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Axiomatic characterization of the center function on graphs with diameter two

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Abstract

In graph theory, the center function identifies a set of vertices in a connected graph that minimizes the maximum distance from any other vertex. Through an axiomatic characterization of the center function, we identify specific axioms that characterize its behaviour on connected graphs. Universal axioms are those axioms satisfied by the center function on all connected graphs. However, on certain graphs, these universal axioms are insufficient to completely characterize the center function. Non-universal axioms, specific to particular graph classes, were introduced to address this limitation. This work focuses on finding an axiomatic characterization of the center function on graphs with diameter two, employing a combination of both universal and non-universal axioms.

1 Introduction

The concept of a consensus process is fundamental in various fields, encompassing the method by which a group arrives at a common conclusion after considering the opinions of all its members. This process is integral to collective decision-making, ensuring that the outcome reflects the views of the entire group rather than just a select few. The wide-reaching implications of consensus processes are evident in disciplines such as social choice theory, biomathematics, computer science, and artificial intelligence, where the need for collective agreement is paramount.

In social choice theory, consensus processes are crucial for aggregating individual preferences into a collective decision. This field explores how societies can make decisions that fairly represent the desires of all individuals involved. The challenge lies in creating a system that captures the collective will while respecting individual preferences. Similarly, in biomathematics, consensus processes model the behaviour of groups, such as cells or animals, and their collective decision-making processes. This application is vital for understanding how biological systems function as a whole despite the diverse behaviours of individual components.

Computer science, particularly in the realm of distributed computing, relies heavily on consensus algorithms. These algorithms are essential for ensuring that multiple agents, such as nodes in a network, can agree on a common decision, even in the presence of failures or malicious behaviour. This ability to reach consensus is critical for maintaining the reliability and security of distributed systems. In artificial intelligence, consensus processes are used in systems that require collective intelligence, where the goal is to aggregate the knowledge or decisions of multiple agents to arrive at a more accurate or robust solution.

One of the most significant contributions to consensus theory comes from Arrow [1], whose approach fundamentally shaped the field. Arrow's Impossibility Theorem, a cornerstone of social choice theory, posits that no rank-order voting system can perfectly reflect the true preferences of individuals in all cases. Another important perspective comes from Holzman [7], who approached consensus problems from the angle of location theory. Holzman proposed that location problems, which typically involve determining an optimal point based on certain criteria, can be viewed as a type of consensus problem. A deeper understanding of consensus processes can be gained by exploring the axiomatic characterization of location functions, particularly the mean, median, and center functions. Axiomatic characterization involves defining a function through a set of axioms, which in turn helps to clarify the nature and role of these functions in consensus processes. The mean, median, and center functions have been rigorously axiomatized, providing a mathematical framework for understanding how these functions serve as tools for achieving consensus in various contexts.

An axiomatic characterization of the median function in tree networks can be seen in [6, 15]. In 1998, McMorris, Mulder, and Roberts considered axiomatic characterization of median function on cube-free median graphs and median graphs [10]. In 2013, Mulder and Novick extended the McMorris, Mulder, and Roberts result to all median graphs by proving that the median function is the only consensus function satisfying the three universal axioms (A), (B), and (C) mentioned below [12]. In addition, axiomatic characterizations of the median function on various graphs can be seen in [2, 3]. McMorris, Mulder, and Ortega established an axiomatic characterization of mean function on finite trees in the discrete case [9]. McMorris et al. in [11] and Mulder et al. in [13] characterized the center function on trees axiomatically.

