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On the toughness of regular graphs and prisms

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Abstract

We contribute results on r-regular graphs that do and do not have toughness $r/2$. Doty and Ferland showed the existence of a 5-regular graph with toughness $5/2$ for all even orders except $n = 18$. Using a computer search we show that there does not exist such a graph for $n = 18$. Also, we provide the first family of 4-regular graphs with toughness 2 that contains claws. For the prism of a graph G, we provide several bounds including a sufficient condition for the prism to have the same toughness as G. In particular, we show that if G has toughness $t \leq \frac{1}{2}$ then its prism has toughness $2t$. Further, the prism of any r-regular r-connected inflation has toughness $r/2$ (despite being $(r+1)$-regular) and in general the prism of any 3-regular graph has toughness at most $3/2$.

1 Introduction

In 1973, Chvátal [6] defined the *toughness* $\tau(G)$ of a graph G to be the minimum value of $|S|/k(G-S)$ where $k(G-S)$ denotes the number of components of $G-S$ and the minimum is taken over all cut-sets $S \subseteq V(G)$. We say that a graph is *t-tough* if $\tau(G) \geq t$. For example, it is immediate that the toughness of a graph is at most half its connectivity, and that every hamiltonian graph has toughness at least 1. The relationship between toughness and properties related to hamiltonicity remains an ongoing area of research. In this paper we instead focus on constructions and bounds on the toughness of a given graph. In this regard, a fundamental bound was given by Matthews and Sumner [15] who showed that if a graph is claw-free, where a *claw* is an induced copy of $K_{1,3}$, then its toughness equals half its connectivity. For a 2006 survey on toughness, see [2].

Most of our results are about regular graphs. An r-regular graph is called *supertough* if it has toughness $r/2$. Chvátal [6] noted that for r even the power of the

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cycle $C_n^{r/2}$ (sometimes called a Harary graph) is supertough; in particular, for each even r there exists a supertough r-regular graph for all possible orders. As regards 3-regular graphs, Chvátal provided examples of supertough 3-regular graphs where the order n is a multiple of 6. These examples are called *inflations*, and the inflation of (multi)graph H can be defined as the graph that results by first subdividing each edge once to form the subdivision graph and then taking the line graph of the result. (Note that if H is r-regular then so is its inflation.) Chvátal also showed that when n is not a multiple of 6, supertough 3-regular graphs do not exist. Later, Jackson and Katerinis [14] showed that being claw-free is also necessary for a 3-regular graph to be supertough.

One possible source of supertough graphs is line graphs, but supertough line graphs exist only for some orders. In a series of papers, Doty and Ferland [7, 8, 9, 10] provided (inter alia) several constructions of 5-regular supertough graphs showing collectively that there exists a 5-regular supertough graph for all even n except possibly for $n = 18$. In Section 2 we confirm the nonexistence for $n = 18$. For odd degree $k \geq 7$, it remains the case that almost nothing is known about orders for which supertough k-regular line graphs do not exist.

For even degrees, the powers of cycles noted by Chvátal as well as line graphs are all claw-free. Doty and Ferland [8] provided the first examples of supertough regular graphs with claws. In Section 3 we provide the first examples of 4-regular supertough graphs with claws.

Another possible method to construct supertough graphs is to use graph products. The *prism* G^p of a graph G is defined as its cartesian product with K_2, or equivalently, taking two copies of G and adding a perfect matching between corresponding vertices. Since the prism of an r-connected r-regular graph is automatically $(r+1)$-connected and $(r+1)$-regular, a prism is a natural candidate for being supertough. The relationship between toughness and the hamiltonicity of the prism has been of interest recently; see for example [11, 19]. But we consider here bounds on the toughness of a prism. The first general bounds were provided by Casablanca, Diánez, and García-Vázquez [4]. Earlier, Chao and Han [5] confirmed a conjecture of Piazza et al. [17] that a cycle permutation graph (that is, any graph formed by taking two disjoint cycles of the same length and adding a perfect matching between them) has toughness at most 4/3 except when the cycles are triangles. (See also [13].)

