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Covering three-tori with cubes

Ilya I. Bogdanov

Moscow Institute of Physics and Technology
Laboratory of Combinatorial and Geometric Structures
Moscow, Russia
ilya.i.bogdanov@gmail.com

Oleg Grigoryan

Yandex School of Data Analysis
Moscow, Russia

Maksim Zhukovskii

The University of Sheffield, School of Computer Science
Sheffield, U.K.
m.zhukovskii@sheffield.ac.uk

Abstract

Let $\mu(\varepsilon)$ be the minimum number of cubes of side ε needed to cover the unit three-torus $[\mathbb{R}/\mathbb{Z}]^3$. We prove new lower and upper bounds for $\mu(\varepsilon)$ and find the exact value for all $\varepsilon \geq \frac{7}{15}$ and all $\varepsilon \in \left[\frac{1}{r+1/(r^2+r+1)}, \frac{1}{r-1/(r^2-1)}\right)$ for any integer $r \geq 3$.

1 Introduction

Let d be a positive integer, and let $\varepsilon \in (0,1)$. Consider the *torus* $T^d := [\mathbb{R}/\mathbb{Z}]^d$ and the set $\mathcal{J}_\varepsilon$ of *(ε-)sub-cubes* of the form $\{(x_1, \dots, x_d) : x_i \in [x_i^0, x_i^0 + \varepsilon]\}$. The question is, what is the minimum number $\mu := \mu(d;\varepsilon)$ of sets $A_1, \dots, A_\mu$ from $\mathcal{J}_\varepsilon$ needed to cover T^d (i.e., $T^d = A_1 \cup \ldots \cup A_\mu$)?

One easily notices that

$$(1/\varepsilon)^d \leq \mu(d;\varepsilon) \leq \lceil 1/\varepsilon \rceil^d \quad \text{and} \quad \mu(d_1 + d_2;\varepsilon) \leq \mu(d_1;\varepsilon) \cdot \mu(d_2;\varepsilon). \tag{1}$$

In [11], it is proven that

$$\mu \geq \lceil 1/\varepsilon \rceil^{(d)}, \tag{2}$$

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where $\lceil x\rceil^{(i)} = \lceil x\lceil x\rceil^{(i-1)}\rceil$ and $\lceil x\rceil^{(1)} = \lceil x\rceil$. Moreover, it is shown that, for $d = 2$, this lower bound is sharp, i.e. $\mu(2;\varepsilon) = \lceil \frac{1}{\varepsilon}\lceil\frac{1}{\varepsilon}\rceil\rceil$.

The behaviour of the function μ for a fixed ε and large d has been studied extensively. Recall that $\mu(d;\varepsilon) \geq (1/\varepsilon)^d$. From a general result by Erdős and Rogers [7] it follows that $1/\varepsilon$ is the correct base of the exponent, i.e. $\mu(d;\varepsilon) = (1/\varepsilon + o(1))^d$. More precisely, they proved that $\mu(d;\varepsilon) = O(d\log d(1/\varepsilon)^d)$. This result has several important generalisations, see, e.g. [8] for coverings by a family homothets, for multiple coverings, and other generalisations. In the discrete case (i.e. $[\mathbb{Z}/t\mathbb{Z}]^d$ is covered by sub-cubes with side $s \in [t] := \{1, 2, \dots, t\}$, $\varepsilon = s/t$) a direct application of the probabilistic method allows one to put away the $\log d$ factor (see [4, Corollary 3.2], [12, Theorem 3.7]). In other words, the bound $\mu(d;\varepsilon) = O(d(1/\varepsilon)^d)$ applies for all rational ε. On the other hand, (1) gives $\mu(d;\varepsilon) = \Omega((1/\varepsilon)^d)$ for all ε (both rational and irrational).

Let us also mention that, for $\varepsilon = \frac{2}{r}$, $r \in \mathbb{N}$, the corresponding packing problem [1] (finding $\nu(d;\varepsilon)$, the maximum number of non-overlapping sub-cubes with side ε inside T^d) is related to the problem of finding the Shannon capacity $c(C_r)$ of a simple cycle on r vertices [6, 13]: $c(C_r) = \sup_{d\geq 1}(\nu(d; 2/r))^{1/d}$ (the same connection works for other rational ε but the respective graphs are not so foreseeable). For even r, $c(C_r) = r/2$ is straightforward. It was shown by Lovász [10] that $c(C_5) = \sqrt{5}$. For larger odd r, the question of finding $c(C_r)$ is still open, see also [2, 3, 14, 15]. A broader problem concerning graph packings has been studied extensively; in particular, [5] establishes conditions for the existence of a perfect packing of a torus by isomorphic copies of its subgraph.

In our paper, we mostly deal with $d = 3$, although some results hold for arbitrary d. Since $\mu(1;\varepsilon) = \lceil\frac{1}{\varepsilon}\rceil$ and $\mu(2;\varepsilon) = \lceil\frac{1}{\varepsilon}\lceil\frac{1}{\varepsilon}\rceil\rceil$, we get that

$$\left\lceil \frac{1}{\varepsilon}\left\lceil \frac{1}{\varepsilon}\left\lceil \frac{1}{\varepsilon}\right\rceil\right\rceil\right\rceil \leq \mu(3;\varepsilon) \leq \left\lceil \frac{1}{\varepsilon}\left\lceil \frac{1}{\varepsilon}\right\rceil\right\rceil \cdot \left\lceil \frac{1}{\varepsilon}\right\rceil. \tag{3}$$

In [11], the authors noticed that the lower bound in (3) is not sharp. For example, $\mu(3;\frac{3}{7}) > \lceil\frac{7}{3}\lceil\frac{7}{3}\lceil\frac{7}{3}\rceil\rceil\rceil = 17$ (but the exact value of $\mu(3; 3/7)$ is unknown). Unfortunately, no general bound better than (3) is known.

In this paper, we have found the exact value of $\mu(3;\varepsilon)$ for $\varepsilon \geq 7/15$. We have also found exact values of $\mu(3;\varepsilon)$ for ε close to $1/r$, $r \in \mathbb{N}$. In Section 1.1, we state new results.