Universal axioms pertain to the axioms satisfied by a location function on every connected graph. For the median function, universal axioms encompass *Anonymity* (A), *Betweenness* (B), and *Consistency* (C). In contrast, *Population Invariance* (PI), *Middleness* (M), *Pre-Consistency* (*Pre-C*), *Quasi-Consistency* (QC), and *Gatedness* (G) are recognized as universal axioms for the center function [4]. The center function on all trees with diameter at most 5, graphs with a dominating vertex and paths can be axiomatically characterized using these universal axioms [4]. However, the center function cannot be characterized across all graph classes using the known universal axioms. Non-universal axioms tailored to specific graph classes have been formulated to address this. By combining these non-universal axioms with the universal axioms, the center function on graph classes such as block graphs, cocktail party graphs, and complete bipartite graphs has been axiomatically characterized [5]. This study aims to unveil an axiomatic characterization of the center function on graphs of diameter two.

2 Preliminaries

Throughout this paper, we consider $G = (V(G), E(G))$ to be an undirected, simple, connected, finite graph with vertex set $V(G)$ and edge set $E(G)$. For u, v in V, the least length of the u, v- path is called the *distance* from u to v, and is denoted by $d(u, v)$. The *diameter* of a graph G denoted by $\text{diam}(G)$ is $\max\{d(u, v) : u, v \in V(G)\}$. For vertices u and v of a graph G, u, v-geodesic is the shortest path joining u and v. The *interval* $I_G(u, v)$ between u and v in G consists of all vertices in u, v-geodesics joining u and v, see [3]. A vertex x in G is called a *dominating vertex* if

$d(v,x) \leq 1, \forall v \in V$, see [14].

For any positive integer k, a *profile* of length k is a non-empty sequence $\mu = (v_1, v_2, \dots, v_k)$ of vertices of V. Vertices may be repeated in a profile. The *carrier set* of μ, denoted by $\{\mu\}$, consists of all distinct vertices in the profile μ. The number of elements in $\{\mu\}$ is denoted by $|\{\mu\}|$. Let $\mu_1 = (u_1, u_2, \dots, u_n)$ and $\mu_2 = (v_1, v_2, \dots, v_m)$ be any two profiles of length n and m respectively. The *concatenation of* μ_1 and μ_2 denoted by $\mu_1\mu_2$ is the profile $(u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_m)$ of length $n+m$. A profile $\mu = (r_1, r_2, \dots, r_k)$ is called a *single-occurrence* profile where each $r_i, 1 \leq i \leq k$, is distinct. Further details can be found in [11].

Let V^* be the set of all profiles of finite length. A *consensus function* on a graph G with vertex set V, defined in [13], is a function $C : V^* \to 2^V - \emptyset$, where 2^V denote the set of all subsets of V. $C((v_1, v_2, \dots, v_n))$ is usually written as $C(v_1, v_2, \dots, v_n)$ where $(v_1, v_2, \dots, v_n)$ is any profile. For any profile μ and vertex u of G, *maximum distance* from u to $\mu = (v_1, v_2, \dots, v_n)$ is defined by $R(u, \mu) = \max\{d(u, v_i) \mid 1 \leq i \leq n\}$. The *center function*, denoted by Cen, is a consensus function defined on G, which gives the set of vertices that minimize the maximum distance to profile μ as output for any input μ. For details of the center function, see [8].

Universal axioms are those axioms that the center function satisfies on every connected graph, whereas non-universal axioms are specific to certain graph classes where the center function satisfies them. For a detailed study of universal axioms for Cen, refer Changat et al. [4]. On graphs with dominating vertices, paths and trees T with diameter, $diam(T) \leq 5$ center function can be characterized using universal axioms [4]. Below are the known universal axioms fulfilled by the center function studied in [4].

Population Invariance (PI): Let μ and δ be two profiles with $\{\mu\} = \{\delta\}$, then $C(\mu) = C(\delta)$.

Observe that (PI) requires that the output solely relies on the vertices themselves without considering their multiplicities or positions within the profile.

Let $P = u_0u_1 \dots u_{2\ell}$ be a path of even length 2ℓ. The *middle* of path P consists of the vertex u_ℓ. If $P = u_0u_1 \dots u_{2\ell+1}$ be a path of odd length $2\ell + 1$, *middle* of P consists of two vertices u_ℓ and $u_{\ell+1}$. Let u and v be any two vertices of graph G. Now the *middle* between u and v, denoted by $\text{Mid}(u, v)$, is defined as the middles of u, v-geodesics if $d(u, v)$ is even and middles of u, v- geodesics and middles of u, v-paths of length $d(u, v) + 1$ if $d(u, v)$ is odd.