In Section 4 we provide bounds on the toughness of prisms and a characterization of when a graph and its prism have equal toughness. Using these ideas, in Section 5 we show that many regular prisms are not supertough, including that the prism of an $r/2$-tough inflation has toughness $r/2$. In Section 6 we consider graphs with small toughness and show that if G has toughness $t \leq \frac{1}{2}$ then its prism has toughness $2t$.

2 Supertough 5-Regular Graphs

Despite constructions of 5-regular supertough graphs being known for every even $n \geq 6$ except $n = 18$, we show the following:

Theorem 2.1 *There is no* 5*-regular supertough graph of order* 18*.*

The proof of Theorem 2.1 is by exhaustive computer search. We do not know a by-hand proof. We made use of the code `genreg` provided by Meringer (on github), and described in [16]. We generated all 2,807,105,250,897 connected 5-regular graphs of order 18, and verified that none of them has the desired toughness. One fact that shortened the calculation was that a 5-regular supertough graph must have independence number at most 4.

Lemma 2.2 *If G is a* 5*-regular supertough graph of order* 18 *then there is no independent set of size* 5*.*

Proof. Suppose J is an independent set of size 5. There are exactly 25 edges from J to $V - J$. Since $|V - J| = 13$, there exists some vertex $v \in V - J$ that has at most one neighbor in J. It follows that $J \cup \{v\}$ induces a subgraph with at least five components. By considering $S = V - (J \cup \{v\})$, it follows that $\tau(G) \leq (18 - 6)/5 < 5/2$, a contradiction. ☐

Thus the C code of Meringer was adapted so that each generated graph was immediately tested for an independent set of size 5, and if so, it was discarded. There were only 624 5-regular graphs with independence number 4. (It seems likely that this collection could be generated in a more efficient manner.) The toughness of each of these graphs was then calculated using some custom Java code.

Doty and Ferland [9] provided a 5-regular graph of order 18 with toughness 12/5. It follows that this value is the maximum possible toughness of such a graph.

3 2-Tough 4-Regular Graphs with Claws

Several authors have opined on the existence of supertough graphs. In the original paper, after noting that the order of a 3-regular supertough graph is a multiple of 3 (except K_4), Chvátal [6] expressed the opinion that this behavior was likely for odd r and order sufficiently large; that is, for any odd r all supertough r-regular graphs of sufficiently large order must have order divisible by r. This was shown to be false by Doty [7]. Noting that the known supertough graphs were claw-free, we wrote about the question for r-regular graphs for larger r: in [12] we conjectured that only claw-free graphs could be supertough, while in [13] we expressed the opposite belief that almost all r-regular graphs are supertough.

It turns out that we were wrong in both cases. Doty and Ferland [8] gave the first example of a supertough r-regular graph with claws, and in [9] they provided an infinite family for $r = 5$. On the other hand, noncomplete supertough r-regular graphs have independence number at most $2n/(r+2)$. (If α denotes the independence number, then taking a vertex cover for S shows that $\tau(G) \leq (n-\alpha)/\alpha$, or equivalently $\alpha \leq n/(\tau(G) + 1)$.) Bollobás [3] showed that the independence number of a random r-regular graph is at least of the order of $n \log r/r$, and hence most regular graphs are not supertough.

In this section we provide a construction of 4-regular supertough graphs that have claws. We start with the comment that computer search reveals there are two supertough 4-regular graphs of order 10 with claws. One of them is drawn in Figure 1. (Note that this is an example of what Alspach and Parsons [1] called a metacirculant.)

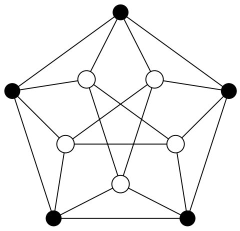

Figure 1: A supertough 4-regular graph

We produce an infinite family as follows. For $m \geq 3$, define a graph J_m on $3m-1$ vertices as follows. Take two disjoint copies of the m-cycle, say with vertices $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$. Then add vertices $C = \{c_1, \ldots, c_{m-1}\}$ and join each c_i to each of $a_i, a_{i+1}, b_i, b_{i+1}$. Finally, add edges a_1b_1 and a_mb_m. The result is 4-regular. The graph J_5 is shown in Figure 2. Note that for $m \geq 4$ the vertices a_1, a_m, b_1, b_m are centers of claws.