1.1 New results

We start with studying, for an arbitrary $d \geq 2$, the relation between the problem of finding $\mu(d;\varepsilon)$ and its discrete version on covering the (discrete) torus $[\mathbb{Z}/t\mathbb{Z}]^d$ by a minimum number $\mu_0(d; s, t)$ of (discrete) sub-cubes with side length s, where t and s are positive integers. See full definitions in Section 2. This relation will be used to derive new results on the value of $\mu(d;\varepsilon)$ for $d = 3$. An abbreviated version of our principal result is as follows.

Theorem 1. *Let $d \geq 2$ and μ be positive integers. Choose a minimum ε_0 such that there is a covering of the torus T^d by μ ε_0-sub-cubes. Then $\varepsilon_0 = s_0/t_0$ where s_0 and t_0 are coprime positive integers, with $t_0 \leq \mu$. Moreover, there exists a covering of $[\mathbb{Z}/t_0\mathbb{Z}]^d$ with μ sub-cubes of side length s_0.*

Remark 1. A minimum ε_0 as in the statement of Theorem 1 exists by a standard compactness argument; the same applies to the statement of Claim 1 that is presented in Section 2.

In Section 2, we will state Claim 1, which is a stronger unabbreviated version of Theorem 1 and show that it implies the following two corollaries, that will be of use in our investigation of the case $d = 3$.

Corollary 1. *Let $d \geq 2$ be an integer, and consider any $\varepsilon \in (0, 1]$; set $r = \lceil 1/\varepsilon \rceil$. Then there exist coprime positive integers s and t with $t \leq r^d$ such that $\mu(d; \varepsilon) = \mu(d; s/t) = \mu_0(d; s, t)$, and $\frac{s}{t} \in \left[\frac{1}{r}, \frac{1}{r-1}\right)$ is the largest discontinuity point of $\mu(d; \cdot)$ not exceeding ε.*
In particular, in order to find $\mu(d; \varepsilon)$, it suffices to determine finitely many values of the form $\mu_0(d; s, t)$.

Corollary 2. *For any rational $\varepsilon = s/t$, where $s \leq t$ are (not necessarily coprime) positive integers, we have $\mu(d; \varepsilon) = \mu_0(d; s, t)$. In particular, $\mu_0(d; s, t)$ depends only on d and s/t.*

We should notice here that Corollary 2 appears already in [11, Theorem 1] with a simpler proof. However, this fact arises naturally in the proof of Theorem 1, so we find it natural to present it here.

In the rest of this section we present new results concerning the value of $\mu(d; \varepsilon)$ for $d = 3$. For an integer $r \geq 2$ and $\varepsilon \in \left[\frac{1}{r}, \frac{1}{r-1/r^2}\right)$, both the lower and the upper bounds in (3) are equal to r^3, so the value of $\mu(3; \varepsilon)$ also equals r^3. We show that the right-neighborhoods of $1/r$ where the function $\mu(3; \cdot)$ is constant can be extended in the following way.

Theorem 2. *Let $r \geq 2$ be an integer. If $\varepsilon \in \left[\frac{1}{r}, \frac{1}{r-1/(r^2-1)}\right)$, then $\mu(3; \varepsilon) = r^3$.*

So, the upper bound in (3) is tight on this larger half-open interval. We do not know whether this interval can be extended to the right.

We also determine the values of $\mu(3; \cdot)$ on certain left neighborhoods of each number of the form $1/r$ as follows.

Theorem 3. *Let r be a positive integer. If $\varepsilon \in \left[\frac{1}{r+1/(r^2+r+1)}, \frac{1}{r}\right)$, then $\mu(3; \varepsilon) = r^3 + r^2 + r + 1$.*

The lower bound in (3) is tight on this half-open interval. On the other hand, (3) yields that $\mu(3; \varepsilon) > r^3 + r^2 + r + 1$ for each $\varepsilon < \frac{1}{r+1/(r^2+r+1)}$, so the interval in this theorem is maximal. Nevertheless, we do not know whether there exists an extension of this interval to the left where the lower bound in (3) is tight. We also notice that the gap between the bounds in (3) is large (of order r^2) on this interval.

Next, we improve the lower bound from (3) in some special cases.

Theorem 4. *Let $r \geq 2$ be an integer, and choose $\xi \in [r]$ satisfying*

$$\xi \geq \xi^2 - (r+1)\left\lfloor \frac{\xi^2}{r+1} \right\rfloor . \tag{4}$$

Let

- $s = r^2 + r + \xi$,
- $t = r^3 + r^2 + 2\xi r + \left\lfloor \frac{\xi^2}{r+1} \right\rfloor$.

Assume that t and s are coprime. Then $\mu(3; s/t) > t$.

One readily checks that, for $\varepsilon = s/t$ with s and t as above, we have $\lceil 1/\varepsilon \rceil^{(2)} = s$ and hence the lower bound in (3) reads $\mu(3; s/t) \geq t$. Thus, Theorem 4 indeed improves upon (3). Condition (4) implies that either $\xi \geq \sqrt{r+1}$ or $\xi = 1$. For instance, in the interval $(1/3, 1/2)$, there are two such values of s/t, namely $s/t \in \{7/16, 8/21\}$.

Finally, we improve the upper bound from (3) in some special cases.

Theorem 5. *Let $r \geq 1$ be an integer, and choose $\xi \in [r]$. Let*

- $s = r^2 + r + \xi$,
- $t = r^3 + r^2 + \xi(r+1)$.

Then $\mu(3; s/t) \leq t$.

For $\varepsilon = s/t$ with s and t as in Theorem 5, the upper bound in (3) reads $\mu(3; s/t) \leq (r^2 + r + 1)(r + 1) = r^3 + r^2 + (r + 1)^2$ which is weaker than the one stated above. Moreover, both Theorem 4 and Theorem 5 imply improvements of the bounds in (3) for values of ε in certain right-neighborhoods of the respective s/t, according to Corollary 1.

We notice that Theorem 3 immediately follows from the lower bound in (3) and Theorem 5 for $\xi = 1$.

We complete this section with the result stating that the above results give tight lower bounds for $\mu(3; \varepsilon)$ for all $\varepsilon \in [1/2, 1)$.