Middleness (M): For any $u, v \in V$, $C(u, v) = \text{Mid}(u, v)$.
Using middleness, outputs for all profiles of length two can be determined.

Faithfulness (F): $C(v) = \{v\}$, for all $v \in V$.

Pre-Consistency $(Pre\text{-}C)$: Let μ and δ be profiles with $C(\mu) \cap C(\delta) \neq \emptyset$, then $C(\mu) \cap C(\delta) \subseteq C(\mu\delta) \subseteq C(\mu) \cup C(\delta)$.

Quasiconsistency (QC): Let μ and δ be profiles with $C(\mu) = C(\delta)$, then $C(\mu\delta) = C(\mu)$.

Notice that quasiconsistency follows from (*Pre-C*). While the preceding list outlines known universal axioms for center functions, it should not be considered definitive. The possibility of additional yet unidentified universal axioms, remains as an open problem.

The following are the non-universal axioms used in [5] to characterize the center function on *block graphs, cocktail party graphs* and *complete bipartite graphs.* To state these axioms we need the following notation. For any entry x_i belonging to a profile μ, $\mu - x_i$ is the profile μ with x_i deleted. For any subset S of vertices, $\text{Gat}(S)$ is the *gated closure* of the set S. This means that $\text{Gat}(S)$ is the smallest subset of the the vertex set V such that $S \subseteq \text{Gat}(S)$ and, for each $v \in V$, there exists $x_v \in \text{Gat}(S)$ such that $x_v \in I(v, y)$ for all $y \in S$. The vertex x_v is the *gate* of v in S. For the first axiom, $\text{Gat}(\mu - x_i) = \text{Gat}(\{\mu\} \setminus \{x_i\})$ and for the second axiom $\text{Gat}(\mu) = \text{Gat}(\{\mu\})$ where μ is any given profile. A detailed description of the non-universal axioms and characterization of Cen on the above graphs using both universal and non-universal axioms can be seen in [5].

Redundancy (R): Let $\mu = (x_1, x_2, \ldots, x_k)$ be any profile. If $x_i \in \text{Gat}(\mu - x_i)$, then $C(\mu - x_i) = C(\mu)$.

Inconclusiveness (Inc): For a profile $\mu = (x_1, x_2, \ldots, x_k)$, if $\cap_{i=1}^{k} C(\mu - x_i) = \emptyset$, then $C(\mu) = \text{Gat}(\mu)$.

Recursive Consistency (RC): For a single-occurrence profile $\mu = (x_1, x_2, \ldots, x_k)$ with $k \geq 3$, if $\cap_{i=1}^{k} C(\mu - x_i) \neq \emptyset$, then $C(\mu) = \cap_{i=1}^{k} C(\mu - x_i)$.

Fullness $(Full)$: $C(V) = V$.

3 Graphs with Diameter Two

The center function defined on any graph with a dominating vertex is characterized by the universal axioms (PI), (M), and (*Pre-C*), see [4]. Such graphs clearly have diameter two. Here, we consider graphs with or without a dominating vertex and diameter two. The *Jewel graph* is an example of a graph having diameter two and without a dominating vertex (see the figure below, for a Jewel graph J_4 on eight vertices), while the *Fan graph* (fan graph is a graph obtained by adding a dominating vertex to the vertices of a path) is a graph with a dominating vertex and hence having diameter two.

We will say that a profile $\pi = (x_1, x_2, \ldots, x_k) \in V^k$ on G is *dominated* if there exists a vertex $x \in V$ such that $d(x_i, x) \leq 1$ for $i = 1, \ldots, k$. On the other hand, we say that π is *undominated* if π is not a dominated profile. Consider the Jewel graph J_4 as shown Figure 1.