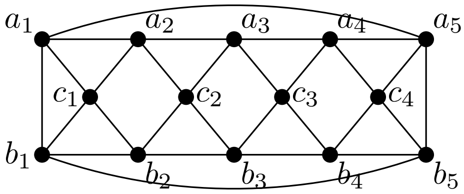

Figure 2: The graph J_5

Lemma 3.1 *(a) The graph J_m has connectivity 4.*
(b) For $m \geq 5$, a cut-set of size 4 is either the neighborhood $N(x)$ for some $x \in A \cup B$ or is of the form $\{a_j, a_{j'}, b_j, b_{j'}\}$ for some $1 \leq j < j' \leq m$ (but not $(j, j') = (1, m)$).
(c) For m odd, the graph J_m has independence number $m-1$.

PROOF. (a,b) Consider a cut-set S of size at most 4. Assume first that S does not contain two vertices of A. Then the remaining vertices of A in $J_m - S$ lie in one component, say Z, and any remaining vertex of C is part of Z. So there must be a component, say X, completely contained in B. Every vertex of $A \cup C$ adjacent to X must be in S. It follows readily that S must also contain two vertices of B and that X is a singleton. That is, $S = N(x)$ for some $x \in B$.

The situation where S does not contain two vertices of B is similar. So suppose that S contains two vertices of both A and B. Then it contains no vertex of C and it is readily argued that the vertices of $S \cap A$ and $S \cap B$ must align. Thus S is of the stated form.

(c) The value $m-1$ is a lower bound because C is an independent set. Suppose there is an independent set X of size m. The graph $J_m - \{a_1, b_m\}$ has a spanning subgraph consisting of $m-1$ triangles: $\{c_1, b_1, b_2\}$, $\{c_2, a_2, a_3\}$, and so on. Thus Xmust contain at least one of $\{a_1, b_m\}$. By a symmetric argument, X must contain one of $\{b_1, a_m\}$; but this is a contradiction of the independence. □

Proposition 3.2 *For $m \geq 3$ and odd, the graph J_m has toughness* 2.

Proof. The value 2 is trivially an upper bound (the neighborhood of any vertex forms a cut-set). So we need to show that the graph J_m is 2-tough. By Lemma 3.1a the graph J_m is 4-connected. For $m = 3$ the graph is claw-free, and so we are done by the result in [15]. So assume $m \geq 5$. Note that each claw in J_m is centered at a vertex of $X = \{a_1, a_m, b_1, b_m\}$.

Suppose $\tau(J_m) < 2$. That is, there is a cut-set S such that $J_m - S$ has more than $|S|/2$ components. Out of all such cut-sets, choose one such that S is as large as possible. Assume the components of $J_m - S$ are $H_1, \ldots, H_k$. Define S_i as the set of vertices of S that are adjacent to H_i and let $P = \sum_i |S_i|$. That is, P is the number of pairs (s, i) such that $s \in S$, $i \in \{1, \ldots, k\}$ and vertex s is adjacent to component H_i. By connectivity, $|S_i| \geq 4$, and thus $P \geq 4k$. A vertex of S not in X can be adjacent to at most two of the H_i, since it is not the center of a claw. At the same time, a vertex of X can be adjacent to at most three of the H_i, since it is not the center of an induced $K_{1,4}$. Let T denote the set of vertices of $S \cap X$ that have neighbors in three components of $J_m - S$. It follows that

$$4k \leq P \leq 2(|S| - |T|) + 3|T| = 2|S| + |T|.$$

Rearranged, the above inequality says that $|S| \geq 2k - |T|/2$. Since the subset X induces a 4-cycle C_X, and every claw in J_m uses two of the edges of C_X, if vertex $v \in T$ then its two neighbors on C_X are not in T. Thus $|T| \leq 2$. Since we are assuming $|S| < 2k$, it follows that $|T| = 2$ and further that $P = 4k$. In particular, $|S_i| = 4$ for all i.