Theorem 6. *We have*

$$\mu(3;\varepsilon) = \begin{cases} 4, & \varepsilon \in [3/4, 1)\,; \\ 5, & \varepsilon \in [2/3, 3/4)\,; \\ 7, & \varepsilon \in [3/5, 2/3)\,; \\ 8, & \varepsilon \in [1/2, 3/5)\,. \end{cases}$$

Notice that, in contrast to $d = 2$, when $\varepsilon \in [1/2, 1)$, the value of $\mu(d;\varepsilon)$ for $d = 3$ achieves the lower bound in (3) if and only if $\varepsilon \in [1/2, 4/7) \cup [3/5, 1)$. As for the next semi-open interval $[1/3, 1/2)$ of values of ε, our results do not imply tight results for the entire range. Nevertheless, for some specific subintervals our results are nearly tight. For instance, Theorems 2, 3, 4, 5 and Corollary 1 imply that

- for $\varepsilon \in \left[\frac{1}{3}, \frac{8}{23}\right)$, $\mu(3;\varepsilon) = 27$ (by Theorem 2);
- for $\varepsilon \in \left[\frac{8}{21}, \frac{5}{13}\right)$, $\mu(3;\varepsilon)$ is constant and lies on $[22, 24]$ (by Theorem 4, Corollary 1, and (3));
- for $\varepsilon \in \left[\frac{7}{16}, \frac{4}{9}\right)$, $\mu(3;\varepsilon)$ is constant and lies on $[17, 21]$ (by Theorem 4, Corollary 1, and (3));
- for $\varepsilon \in \left[\frac{4}{9}, \frac{7}{15}\right)$, $\mu(3;\varepsilon) \in [16, 18]$ (by Theorem 5 and (3));
- for $\varepsilon \in \left[\frac{7}{15}, \frac{1}{2}\right)$, $\mu(3;\varepsilon) = 15$ (by Theorem 3).

Notation. For convenience, for any $a, b \in \mathbb{R}/\mathbb{Z}$, we denote by $\|a - b\|$ the smallest $\nu \in [0, 1)$ such that $a = b \pm \nu$. Similarly, for every $a, b \in \mathbb{Z}/t\mathbb{Z}$, we denote by $\|a - b\|_t$ the smallest $\nu \in \{0, 1, \ldots, t - 1\}$ such that $a = b \pm \nu$.

Organisation. The rest of the paper is organised as follows. The proofs of Theorem 1 and its corollaries are given in Sections 2–3. In particular, Section 2 states Claim 1, a stronger version of Theorem 1, and then derives Corollaries 1 and 2. Section 3 proves Claim 1. The proofs of Theorems 2 and 4 are presented in Section 4. In Section 5, we give a complete proof of Theorem 5 for all values of ξ that, in turn, implies Theorem 3. Section 6 describes the constructions that prove tightness of the lower bounds given by (3) and Theorem 2 in the range $\varepsilon \in [1/2, 1)$ and, thus, prove Theorem 6. Finally, Section 7 is devoted to a discussion of remaining challenges.

2 Integer lattices

This section, along with the subsequent one, is devoted to the discussion of the discrete version of the problem and a connection between the two versions. That allows one to reduce continuously many problems of finding the values $\mu(d;\varepsilon)$ to countably many discrete problems.

Let $d \geq 2$ be an integer. Let $s \leq t$ be positive integers. Consider the *(discrete) torus* $[\mathbb{Z}/t\mathbb{Z}]^d$. We say that a *sub-cube of side length s* in this torus is a set of the form

$$\{(x_1, \ldots, x_d) : x_i^0 \leq x_i \leq x_i^0 + s - 1 \mod t\}, \qquad x_i^0 \in \mathbb{Z}/t\mathbb{Z}.$$

Throughout the paper, the node $(x_1^0, \ldots, x_d^0)$ of the above sub-cube will be referred to as its *base vertex*, and we use the same notion in the continuous setting. Let $\mu_0(d; s, t)$ be the minimum number of such sub-cubes needed to cover the torus.

Clearly, $\mu_0(d; s, t) \geq \mu(d; s/t)$, since any covering of the discrete torus $[\mathbb{Z}/t\mathbb{Z}]^d$ with μ sub-cubes of side length s immediately yields a corresponding covering of T^d with μ $\frac{s}{t}$-sub-cubes. Conversely, the result we are about to prove shows that, in a sense, all coverings of the continuous torus by ε-sub-cubes may be reduced to the discrete ones.

In fact, we prove a bit stronger statement than Theorem 1 itself.

Claim 1. *Let μ be a positive integer. Choose a minimum ε_0 such that there is a covering of the torus T^d by μ ε_0-sub-cubes. Then $\varepsilon_0 = s_0/t_0$ where s_0 and t_0 are coprime positive integers with $t_0 \leq \mu$.*

Moreover, for every rational $\varepsilon = s/t \geq \varepsilon_0$, there exists a covering of T^d with μ ε-sub-cubes such that all the coordinates x_i^j of their base vertices are multiples of $1/t$.

We finish this section with deducing Corollaries 1 and 2 from Claim 1. The next section is devoted to the proof of Claim 1.

To prove Corollary 1, put $\mu := \mu(d; \varepsilon)$ and notice that $\mu \leq \lceil 1/\varepsilon \rceil^d = r^d$ by (1). Choose the smallest ε_0 such that $\mu(d; \varepsilon_0) = \mu$; since $\mu(d; \cdot)$ is obviously non-increasing, it is constant on $[\varepsilon_0, \varepsilon]$, and hence ε_0 is indeed its largest discontinuity point not exceeding ε. By Claim 1, $\varepsilon_0 = s_0/t_0$, where s_0 and t_0 are coprime integers with $t_0 \leq \mu \leq r^d$, and $\mu = \mu(d; \varepsilon_0) = \mu_0(d; s_0, t_0)$. Finally, since $\frac{1}{r-1} > \varepsilon$ and $\mu \leq r^d < \mu(d; \varepsilon')$ for every $\varepsilon' < \frac{1}{r}$ by (1), we have $\frac{1}{r-1} > \frac{s_0}{t_0} \geq \frac{1}{r}$. This settles the first part of the Corollary.

Next, for every discontinuity point $\frac{s}{t} \in \left[\frac{1}{r}, \frac{1}{r-1}\right)$, its denominator t does not exceed $\mu\left(d; \frac{1}{r}\right) = r^d$. Therefore, for every integer $r \geq 2$, this semi-open interval contains at most $\frac{r^{d-1}(r^d+1)}{2(r-1)} + r^d$ candidates on the role of a discontinuity point. In particular, if one determines the values $\mu_0(d; s, t)$ for all such candidates s/t, that would determine all values of the form $\mu(d; \varepsilon)$ with $\varepsilon \in \left[\frac{1}{r}, \frac{1}{r-1}\right)$.