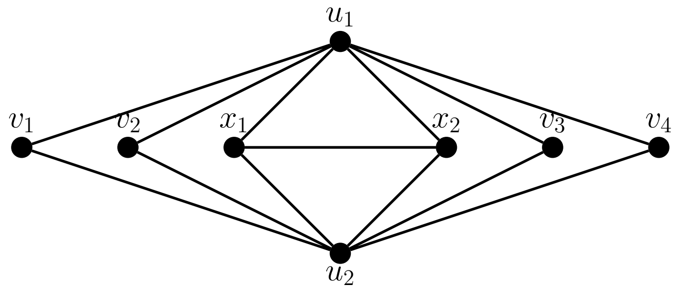

Figure 1: Jewel graph J_4

The profile $(v_1, v_2, x_1, x_2, v_3, v_4)$ is dominated since each profile element is distance one from u_1 and u_2. The profile (u_1, u_2, x_1, v_2) is an undominated profile.
Suppose π is a dominated profile on G such that $|\{\pi\}| \geq 2$. The assumption $|\{\pi\}| \geq 2$ implies that there does not exist a vertex $y \in V$ such that $d(x_i, y) = 0$ for $i = 1, \ldots, k$. Therefore $R(y, \pi) \geq 1$ for all $y \in V$ and so $\text{Cen}(\pi) = \{x \in V : d(x_i, x) \leq 1 \text{ for } i = 1, \ldots, k\}$.

Lemma 3.1. *Let C be a consensus function defined on a finite connected graph G such that C satisfies (PI), (M) and (Pre-C). Then for any dominated profile $\pi = (x_1, x_2, \ldots, x_k)$ with $|\{\pi\}| \geq 2$ on G, $C(\pi) = \text{Cen}(\pi)$.*

Proof. Let C be a consensus function on G satisfying (PI), (M), and (*Pre-C*). Let $\pi = (x_1, x_2, \ldots, x_k) \in V^k$ be a dominated profile on G such that $|\{\pi\}| \geq 2$. By (PI), we need to consider only single-occurrence profiles. If $k = 2$, then $C(\pi) = \text{Cen}(\pi)$ by (M). From now on, we will assume that $k \geq 3$. Since π is a single-occurrence profile, $d(x_i, x_j) \geq 1$ for all $i \neq j$ in $\{1, 2, \ldots, k\}$. Using the fact that π is dominated, there exists a $x \in V$ such that $d(x_i, x) \leq 1$ for $i = 1, 2, \ldots, k$. Therefore, for all $i \neq j$ in $\{1, 2, \ldots, k\}$, $d(x_i, x_j) \leq d(x_i, x) + d(x, x_j) \leq 2$. So $d(x_i, x_j) = 1$ or $d(x_i, x_j) = 2$. In either case, $x \in \text{Mid}(x_i, x_j)$ and so $x \in C(x_i, x_j)$ by (M). Since $\text{Cen}(\pi) = \{x \in V : d(x_i, x) \leq 1 \text{ for } i = 1, 2, \ldots, k\}$, it follows from (PI) and (*Pre-C*) that $\text{Cen}(\pi) \subseteq \bigcap_{j=2}^{k} C(x_1, x_j) \subseteq C(x_1, x_2, \ldots, x_k)$. Thus $\text{Cen}(\pi) \subseteq C(\pi)$.
If $y \in V$ does not belong to $\text{Cen}(\pi)$, then there exists $i \in \{1, 2, \ldots, k\}$ such that $d(x_i, y) \geq 2$. Since C satisfies (PI), we may assume without loss of generality that $i = 1$. Now $d(x_1, y) \geq 2$ along with (M) implies that $y \notin \bigcup_{j=2}^{k} \text{Mid}(x_1, x_j) = \bigcup_{j=2}^{k} C(x_1, x_j)$. Since $\text{Cen}(\pi)$ is a nonempty subset of $\bigcap_{j=2}^{k} C(x_1, x_j)$ it follows from (PI) and *(Pre-C)* that