Suppose one of the components H_i is nontrivial. By Lemma 3.1b there are two possibilities for S_i. If S_i is of the form $N(x)$ for some $x \in A \cup B$, then since $J_m - (\{x\} \cup N(x))$ is connected, the component H_i must be $J_m - (\{x\} \cup N(x))$. Thus in this case there are exactly two components, namely H_i and $\{x\}$, which contradicts our assumption that $|S| < 2k$. So, by Lemma 3.1b we may assume that H_i is a component that results when the set $\{a_j, a_{j'}, b_j, b_{j'}\}$ is removed. Such a subgraph has a vertex c_ℓ of degree 2. Consider the set S' obtained by adding to S the two neighbors of c_ℓ in H_i. Then $k(J_m - S') \geq k(J_m - S) + 1 > |S|/2 + 1 = |S'|/2$, a contradiction of the maximality of S. That is, every component H_i is an isolated vertex. In other words, S is a vertex cover of J_m. But by Lemma 3.1c, the independence number of J_m is only $m-1$, and hence its vertex cover number is $2m$. Thus $|S|/k \geq 2m/(m-1) > 2$, a contradiction. □

The above construction was found by starting with a line graph and performing a local adjustment; maybe this idea works in general. It is unclear what happens if one insists that *every* vertex is in a claw.

4 General Lower Bounds for Prisms

In this section we establish some bounds on the toughness of prisms and consider their sharpness. For the prism G^p we refer to the two copies of G as fibers and denote them by G_1 and G_2.

It is tempting to conjecture that the toughness of a prism is at least the toughness of the original graph, but that is not true. The key point is that one can use as subset of a fiber a set that is not a cut-set. The first result in this regard was presented by Casablanca, Diánez, and García-Vázquez [4] without proof. We add the proof.

Theorem 4.1 *[4] Let G be a connected graph of order n and independence number $\alpha(G)$. Then*

$$\tau(G^p) \geq \min\left\{\tau(G), \frac{n}{\alpha(G)+1}\right\}.$$

Proof. Let S be a cut-set of the prism where S_i is the subset of S contained in fiber G_i. Let k_i be the number of components that have **at least** one vertex in G_i, and note that $k_1, k_2 \leq \alpha(G)$. Let k^p denote $k(G^p - S)$. There are three cases.

(i) Both S_1 and S_2 are cut-sets of G. Then

$$\frac{|S|}{k^p} \geq \frac{|S_1|+|S_2|}{k_1+k_2} \geq \min\left\{\frac{|S_1|}{k_1}, \frac{|S_2|}{k_2}\right\} \geq \tau(G). \tag{1}$$

(ii) Neither S_1 nor S_2 is a cut-set of G. Then $G^p - S$ has two components. Further, $|S| \geq n$, since all edges between the two fibers must be severed. In this case $|S|/k^p \geq n/(\alpha(G)+1)$.

(iii) Assume S_1 say is a cut-set of G but S_2 is not. If there is a component of $G^p - S$ completely contained in G_2, then all edges between the fibers are severed. So $|S| \geq n$ and $k^p = k_1 + 1$ and thus $|S|/k^p \geq n/(\alpha(G)+1)$. If there is no such component, then $k^p = k_1$. Further, $|S_2| > 0$ since there must be a component completely contained in G_1. Thus $|S|/k^p > |S_1|/k_1 \geq \tau(G)$ □

At the same time, we observe that one of the terms in the lower bound of Theorem 4.1 is also an upper bound:

Theorem 4.2 *Let G be a connected graph of order n and independence number $\alpha(G)$. Then*

$$\tau(G^p) \leq \frac{n}{\alpha(G)+1}.$$

In particular, if $\tau(G) \geq n/(\alpha(G)+1)$, then $\tau(G^p) = n/(\alpha(G)+1)$.