To prove Corollary 2, assume that $\varepsilon = s/t$ is rational, set $\mu := \mu(d; s/t)$, and choose ε_0 as above. The second part of Claim 1 implies that there exists a covering of T^d with μ ε-sub-cubes such that all coordinates of their base vertices are integer multiples of $1/t$. This covering induces a covering of $[\mathbb{Z}/t\mathbb{Z}]^d$ by μ sub-cubes of side length s, thus showing that $\mu_0(d; s, t) \leq \mu(d; s/t)$. In view of the converse inequality $\mu(d; s/t) \leq \mu_0(d; s, t)$ mentioned above, this yields the result.

3 Proof of Claim 1

Consider now any ε such that there is a covering $\mathcal{A} = \{A_1, A_2, \ldots, A_\mu\}$ of the torus T^d by μ ε-sub-cubes

$$A_j := \left\{(x_1, \ldots, x_d) : x_i \in \left[x_i^j, x_i^j + \varepsilon\right]\right\}, \quad j \in [\mu].$$

We will modify this covering in two steps. The result of Step 1 will, in particular, establish the second part of Claim 1. In Step 2, we will see that, if ε is not of the form $\varepsilon = s/t$ with $t \le \mu$ and s, t coprime, then there is a covering with μ ε'-sub-cubes for some $\varepsilon' < \varepsilon$, thus establishing the first part of Claim 1.

Let $i \in [d]$. Consider in $\mathbb{R}/\mathbb{Z}$ the segments $\left[x_i^j, x_i^j + \varepsilon\right]$, $j \in [\mu]$. Let us introduce a graph $G_i(\mathcal{A})$ with vertex set $[\mu]$. Let vertices $j_1, j_2 \in [\mu]$ be adjacent in $G_i(\mathcal{A})$ if and only if the sets $\left\{x_i^{j_1}, x_i^{j_1} + \varepsilon\right\}$, $\left\{x_i^{j_2}, x_i^{j_2} + \varepsilon\right\}$ are not disjoint (i.e., the respective segments have at least one common endpoint).

Step 1. We show that there exists a covering $\widetilde{\mathcal{A}}$ of T^d by μ ε-sub-cubes such that $G_i(\widetilde{\mathcal{A}})$ is connected, for every $i \in [d]$. For that purpose, we shift some sub-cubes as follows.

Assume that, in $G_i(\mathcal{A})$, there are several connected components $H_1, \dots, H_\ell$, $\ell \ge 2$. Choose indices $j \in H_1$ and $j' \in [\mu] \setminus H_1$, and choose endpoints a and b of the jth and j'th segments, respectively (i.e., $a \in \{x_i^j, x_i^j + \varepsilon\}$ and $b \in \{x_i^{j'}, x_i^{j'} + \varepsilon\}$), such that the distance $\rho = \|a - b\|$ is minimal over the choice of j, j', a, and b; since j and j' are in different components, we have $\rho > 0$.

Without loss of generality, $\rho = a - b$. Now let us shift all segments labeled by H_1 leftwards by distance ρ (with the sub-cubes shifted accordingly). By the choice of ρ, all points of T^d remain covered — it follows from the fact that every circle $\{\alpha_1\} \times \cdots \times \{\alpha_{i-1}\} \times (\mathbb{R}/\mathbb{Z}) \times \{\alpha_{i+1}\} \times \cdots \times \{\alpha_\mu\} \subset T^d$ was completely covered by the intersections with the sub-cubes before the shift, and neither the cyclic order of those intersections nor their intersection scheme (i.e. the set of pairs of intersecting segments) has been changed upon the shift.

Therefore, we get a covering $\mathcal{A}_1$ of T^d, where the graph $G_i(\mathcal{A}_1)$ has at most $\ell - 1$ components. If $G_i(\mathcal{A}_1)$ is not connected, we perform the same procedure with $\mathcal{A}_1$ and obtain a covering $\mathcal{A}_2$ with $G_i(\mathcal{A}_2)$ having at most $\ell - 2$ components. Proceeding in this way, we reach a covering $\widehat{\mathcal{A}}$ where $G_i(\widehat{\mathcal{A}})$ is connected.

Applying the same procedure for every $i \in [d]$, we get a desired covering $\widetilde{\mathcal{A}}$.

Now, if $\varepsilon = s/t$ is rational, we may assume that $\mathbf{x}^1 = (x_1^1, \dots, x_d^1) = (0, 0, \dots, 0)$. By connectedness of all graphs $G_i(\widetilde{\mathcal{A}})$, all coordinates of the vertices of the sub-cubes are multiples of $1/t$; thus the second part of Claim 1 is established.

Step 2. Now we may assume that for every $i \in [d]$, the graph $G_i(\mathcal{A})$ is connected. Suppose that there is no integer $t \in [\mu]$ such that $t\varepsilon \in \mathbb{N}$; we will show that there exists an $\varepsilon' < \varepsilon$ such that T^d can be covered by μ ε'-sub-cubes, thus proving the first part of Claim 1.

Fix $i \in [d]$ and define a relation $<_i$ on the set of segments $[x_i^j, x_i^j + \varepsilon]$ as the transitive closure of the following elementary comparisons:

$$\text{if } x_i^{j_1} + \varepsilon = x_i^{j_2} \mod 1, \text{ then } [x_i^{j_1}, x_i^{j_1} + \varepsilon] <_i [x_i^{j_2}, x_i^{j_2} + \varepsilon].$$

Since $q\varepsilon \notin \mathbb{N}$ for any $q \in [\mu]$, we get that $<_i$ is a partial order (recall that some numbers of the form x_i^j may coincide).

Choose now a minimal segment $[x_i^{j_i}, x_i^{j_i} + \varepsilon]$ with respect to $<_i$. Shifting all ith coordinates leftwards by $x_i^{j_i}$, we may assume that $x_i^{j_i} = 0$. Performing such shifts along all coordinates, we arrive at the situation where all coordinates of all vertices of the sub-cubes lie in the set

$$I_\varepsilon = \bigl\{k\varepsilon \mod 1 \colon k = 0, 1, \dots, \mu\bigr\}.$$

Notice that I_ε consists of $\mu + 1$ distinct numbers; let γ be the minimum distance between elements of I_ε (modulo 1).

