3	4	0	1	2	3	4	0	1	2
9	1	2	3	4	0	1	2	3	4	0

Figure 2: $\mathrm{ML}_2(10, 2)$ formed by the construction above

Lemma 3.3. *For a pair $(r, c) \in \{0, \ldots, n-1\}^2$, let $s(r, c)$ denote the symbol in cell (r, c) of the $n \times n$ array formed by Construction 3.2.*

(a) Suppose that $r = p + qh$ for some $p \in \{0, \ldots, h-2\}$ and $q \in \{0, \ldots, g-1\}$. Then $s(r+1, c) \equiv s(r, c) - d$ mod m.

(b) Suppose that $r = h - 1 + qh$ for some $q \in \{0, \ldots g-2\}$. Then $s(r+1, c) \equiv s(r, c) - (d+1)$ mod m.

(c) Suppose that $c < n - 1$. Then $s(r, c+1) \equiv s(r, c) + 1$ mod m.

Remark. Lemma 3.3 indicates how our construction was discovered. We start with the first row consisting of k repetitions of | 0 | 1 | $\cdots$ | $m-1$ | and look for cyclic permutations of successive rows that will produce an array that satisfies our other two requirements: column multi-latinity and the persistent Sudoku condition with respect to d. It might be useful to catalog variants in this family of constructions. Our task in this paper was to produce just one construction that could be economically described.

Proof of Lemma 3.3

(a) Because $p < h-1$, $r+1 = p+1+qh$ is the unique representation of $r+1$ promised in Lemma 3.1. Therefore

$$\begin{aligned} s(r+1,c) &\equiv c-(q+(p+1)d) \bmod m \\ &= c-(q+pd)-d \equiv s(r,c)-d \bmod m. \end{aligned}$$

(b) $r+1 = h-1+qh+1 = 0+(q+1)h$. Therefore,

$$\begin{aligned} s(r+1,c) &\equiv c-(q+1+0\cdot d) \bmod m \\ &= c-(q+(h-1)d)-1+(h-1)d \\ &\equiv s(r,c)-(d+1)+hg\tfrac{d}{g} \bmod m \\ &= s(r,c)-(d+1)+n\tfrac{d}{g} \\ &= s(r,c)-(d+1)+mk\tfrac{d}{g} \\ &\equiv s(r,c)-(d+1) \bmod m. \end{aligned}$$

(c) $r = p+qh$ for some $(p,q) \in \{0,\dots,n-1\} \times \{0,\dots g-1\}$; so

$$\begin{aligned} s(r,c+1) &\equiv c+1-(q+pd) \bmod m \\ &\equiv c-(q+pd)+1 \\ &\equiv s(r,c)+1 \bmod m. \end{aligned}$$

□

Lemma 3.4. *Construction 3.2 results in an array in ML(n,k).*

Proof. Let A be the array formed by Construction 3.2. By Lemma 3.3(c), each symbol in $\{0,\dots,m-1\}$ appears exactly k times in each row of A, so it suffices to show that the same holds for each column. Again by Lemma 3.3(c), each column is a translate in $(\mathbb{Z}_m)^n$ of column 0. Therefore, it suffices to show that each symbol in $\{0,\dots,m-1\}$ occurs exactly k times in column 0.

In row $r = p+qh$ of column 0 we have symbol $s(r,0) \equiv 0-(q+pd) = -(q+pd) \bmod m$. Therefore, it suffices to show that as the ordered pair (p,q) varies over the n elements of $\{0,\dots,h-1\} \times \{0,\dots,g-1\}$, $q+pd \bmod m$ varies over the congruence classes $0,\dots,m-1$ of $\mathbb{Z}_m$ exactly k times. Since $n = mk$, we can show this by showing that as (p,q) varies, $q+pd \bmod n$ varies over the congruence classes of $\mathbb{Z}_n$.

Let $d' = \frac{d}{g}$. Since $g = \gcd(n,d)$, n and d' are relatively prime. Therefore, if $a_1,\dots,a_z$ are in distinct congruence classes mod n, then so are $a_1d',\dots,a_zd'$.

As p ranges over $0,\dots,h-1$, pg ranges over $0,g,\dots,hg-g = n-g$, which represent h distinct congruence classes mod n. These congruence classes are precisely the elements of the additive cyclic subgroup H of $\mathbb{Z}_n$ generated by the congruence class of g.

Therefore, as p varies over $0, \dots, h-1$, $pd = (pg)d'$ varies over h distinct congruence classes mod n, and since each of these integers is a multiple of g, each of the congruence classes is in H.

Letting g stand for both an integer and the generator of H in $\mathbb{Z}_n$, we have that the cosets of H in $\mathbb{Z}_n$ are H, $1+H, \dots, (g-1)+H$, and thus the congruence classes of the n integers $q+pd$, $q \in \{0, \dots, g-1\}$ and $p \in \{0, \dots, h-1\}$ are the n distinct congruence classes mod n. □

Note that the proof of Lemma 3.4 in no way uses the assumption that $m > d^2$. So, the construction executed with reference to arbitrary positive integers m, k and d will produce an array in $\mathrm{ML}(mk, k)$.

Proof of Theorem 2.4

It remains only to show that if $n = mk$, m, k, and d satisfy the hypotheses of Theorem 2.4, and A is formed from Construction 3.2, then in A no $B_d(a, b) = D(a, a+d-1, b, b+d-1)$ contains any symbol from $\{0, \dots, m-1\}$ more than once.

Two cells (r, c) and $(r+x, c+y)$ are in the same B_d if and only if $0 \le |x|$, $|y| \le d-1$. Assuming this and $|x|+|y| \ge 1$, so that the cells are different, it suffices to show that $s(r+x, c+y) \ne s(r, c)$ mod m. We may also assume that not both of x and y are non-positive.

Applying Lemma 3.3, we have

$$\begin{aligned} s(r+x,c+y) - s(r,c) &= s(r+x,c+y) - s(r+x,c) \\ &\quad + s(r+x,c) - s(r,c) \\ &\equiv y - (xd \pm \alpha) \bmod m, \end{aligned}$$

where $\alpha \in \{0, 1\}$ is the number of integers z between 0 and x such that $r + z = h - 1 + qd$. Let $\delta(x, y) = y - (xd \pm \alpha)$. To finish the proof, it will suffice to show that $1 \le |\delta(x, y)| \le m-1$. Note that if $x > 0$ then $\delta(x, y) = y - (xd + \alpha)$.

If $x = 0$ then $0 < y \le d-1$ and $\alpha = 0$. Since $m-1 \ge d^2 > d$ (since $d > 1$), $1 \le |y - (0 \cdot d \pm \alpha)| = y \le d-1 < m-1$.

If $0 < x \le d-1$, we have

$$\begin{aligned} -(m-1) \le -d^2 &= -(d-1) - [(d-1)d+1] \\ &\le y - [xd+\alpha] = \delta(x,y) \\ &\le d-1-d = -1, \\ &\Rightarrow 1 \le |\delta(x,y)| \le m-1. \end{aligned}$$

If $-(d-1) \le x < 0$ then $0 < y \le d-1$ and

$$\delta(x,y) = y - [xd - \alpha] \le d-1+(d-1)d+1 = d^2 \le m-1$$

while, also, $\delta(x, y) \geq 1 + d > 0$. Thus, in every case,

$$0 < |\delta(x, y)| \leq m - 1.$$

□

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(Received 1 July 2025; revised 24 Jan, 20 March 2026)