Proof. Let A be a maximum independent set of G. Then define in the prism the set S to be the union of A from G_1 and the complement of A from G_2. Thus $|S| = n$. In the subgraph $G^p - S$, each remaining vertex of G_2 is in a component by itself, and there is at least one component containing vertices of G_1. Thus $k(G^p - S) \geq \alpha(G)+1$ and therefore $\tau(G^p) \leq n/\alpha(G) + 1$. □

From Theorem 4.1 one can deduce that the toughness of the prism is at least $\frac{2}{3}\tau(G)$ for noncomplete G. This follows since the toughness of G is less than $n/\alpha(G)$, and thus $\tau(G^p)/\tau(G) \geq \alpha(G)/(\alpha(G)+1)$. The ratio $\frac{2}{3}$ is asymptotically sharp. For example, consider the graph $K_n - e$. Then its toughness is $(n-2)/2$, while its prism has toughness $n/3$.

One can also ask when does graph G and its prism have the same toughness. It is immediate that if G is disconnected then both have toughness 0. And by Theorem 4.2, equality holds if $\tau(G) \geq n/(\alpha+1)$. We say a graph G has *complementary tough sets* if there exist tough sets whose union is $V(G)$. For example, a bipartite graph with toughness 1 has complementary tough sets, namely the two partite sets.

Theorem 4.3 *For connected graph G with $\tau(G) < n/(\alpha+1)$, it holds that $\tau(G^p) = \tau(G)$ if and only if G has complementary tough sets.*

PROOF. Assume G has complementary tough sets, say X_1 and X_2. We form a cut-set S of G^p by taking the vertices of X_1 in G_1—call the set S_1—and the vertices of X_2 in G_2—call the set S_2. Since $X_1 \cup X_2 = V(G)$, each component of $G^p - S$ is completely contained within one of the fibers and indeed $G^p - S$ has $k(G_1 - S_1) + k(G_2 - S_2) = k(G - X_1) + k(G - X_2)$ components. It follows that $\tau(G^p) \leq (|X_1|+|X_2|)/(k(G-X_1)+k(G-X_2)) = \tau(G)$. (Note that if a, b, c, d are real numbers such that $a/c = b/d = x$, then $(a+b)/(c+d) = x$.) By Theorem 4.1 we have equality.

Assume $\tau(G^p) = \tau(G)$. Let S be a tough set of G^p. Since $\tau(G^p) < n/(\alpha(G)+1)$, by the proof of Theorem 4.1 it must be that the restrictions S_1 and S_2 of S are both cut-sets in G_1 and G_2. Thus from the inequality chain (1), it is necessary that $|S_1|/k(G_1 - S_1) = |S_2|/k(G_2 - S_2) = \tau(G)$. In particular, the projections X_1 and X_2 of S_1 and S_2 onto G are both tough sets. Further, we need $k(G^p - S) = k(G_1 - S_1) + k(G_2 - S_2)$. Thus each component of $k(G^p - S)$ is completely contained within a fiber, or in other words, every edge of G^p joining the fibers has an end in S. It follows that $X_1 \cup X_2 = V(G)$. □

As an application of this result, we note for example that it applies to the graph J_m described earlier:

Proposition 4.4 *For odd m the graph J_m has complementary tough sets and so $\tau(J_m^p) = 2$.*

PROOF. Since Proposition 3.2 showed that $\tau(J_m) = 2$, we need two sets X_1 and X_2 whose union is the entire vertex set and $|X_1|/k(J_m - X_1) = |X_2|/k(J_m - X_2) = 2$. Recall that the vertices of one cycle of J_m are $A = \{a_1, \ldots, a_m\}$, the other $B = \{b_1, \ldots, b_m\}$, and the remaining vertices $C = \{c_1, \ldots, c_{m-1}\}$ such that vertex c_i is adjacent to vertices a_i, a_{i+1}, b_i, and b_{i+1}. Define

$$X_1 = A \cup B - \{a_1, b_m\}, \qquad X_2 = \{a_1, a_3, \ldots, a_{m-2}\} \cup \{b_3, b_5, \ldots, b_m\} \cup C.$$