$$C(x_1, x_2, \ldots, x_k) \subseteq \bigcup_{j=2}^{k} C(x_1, x_j).$$

Therefore, $y \notin C(x_1, x_2, \ldots, x_k)$. Since y was an arbitrary vertex not belonging to $\text{Cen}(\pi)$, it follows that $C(\pi) \subseteq \text{Cen}(\pi)$. Hence, $C(\pi) = \text{Cen}(\pi)$ and we are done. □

We now introduce a new axiom.

Extreme Fullness (EF) A consensus function C on a graph $G = (V, E)$ satisfies extreme fullness if, for any undominated profile π on G, $C(\pi) = V$.

Lemma 3.2. *Let G be a graph of diameter two without any dominating vertex. Then, for any undominated profile π defined on G,* $\mathrm{Cen}(\pi) = V$.

Proof. Since the graph does not contain a dominating vertex and its diameter is two, $R(v, \pi) = 2$, for all $v \in V$. Hence $\mathrm{Cen}(\pi) = V$. □

We are now ready to state and prove our main result.

Theorem 3.3. *Let G be a graph with diameter two. Then the only function on G satisfying $(PI), (M)$, (Pre-C), and (EF) is the center function.*

Proof. Let C be a consensus function satisfying (PI), (M), $(Pre\text{-}C)$, and (EF). If π is a dominated profile, then by Lemma 3.1, $C(\pi) = \mathrm{Cen}(\pi)$. On the other hand if π is an undominated profile, then by Lemma 3.2 and (EF), $C(\pi) = \mathrm{Cen}(\pi)$. Conversely if $C = \mathrm{Cen}$, then C satisfies (PI), (M), $(Pre\text{-}C)$, and (EF). □

The previous theorem is similar to the following two results given in [5].

Theorem 3.4. *Let L be a consensus function on a cocktail party graph G. Then* $L = \mathrm{Cen}$ *if and only if L satisfies $(PI), (M)$, (Pre-C), and $(Full)$.*

Theorem 3.5. *Let L be a consensus function on a complete bipartite graph $K_{m,n}$ with $m, n \geq 2$. Then* $L = \mathrm{Cen}$ *if and only if L satisfies $(PI), (M)$, (Pre-C), and L satisfies (Inc) for profiles π with $|\{\pi\}| \geq 3$.*

Since cocktail party graphs and complete bipartite graphs $K_{m,n}$ with $m, n \geq 2$ are graphs with diameter 2, our main result implies that we can replace $(Full)$ in Theorem 3.4 and (Inc) for profiles π with $|\{\pi\}| \geq 3$ in Theorem 3.5 with the extreme fullness condition (EF). In this way, we can view our main result as a generalization of Theorems 3.4 and 3.5.

4 Independence of Axioms

We will now check the independence of the axioms.

Example 4.1 (Not (M))**.** Let $G = (V, E)$ be a graph of diameter two without any dominating vertex. Define a consensus function C on G as $C(\pi) = V$ for any profile π.

C satisfies (PI), $(Pre\text{-}C)$ and (EF), follows from the definition of C. Let x and y be two vertices of G such that $d(x, y) = 2$. Then for any edge xu in G, $y \notin \mathrm{Mid}(x, u)$. But $C(x, u) = V$. Hence (M) is violated.