Choose a positive $\delta < \gamma/\mu$ and put $\varepsilon' = \varepsilon - \delta$. Then, for every $k_1, k_2 \in \{0, 1, \dots, \mu\}$ we have

$$\{k_1\varepsilon\} < \{k_2\varepsilon\} \quad \text{if and only if} \quad \{k_1\varepsilon'\} < \{k_2\varepsilon'\},$$

where $\{\cdot\}$ stands for the fractional part. Informally speaking, the sets I_ε and $I_{\varepsilon'}$ again have the same cyclic order and the same intersection scheme.

Now choose the ε'-sub-cubes

$$A'_j = \bigl\{(x_1, \dots, x_d) \colon x_i \in \bigl[(x')^j_i, (x')^j_i + \varepsilon'\bigr]\bigr\}, \quad j \in [\mu],$$

as follows: if $x^j_i = \{k\varepsilon\}$, then $(x')^j_i = \{k\varepsilon'\}$. The above relation shows that for every $a' \in [0, 1)$ there exists $a \in [0, 1)$ such that

$$a' \in [(x')^j_i, (x')^j_i + \varepsilon'] \quad \text{if and only if} \quad a \in [x^j_i, x^j_i + \varepsilon].$$

Therefore, for any point $x' = (x'_1, \dots, x'_d) \in T^d$ there exists a point $x = (x_1, \dots, x_d) \in T^d$ such that x' is covered by A'_j if and only if x is covered by A_j. Thus, since the A_j form a covering of T^d, so do the A'_j, and we have constructed a covering by μ ε'-sub-cubes, as desired.

4 Proofs of Theorems 2 and 4

Introduce the following conditions on positive integers s and t:

(i) $\lfloor \frac{t}{s} \rfloor = r$, $\lceil \frac{t}{s} \rceil = r + 1$, and $\frac{t}{s} \geq \frac{3}{2}$;

(ii) $\lceil \frac{t}{s} \lceil \frac{t}{s} \rceil \rceil = s$ and, consequently, $\lceil \frac{t}{s} \lceil \frac{t}{s} \lceil \frac{t}{s} \rceil \rceil \rceil = t$;

(iii) $s \geq (r + 1)\Delta$, where $\Delta := s^2 - t(r + 1)$;

(iv) $t \geq r(s + \Delta)$;

(v) at least one of the inequalities (iii) and (iv) is strict.

Notice here that the conditions (i)–(v) yield $t > s + \Delta$; indeed, in view of (iv), this could fail only if $r = 1$ and $t = s + \Delta$. But then, due to (v), we should have $s > 2\Delta$ and hence $\frac{t}{s} = 1 + \frac{\Delta}{s} < \frac{3}{2}$, which contradicts (i).

In fact, Theorems 2 and 4 are particular cases of the following lemma. We start with proving the Lemma, and then we derive both theorems from it.

Lemma 1. *If t, s are coprime positive integers satisfying the conditions* (i)–(v)*, then* $\mu(3; s/t) > t$.

Proof. Due to Corollary 2, it suffices to show $\mu_0(3; s, t) > t$. Assume, to the contrary, that there is a covering of $[\mathbb{Z}/t\mathbb{Z}]^3$ with t sub-cubes $C_1, C_2, \ldots, C_t$ of side s.

For $i \in [t]$, let $\mathbf{x}^0(i) = (x_1^0(i), x_2^0(i), x_3^0(i))$ be the base vertex of the sub-cube C_i. For every $\alpha \in \mathbb{Z}/t\mathbb{Z}$, define the *$\alpha$th layer* $\mathcal{S}_\alpha$ as the intersection of the torus with the hyperplane $x_3 = \alpha$, i.e., $\mathcal{S}_\alpha = [\mathbb{Z}/t\mathbb{Z}]^3\big|_{x_3=\alpha}$. Each such intersection is a 2-torus covered with the *squares*, i.e., the (nonempty) intersections of the sub-cubes with $\mathcal{S}_\alpha$.

Let us denote the number of squares covering the layer $\mathcal{S}_\alpha$ by $f(\alpha)$. Since $\mu(2; \varepsilon) = \left\lceil \frac{1}{\varepsilon} \left\lceil \frac{1}{\varepsilon} \right\rceil \right\rceil$ (see [11]), for every α we get $f(\alpha) \geq \mu(2; s/t) = s$ due to (ii). On the other hand, $\sum_{\alpha \in \mathbb{Z}/t\mathbb{Z}} f(\alpha) = st$ because each sub-cube meets exactly s layers. This means that $f(\alpha) = s$ for every $\alpha \in \mathbb{Z}/t\mathbb{Z}$. In particular, for every $i \in [t]$ we have $f(x_3^0(i) + s - 1) = f(x_3^0(i) + s)$. In other words, the last layer of the sub-cube C_i is covered by the same number of squares as the next layer. It is only possible when there is $j \in [t]$ such that $x_3^0(j) = x_3^0(i) + s$; informally speaking, the sub-cube C_j 'starts' exactly when C_i 'finishes'. In this situation, we say that the sub-cube C_j is a *successor* of the sub-cube C_i.

Since s and t are coprime and we have exactly t sub-cubes, for every $\alpha \in \mathbb{Z}/t\mathbb{Z}$ there is a unique sub-cube C_i with $x_3^0(i) = \alpha$ (surely, the same holds for any other coordinate). Renumbering the sub-cubes C_i, we assume that $x_3^0(i) = i$ for all $i \in [t]$; we assume that the numeration of the sub-cubes is cyclic, i.e., $C_{i+t} = C_i$; so, further we assume that the indices i run over $\mathbb{Z}/t\mathbb{Z}$. Now, each sub-cube C_i has a unique successor C_{i+s}.

For $\gamma \in \{1, 2, 3\}$, we say that a *γ-column* is a set of t elements in $[\mathbb{Z}/t\mathbb{Z}]^3$ differing only in the γth coordinate. Due to (i), we have $\mu(1; s/t) = \lceil t/s \rceil = r + 1$, so each γ-column meets at least $r + 1$ sub-cubes.