It is immediate that $|X_1| = |X_2| = 2(m-1)$ and their union is the whole vertex set. Each vertex of C is in a separate component of $J_m - X_1$ for a total of $m-1$ components. In the subgraph $J_m - X_2$, each odd cycle is broken into $(m-1)/2$ pieces for a total of $m-1$ components. □

5 Prisms of Regular Graphs

As noted in the introduction, the inflation of a (multi)graph H can be defined as the line graph of the subdivision graph $S(H)$ obtained by subdividing each edge exactly once, and Chvátal used the inflation of a graph to produce graphs that are supertough. So one might hope that the prism of an inflation is also supertough. However, despite the fact that an inflation has small tough sets (for example the neighborhood of any vertex), the following holds:

Theorem 5.1 *For $r \geq 2$ an r-connected r-regular inflation G has complementary tough sets.*

Proof. Assume G is the line graph of $S(H)$ where (multi)graph H has order m. Note that G has order $n = rm$. Since G is 2-connected, it follows that the graph $S(H)$ and hence the (multi)graph H is 2-edge-connected. By Robbins' Theorem [18] one can orient the edges of H such that the result is strongly connected. In particular, every vertex is the head and the tail of at least one arc.

Now for each oriented edge e in H take the edge in $S(H)$ corresponding to the head, and let X denote the resultant set of edges. Note that every vertex of H is in a different component of $S(H) - X$ and each of these components is nontrivial. It follows that $k(G-X) = m$ while $|X| = n/2$. Thence $|X|/k(G-X) = (n/2)/(n/r) = r/2$ and X is a tough set for G. Similarly, the complement of X formed from taking each tail is a tough set. Thus G has complementary tough sets. □

Corollary 5.2 *If G is a noncomplete r-connected r-regular inflation, then $\tau(G^p) = r/2$.*

Proof. Say G has order n. Since G is claw-free, $\tau(G) = r/2$. Since G is r-colorable and the vertices can be partitioned into disjoint copies of K_r, it follows that $\alpha(G) = n/r$ and so $n/(\alpha(G)+1) = r/(1 + r/n)$. Hence $\tau(G) < n/(\alpha(G)+1)$. Thus the corollary follows from Theorems 4.3 and 5.1. □

We observe next that the upper bound of Corollary 5.2 generalizes to all cubic graphs:

Theorem 5.3 *If graph $G \neq K_4$ has maximum degree at most 3, then its prism has toughness at most $3/2$.*

Proof. Consider a proper 3-coloring (A_1, A_2, A_3) of G. Let X_1 be the subset of A_3 whose vertices have at most one neighbor in A_1. Then define set S of the prism as: all vertices of $A_1 \cup X_1$ from the first fiber, and all vertices of $A_2 \cup (A_3 - X_1)$ from the second fiber. Note that $|S| = n$ where n is the order of G.

Consider the subgraph $G^p - S$. Note that each vertex of A_2 in the first fiber G_1 is in a separate component, since each vertex of $A_3 - X_1$ has at most one neighbor in A_2. Similarly with each vertex of A_1 in the second fiber G_2. Hence $k(G^p - S) \geq |A_1| + |A_2|$. We can choose this quantity to be at least $\frac{2}{3}n$. It follows that $\tau(G^p) \leq n/(2n/3) = 3/2$. □

For maximum degree 4 the proof of Theorem 5.3 does not immediately generalize. If a vertex has at most one neighbor in each of two of the colors, then it is good; but it is unclear how to handle vertices that have two neighbors of each of two colors. It is unclear whether the toughness of prism of 4-regular noncomplete graph can be more than 2. For some natural candidates, such as the supertough 4-regular graphs J_m described earlier, their prism has toughness 2 (Proposition 4.4).

However, Theorem 5.3 does not generalize to 5-regular graphs. For example, there are two supertough 5-regular graphs of order 12, namely the icosahedron and the line graph of $K_{3,4}$, and it can be shown by computer that both of these have a prism whose toughness is 8/3. But it is an open question whether the prism of a 5-regular graph can be supertough.

6 Upper Bounds and Prisms of Graphs with Small Toughness

Casablanca, Diánez, and García-Vázquez in [4] provided an upper bound on the toughness of a prism. We add the proof.