Example 4.2 (Not (*Pre-C*))**.** Let $G = (V, E)$ be a graph of diameter two such that G does not contain a dominating vertex. This implies that $|V| \geq 5$. Let xyz be an induced path, and any profile π on G satisfying $\{\pi\} = \{x, y, z\}$ has y as a dominating vertex. Define the consensus function C on G as follows. For any profile π,

$$C(\pi) = \begin{cases} \{x\} & \text{if } \{\pi\} = \{x, y, z\}, \\ \text{Cen}(\pi) & \text{otherwise.} \end{cases}$$

By definition of C and the fact that Cen satisfies (PI), C satisfies (PI). Also, C satisfies (M) and (EF) since Cen satisfies these properties. If $\pi = (x, y)$ and $\rho = (y, z)$, then $y \in C(\pi) \cap C(\rho)$ and $y \notin C(\pi\rho)$ since $\{\pi\rho\} = \{x, y, z\}$ and so $C(\pi\rho) = \{x\}$. Thus, C does not satisfy (*Pre-C*).

Example 4.3. (Not (PI)) Let $G = (V, E)$ be a graph of diameter two and let xy be an edge of G. Define the consensus function C on G as follows. For any profile $\pi = (v_1, v_2, \ldots, v_n)$,

$$C(\pi) = \begin{cases} \{x\} & \text{if } v_1 = v_2 = x \text{ and } \{\pi\} = \{x, y\}, \\ \text{Cen}(\pi) & \text{otherwise.} \end{cases}.$$

Notice that for any profile $\pi, C(\pi) \subseteq \text{Cen}(\pi)$.

Proposition 4.4. *C satisfies $(M), (EF)$ and (Pre-C), but not (PI).*

Proof. Observe that $C(x, x, y) = \{x\}$ and $y \in C(x, y)$ since $C(x, y) = \text{Cen}(x, y)$. Therefore, $C(x, x, y) \neq C(x, y)$ and so C does not satisfy (PI). The consensus function C satisfies (M) and (EF) based on the fact that Cen satisfies these properties and the fact that any profile π satisfying $\{\pi\} = \{x, y\}$ is a dominated profile.

Our next goal is to show that C satisfies (*Pre-C*). Let

$$S = \{\pi = (w_1, w_2, \ldots, w_n) \colon w_1 = w_2 = x \text{ and } \{\pi\} = \{x, y\}\}$$

and suppose $C(\pi) \cap C(\rho) \neq \emptyset$ for some profiles π and ρ. If $\pi\rho \in S$, then either $\{\pi\} = \{x\}$ or $\pi \in S$. In either case, $C(\pi\rho) = C(\pi) = \{x\}$ and we are done. From now on, we may assume $\pi\rho \notin S$. This implies that $C(\pi\rho) = \text{Cen}(\pi\rho)$. Using the fact that Cen satisfies (*Pre-C*) we get $C(\pi) \cap C(\rho) \subseteq \text{Cen}(\pi) \cap \text{Cen}(\rho) \subseteq \text{Cen}(\pi\rho) = C(\pi\rho)$. To finish our proof, we need to show that $C(\pi\rho) \subseteq C(\pi) \cup C(\rho)$. If there exists a vertex v in $C(\pi) \cap C(\rho)$ such that $v \neq x$, then none of $\pi, \rho, \pi\rho$ belong to S and so $C(\pi\rho) \subseteq C(\pi) \cup C(\rho)$ holds since Cen satisfies this containment. So from now on we will assume that $C(\pi) \cap C(\rho) = \{x\}$.

If π is an undominated profile, then $\pi \notin S$ and so $C(\pi) = \text{Cen}(\pi) = V$. Consequently, $C(\pi\rho) \subseteq C(\pi)$. Similarly, if ρ is an undominated profile, then $C(\pi\rho) \subseteq C(\rho)$. Therefore, we may assume π and ρ are dominated profiles. Therefore, $x \in C(\pi) \cap C(\rho) \subseteq \text{Cen}(\pi) \cap \text{Cen}(\rho)$ implies that $R(x, \pi) \leq 1$ and $R(x, \rho) \leq 1$. Since $R(x, \pi\rho) = max\{R(x, \pi), R(x, \rho)\} \leq 1$, it follows that $\pi\rho$ is an undominated profile.