Claim 2. *Assume that a layer $\mathcal{S}$ is covered with squares $A_1, A_2, \ldots, A_s$. Fix $\gamma \in \{1, 2\}$. Then there are no more than Δ γ-columns crossing at least $r + 2$ squares in the covering.*

Proof. Let h be the number of γ-columns crossing at least $r + 2$ squares in the covering. Consider the pairs of the form (U, A_i), where $i \in [s]$ and U is a γ-column in $\mathcal{S}$ that meets A_i. The number of such pairs is exactly s^2 since every square A_i meets s columns. Since each γ-column crosses at least $r + 1$ squares, and h of them cross at least $r + 2$ squares, we obtain

$$s^2 \geq h(r + 2) + (t - h)(r + 1) = t(r + 1) + h,$$

so $h \leq s^2 - t(r + 1) = \Delta$. □

Notice here that $0 \leq \Delta \leq s$ due to (i), (ii), and (iii). In particular,

$$\Delta = s^2 - t(r + 1) \overset{\text{(ii)}}{=} s \left\lceil \frac{t}{s} \left\lceil \frac{t}{s} \right\rceil \right\rceil - t(r + 1) \geq t \left\lceil \frac{t}{s} \right\rceil - t(r + 1) \overset{\text{(i)}}{=} 0.$$

Claim 3. *For every* $i \in \mathbb{Z}/t\mathbb{Z}$ *and every* $\gamma \in \{1,2\}$*, the intersection of the sets of* γ*-coordinates of* C_i *and* C_{i+s} *has cardinality at least* $s-\Delta$*, i.e.*

$$\left|\{x^0_\gamma(i),\dots,x^0_\gamma(i)+s-1\}\cap\{x^0_\gamma(i+s),\dots,x^0_\gamma(i+s)+s-1\}\right|\geq s-\Delta.$$

Proof. The layer $\mathcal{S}_s$ meets sub-cubes C_1, C_2, …, C_s, while $\mathcal{S}_{s+1}$ meets sub-cubes C_2, C_3, …, C_{s+1}. Let A_ℓ be the projection of C_ℓ onto $\mathcal{S} := \mathcal{S}_s$, for $\ell \in [s+1]$.

Consider the covering of $\mathcal{S}$ by $A_1, A_2, \dots, A_s$. By Claim 2, among the $(3-\gamma)$-columns crossing A_1, there are at least $s-\Delta$ ones crossing exactly $r+1$ squares in the covering. This means that each of those columns is not covered completely by $A_2, A_3, \dots, A_s$, as $\mu(1; s/t) > r$. Hence each of those columns needs to cross A_{s+1}, since $\mathcal{S}$ is covered by $A_2, A_3, \dots, A_{s+1}$ as well. This finishes the proof for $i=1$; the proof for other values of i is obtained by shifting the indices. □

Notice that either the intersection of the sets of γ-coordinates of C_i and C_{i+s} has cardinality $s - \|x^0_\gamma(i) - x^0_\gamma(i+s)\|_t$, or those two sets cover $\mathbb{Z}/t\mathbb{Z}$; in the latter case, since $t > s+\Delta$, each set contains more than Δ elements not contained in the other. Therefore, Claim 3 yields that for each $i \in \mathbb{Z}/t\mathbb{Z}$ and $\gamma \in \{1,2\}$ we have $\|x^0_\gamma(i) - x^0_\gamma(i+s)\|_t \leq \Delta$. Therefore,

$$\|x^0_\gamma(i) - x^0_\gamma(i+rs)\|_t \leq \sum_{j=1}^{r} \|x^0_\gamma(i+(j-1)s) - x^0_\gamma(i+js)\|_t \leq r\Delta.$$

Now we distinguish two cases.

Case 1. The above inequality is sometimes strict, i.e., there exist $\gamma \in \{1,2\}$ and $i \in \mathbb{Z}/t\mathbb{Z}$ such that

$$\|x^0_\gamma(i) - x^0_\gamma(i+rs)\|_t < r\Delta.$$

Without loss of generality, we assume $i=0$. For $\kappa = 1,2$, we denote

$$N_\kappa := \left|\{x^0_\kappa(0),\dots,x^0_\kappa(0)+s-1\}\cap\{x^0_\kappa(rs),\dots,x^0_\kappa(rs)+s-1\}\right|;$$

in other words, N_κ is the number of common κ-coordinates in C_0 and C_{rs}. Then we have

$$N_\gamma > s - r\Delta \quad \text{and} \quad N_{3-\gamma} \geq s - r\Delta. \tag{5}$$

We show that there exists $\kappa \in \{1,2\}$ such that

$$N_\kappa > s(r+1) - t \quad \text{and} \quad N_{3-\kappa} > \Delta. \tag{6}$$

Indeed, notice that (iii) and (iv) yield

$$s - r\Delta \geq \Delta \qquad \text{and} \tag{7}$$

$$s - r\Delta = s(r+1) - r(s+\Delta) \geq s(r+1) - t, \tag{8}$$

respectively; moreover, one of the inequalities (7) and (8) is strict due to (v). Therefore, if (7) is strict, then (6) holds for $\kappa = \gamma$; otherwise, (8) is strict, and (6) holds for $\kappa = 3-\gamma$.

Consider $\mathcal{S} := \mathcal{S}_0$, and let A_ℓ be the projection of C_ℓ onto $\mathcal{S}$, for $\ell \in \{t+1-s, t+2-s, \dots, t\}$. Notice that $\mathcal{S}$ is covered by the A_ℓ, where ℓ runs through the same range.

Recall that $t - s + 1 \leq rs < t$ by (i). In $\mathcal{S}$, consider the κ-columns crossing both A_{rs} and $A_0 (= A_t)$; there are $N_{3-\kappa} > \Delta$ such κ-columns. By Claim 2, at least one of those is covered by exactly $r+1$ squares in the covering. But the intersections of the column with two of those squares, namely A_t and A_{rs}, have N_κ common elements, hence those $r+1$ squares cover at most

$$s(r+1) - N_\kappa < s(r+1) - (s(r+1) - t) = t$$

elements in the column. Therefore, this column is not covered completely; this is a contradiction.

Case 2. Conversely, assume now that, for every $\gamma \in \{1, 2\}$ and $i \in \mathbb{Z}/t\mathbb{Z}$,

$$\|x^0_\gamma(i) - x^0_\gamma(i + rs)\|_t = \sum_{j=1}^{r} \|x^0_\gamma(i + (j-1)s) - x^0_\gamma(i + js)\|_t = r\Delta. \tag{9}$$

Without loss of generality, we assume also that $\mathbf{x}^0(0) = (0, 0, 0)$.