Theorem 6.1 *[4] Let G be a connected graph of order n and minimum degree $\delta(G)$. Then*

$$\tau(G^p) \leq \min\left\{2\tau(G), \frac{\delta(G)+1}{2}\right\}.$$

Proof. To show that $2\tau(G)$ is an upper bound, take the tough set for G and use it in both fibers. To show that $(\delta(G)+1)/2$ is an upper bound, take S to be all neighbors of one vertex of minimum degree. □

There are graphs where $\tau(G^p) = 2\tau(G)$. Consider for example a path: this has toughness $\frac{1}{2}$, but its prism is hamiltonian and so has toughness 1. Further, a consequence of the main result of Ellingham et al. [11] is that the prism of every P_4-free graph with toughness $\frac{1}{2}$ has toughness 1. In fact, this doubling is true of all graphs with toughness at most $\frac{1}{2}$, as we will show.

For a vertex v in G^p, we denote its neighbor in the other G-fiber by v'. Given a tough set S of G^p, we say vertex $v \in S$ is *mirrored* if $v' \in S$; otherwise v is *split*. We say S is mirrored if every vertex in S is mirrored.

Theorem 6.2 *If $\tau(G^p) < 1$, then every tough set of G^p is mirrored.*

Proof. If G is disconnected, then so is G^p and the only tough set is the empty set. So we may assume G is connected. Suppose there exists a tough set S of G^p that is not mirrored.

We construct a set M containing (a) some of the components of $G^p - S$, and (b) some of the split vertices. Start with a component C_0 that contains at least one vertex v where v' is a (split) vertex of S. Add to M the component C_0 and all (split) vertices v' of S where v is in C_0.

If there is a component C not in M that is adjacent to a split vertex x in M, then enlarge M as follows. Add C to M; also add all (split) vertices v' of S where v is in C. Repeat until there does not exist such a component C.

Claim. *The number of split vertices in M is at least the number of components in M.*

Proof. After the initial step there is one component in M and at least one split vertex in M. So the property is true at this stage. Consider a step where a component C is added that is adjacent to split vertex x in M; say vertex w of C is adjacent to x. The vertex x was added to M in the same step as the component containing x'. Thus C does not contain x'. In particular, since vertex w' is a common neighbor of w and x', it follows that w' must be in S. Since every split vertex y in M is added with the component containing y', it follows that w' is not yet in M and is added at this step. That is, at each step we add one component and at least one split vertex. The claim follows. □

If all components of $G^p - S$ are in M, then by the claim, we have $|S| \geq k(G^p - S)$ and so $\tau(G^p) \geq 1$, a contradiction. So assume not all components of $G^p - S$ are in M. From this, it follows that there must be vertices of S that are not in M; let S' be the set of all such vertices. Note that the subgraph H_M of G^p induced by the vertices and components of M is connected. Since H_M is maximal with respect to components of $G^p - S$ adjacent to it, it must be that S' is a cut-set of G^p that leaves H_M as a component. Thus $k(G^p - S') > k(G^p - S) - |S - S'|$. Since $|S| < k(G^p - S)$, it follows that $|S'|/k(G^p - S') < |S|/k(G^p - S)$. This is a contradiction of S being a tough set. □

Theorem 6.3 *If G is a graph with toughness t, then the toughness of G^p is at least* $\min\{2t, 1\}$.

Proof. Assume G^p has toughness less than 1. Then by the above theorem a tough set S is mirrored. Let $\bar{S}$ denote the projection of S onto G. It follows that $|S|/k(G^p - S) = 2|\bar{S}|/k(G - \bar{S}) \geq 2t$. □

The following results follow immediately from Theorem 6.3.

Corollary 6.4 *(a) If* G *is a graph with toughness* $t \leq 1/2$*, then* $\tau(G^p) = 2t$*.*
(b) If G *is bipartite with toughness* t*, then* $\tau(G^p) = \min\{2t, 1\}$*.*

Proof. For any graph, $2t$ is an upper bound by Theorem 6.1. Further, the prism of a bipartite graph is bipartite and so has toughness at most 1. □

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