If $R(x, \pi\rho) = 0$, then $R(x, \pi) = R(x, \rho) = 0$ and $C(\pi) = C(\rho) = C(\pi\rho) = \{x\}$. If $R(x, \pi\rho) = 1$, then $R(x, \pi) = 1$ or $R(x, \rho) = 1$. If $\pi \in S$, then $\{\rho\} \neq \{x\}$ and $\rho \notin S$ since $\pi\rho \notin S$. In this case,

$$C(\pi\rho) = \text{Cen}(\pi\rho) = \{v \in V : R(v, \pi\rho) = 1\}$$

and

$$C(\rho) = \text{Cen}(\rho) = \{v \in V : R(v, \rho) = 1\}.$$

Thus, $R(y, \rho) \neq 0$ for all $y \in V$. Therefore, $R(v, \pi\rho) = 1$ implies $R(v, \rho) = 1$ and so $C(\pi\rho) \subseteq C(\rho)$. Our final case is when $\pi \notin S$. It is possible that $\{\pi\} = \{x\}$. Since $\pi\rho \notin S$, $\{\pi\} = \{x\}$ implies that $\rho \notin S$. Since none of $\pi, \rho, \pi\rho$ belong to $S, C(\pi\rho) \subseteq C(\pi) \cup C(\rho)$, holds since Cen satisfies this containment. Now suppose $\pi \notin S$ and $\{\pi\} \neq \{x\}$. Then

$$C(\pi) \subseteq \text{Cen}(\pi) = \{v \in V : R(v, \pi) = 1\}.$$

Since $R(v, \pi\rho) = 1$ implies $R(v, \pi) = 1$ it follows that $C(\pi\rho) \subseteq C(\pi)$ and we are done. □

Example 4.5. (Not (EF)) Let $G = (V, E)$ be a graph of diameter two such that G does not contain a dominating vertex. Let A be a non-empty proper subset of V. Define a consensus function C_A on G as follows. For any profile π,

$$C_A(\pi) = \begin{cases} A & \text{if } \pi \text{ is is an undominated profile.} \\ \text{Cen}(\pi) & \text{otherwise.} \end{cases}$$

Proposition 4.6. *C_A satisfies $(PI), (M)$ and (Pre-C), but not (EF).*

Proof. The consensus function C_A satisfies (PI) and (M) based on the definition of C_A and the fact that Cen satisfies these properties. Since G does not contain a dominating vertex, there exists an undominated profile π. Note that $C_A(\pi) = A$ and $A \neq V$. So C_A does not satisfy (EF).
Our next goal is to show that C_A satisfies (*Pre-C*). Suppose $C_A(\pi) \cap C_A(\rho) \neq \emptyset$ for some profiles π and ρ. If $\pi\rho$ is a dominated profile, then both π and ρ are dominated profiles. In this case, $C_A(\pi) = \text{Cen}(\pi), C_A(\rho) = \text{Cen}(\rho)$, and $C_A(\pi\rho) = \text{Cen}(\pi\rho)$. So C_A satisfies $C_A(\pi) \cap C_A(\rho) \subseteq C_A(\pi\rho) \subseteq C_A(\pi) \cup C_A(\rho)$ since Cen satisfies these containments. If $\pi\rho$ is an undominated profile, then, since $C_A(\pi) \cap C_A(\rho) \neq \emptyset$, at least one of π and ρ is an undominated profile. Since C_A satisfies (PI), we may assume without loss of generality that π is undominated. Therefore, $C_A(\pi\rho) = C_A(\pi) = A$ and we are done. □

5 Conclusion

The main result of this paper, Theorem 3, is a new characterization of the center function on graphs of diameter two. Of the four axioms used in this characterization, extreme fullness (EF) is the only non-universal axiom. In fact, if $G = (V, E)$ is a graph of diameter greater than two, then the center function Cen on G does not satisfy (EF). This leads to the following question. Is it possible to extend Theorem 3 to graphs of diameter greater than two by replacing (EF) with one or more universal and/or non-universal axiom(s)?

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