For a moment, fix $\gamma \in \{1, 2\}$. By the triangle inequality, (9) is only possible when, for every $i \in \mathbb{Z}/t\mathbb{Z}$,

$$\|x^0_\gamma(i) - x^0_\gamma(i + s)\|_t = \Delta.$$

Moreover, all expressions of the form $x^0_\gamma(i) - x^0_\gamma(i + s)$ with $i \in \mathbb{Z}/t\mathbb{Z}$ are equal. Indeed, otherwise we would have $x^0_\gamma(i) - x^0_\gamma(i + s) = x^0_\gamma(i) - x^0_\gamma(i - s)$ for some $i \in \mathbb{Z}/t\mathbb{Z}$ (as s and t are coprime), so that $x^0_\gamma(i + s) = x^0_\gamma(i - s)$. This is impossible, since $i + s \neq i - s$ in $\mathbb{Z}/t\mathbb{Z}$ (because $t > 2$), and all γ-coordinates of the base vertices should be distinct.

Thus, both sequences $x^0_1(i)$ and $x^0_2(i)$, $i = 0, 1, \dots, t-1$, are arithmetic progressions, each with difference $\pm\Delta$. Since $x^0_1(0) = x^0_2(0) = 0$, we conclude that either $x^0_1(i) = x^0_2(i)$ holds for all $i \in \mathbb{Z}/t\mathbb{Z}$, or $x^0_1(i) = -x^0_2(i)$ holds for all $i \in \mathbb{Z}/t\mathbb{Z}$. Now we can repeat the whole argument switching the second and the third coordinate. After switching, we still cannot fall into Case 1, as it yields a contradiction. So we get $x^0_1(i) = \pm x^0_3(i)$ for all $i \in \mathbb{Z}/t\mathbb{Z}$, with a fixed sign choice. Hence, we may reflect the whole picture in some planes in order to get $x^0_1(i) = x^0_2(i) = x^0_3(i)$ for all $i \in \mathbb{Z}/t\mathbb{Z}$. In other words, all base vertices of the sub-cubes have the form (i, i, i) with $i \in \mathbb{Z}/t\mathbb{Z}$.

In order to get the final contradiction, it remains to notice that the point $(0, s, -s)$ is not covered by any of the C_i. Indeed, otherwise some set $\{i, i+1, \dots, i+s-1\} \subseteq \mathbb{Z}/t\mathbb{Z}$ contains all three residues $0, s, -s$. Choose a representative $i \in \mathbb{Z}$ so that the set of integers $\{i, i+1, \dots, i+s-1\}$ contains 0. Then it contains neither s nor $-s$, so it should contain both $t-s$ and $s-t$, and hence $(t-s) - (s-t) \leq s-1$, or $3s > 2t$, which contradicts (i). □

Proof of Theorem 4. We show that the parameters in the theorem satisfy (i)–(v); the theorem follows then from Lemma 1.

To prove (i), write

$$0 < t - rs = \xi r + \left\lfloor \frac{\xi^2}{r+1} \right\rfloor \leq \xi r + r - 1 \leq r^2 + r - 1 < s.$$

Notice that

$$\Delta = s^2 - t(r+1) = \xi^2 - \left\lfloor \frac{\xi^2}{r+1} \right\rfloor (r+1) \in [0, \xi], \tag{10}$$

where the last inclusion follows from (4). Therefore,

$$\left\lceil \frac{t}{s} \left\lceil \frac{t}{s} \right\rceil \right\rceil = \left\lceil \frac{t(r+1)}{s} \right\rceil = \left\lceil s - \frac{\Delta}{s} \right\rceil = s$$

which yields (ii).

Moreover, (10) implies that

$$(r+1)\Delta \leq \xi(r+1) \leq r^2 + r < r^2 + r + \xi = s.$$

This verifies (iii) with a strict sign (hence (v) holds as well).

Finally, (10) implies that

$$(s + \Delta)r \leq (s + \xi)r = (r^2 + r + 2\xi)r = r^3 + r^2 + 2r\xi \leq t$$

which establishes (iv). □

Proof of Theorem 2. For convenience, we replace r with $r+1$ (i.e., we prove that $\mu(3; \varepsilon) = (r+1)^3$ for any $r \in \mathbb{N}$ and any $\varepsilon \in \left[\frac{1}{r+1}, \frac{(r+1)^2-1}{(r+1)^3-r-2}\right)$).

Let $s = (r+1)^2$, $t = (r+1)^3 - 1$. Notice that the fractions

$$\frac{1}{r+1} \quad \text{and} \quad \frac{(r+1)^2 - 1}{(r+1)^3 - r - 2}$$

are Farey neighbours, as

$$(r+1) \cdot ((r+1)^2 - 1) - 1 \cdot ((r+1)^3 - r - 2) = 1$$

(for properties of Farey sequences, see [9, Chapter III]). Therefore, the rational number between them with the smallest denominator is

$$\frac{1 + (r+1)^2 - 1}{(r+1) + (r+1)^3 - r - 2} = \frac{(r+1)^2}{(r+1)^3 - 1},$$

and the next smallest denominator is already greater than $(r+1)^3$. By Corollary 1, in order to prove Theorem 2, it suffices to show that $\mu(3; s/t) \geq (r+1)^3$. For that purpose, we show that the numbers s and t satisfy the conditions of Lemma 1.

Clearly, s and t are coprime. The conditions (i)–(ii) are straightforward to check. Next, since $\Delta = s^2 - t(r+1) = r+1$, we have

$$(r+1)\Delta = (r+1)^2 = s$$

which yields (iii) (with the equality sign). It remains to show that (iv) holds and is strict:

$$r(s + \Delta) = r(r+1)(r+2) = (r+1)((r+1)^2 - 1) < t. \qquad \square$$

5 Proof of Theorem 5

Let $r \geq 1$ be an integer, $\xi \in [r]$, $s = r^2 + r + \xi$, $t = r^3 + r^2 + \xi(r + 1)$. We construct a covering of $[\mathbb{Z}/t\mathbb{Z}]^3$ consisting of t sub-cubes of side s.

The main idea here is the same as in [11]: to take each next sub-cube shifted relative to the previous one by a fixed integer vector v. We take

$$v = (1, r, r^2 + r + 1).$$

Thus, we need to prove that the sub-cubes C_i with base vertices $(i, ri, (r^2+r+1)i)$, $i \in \mathbb{Z}/t\mathbb{Z}$, cover the torus $[\mathbb{Z}/t\mathbb{Z}]^3$. Since each layer $\{j\} \times [\mathbb{Z}/t\mathbb{Z}]^2$ intersects s consecutive sub-cubes C_{j-i} with $i \in \{0, 1, \ldots, s-1\}$, the following claim finishes the proof of Theorem 5.

Claim 4. *The squares of side length s with base vertices $(ri, (r^2 + r + 1)i)$, $i \in \{0, 1, \ldots, s-1\}$, cover $[\mathbb{Z}/t\mathbb{Z}]^2$.*

Proof. Denote by B_i, $i \in \{0, 1, \ldots, s-1\}$, the square with the base vertex $(ri, (r^2 + r+1)i)$. In what follows, we assume that i runs over $\mathbb{Z}/s\mathbb{Z}$, i.e., $i+1 = 0$ for $i = s-1$.

Let C be an arbitrary column consisting of all points with a fixed first coordinate. We show that C is covered completely by the B_i.

Notice that the first coordinate of the relative shift of B_{i+1} with respect to B_i is $x_1^0(i + 1) - x_1^0(i) = r$ for $i \neq s - 1$ and

$$x_1^0(0) - x_1^0(s - 1) = t - r(s - 1) = r + \xi$$

for $i = s-1$. Notice that both r and $r+\xi$ are smaller than s, so the projections of any two cyclically consecutive squares B_i and B_{i+1} onto the first axis overlap, with that of B_{i+1} shifted to the right with respect to B_i. Choose now an index j such that B_j, but not B_{j-1}, overlaps with C (so B_{j-1} is 'cyclically to the left' of C), and let p be the largest integer such that each of the squares B_j, B_{j+1}, …, B_{j+p} meets C. Then B_{j+p+1} is 'cyclically to the right' of C. This implies that $x_1^0(j+p+1) - x_1^0(j-1) > s$. Therefore,

$$r^2 + r + \xi + 1 = s + 1 \leq x_1^0(j + p + 1) - x_1^0(j - 1) \leq (p + 1)r + (r + \xi),$$

which yields $p \geq r$. So, C definitely meets the squares $B_j, B_{j+1}, \ldots, B_{j+r}$ (and, perhaps, some more).

Now, the second coordinate of the relative shift of B_{i+1} with respect to B_i is $x_2^0(i + 1) - x_2^0(i) = r^2 + r + 1$ for $i \neq s - 1$ and

$$x_2^0(0) - x_2^0(s - 1) = t(r + 1) - (r^2 + r + 1)(s - 1) = \xi r + 1 < r^2 + r + 1$$

for $i = s - 1$. This yields that the parts of C covered with two cyclically consecutive squares are either tangent or even overlapping. So the interval covered by the $r + 1$ cyclically consecutive squares consists of at least $\min\{t, s + (r-1)(r^2+r+1) + (r\xi + 1)\} = \min\{t, t + r\} = t$ points, and hence the whole column is covered. □

6 Proof of Theorem 6

We start with the lower bounds for $\mu(3;\varepsilon)$ agreeing with the statement of the theorem. For $\varepsilon \in [1/2, 4/7) \cup [3/5, 1)$, such lower bounds follow from (3). For $\varepsilon \in [4/7, 3/5)$ the lower bound follows from Theorem 2.

In order to show that those bounds are achieved, it suffices to show that $\mu_0(3;3,4) \leq 4$, $\mu_0(3;2,3) \leq 5$, and $\mu_0(3;3,5) \leq 7$. The corresponding examples are provided by the following sets of base vertices:

$$\begin{aligned} &(0,0,0),\ (1,1,1),\ (2,2,2),\ (3,3,3), && \text{for } s/t = 3/4;\\ &(0,0,0),\ (1,1,1),\ (1,2,2),\ (2,1,2),\ (2,2,1), && \text{for } s/t = 2/3;\\ &(0,0,0),\ (1,1,3),\ (1,3,1),\ (2,4,4),\ (3,0,2),\ (3,2,0),\ (4,3,3), && \text{for } s/t = 3/5. \end{aligned}$$

The theorem is proved.

7 Further questions

This paper represents the first step toward determining the value of $\mu(d;\varepsilon)$ for $d = 3$. We found the exact value for $\varepsilon \geq 7/15$, proved non-trivial bounds for certain other ε, and established a connection with the discrete version of this problem which, in particular, implies that for every $\varepsilon \in (0, 1)$ it is sufficient to solve constantly many instances of the discrete version. Although the discrete version (in other words, determining $\mu(3; s/t)$ for positive integers $s < t$) is clearly computable, and computer-assisted results can be obtained for certain values of ε not addressed in this paper — even using a greedy approach — we leave open the question of whether an efficient solution exists.

We believe that, even though slight extensions of our results with current methods are possible, they are not sufficient to solve the problem entirely. Even getting the precise value of $\mu(3;\varepsilon)$ for any $\varepsilon \in [1/3, 1)$ seems a very challenging problem. One particularly interesting case where we only know that $\mu(3;\varepsilon) \in [18, 21]$ is $\varepsilon = 3/7$, which was also discussed earlier in [11], as mentioned in the Introduction.

The assertions of Theorem 2 and Theorem 3 imply that the function $\mu(3;\varepsilon)$ has a jump of size $\Theta(r^2)$ at $\varepsilon = \frac{1}{r}$, which is maximum possible in the interval $\varepsilon \in \left[\frac{1}{r}, \frac{1}{r-1}\right)$ since the difference between the upper and lower bounds in (3) is at most r^2. We believe that $\mu(3;\varepsilon)$ has certain other large jumps inside this interval as well (recall that the total number of jumps is at most $O(r^2)$, as $(r-1)^3 < \mu(3;\varepsilon) \leq r^3$ in this interval). In particular, Theorem 6 shows that there is a jump of size $r = 2$ at $\varepsilon = 2/3$ on the 'middle' of the considered interval. It would be interesting to investigate the number of jumps in $\left[\frac{1}{r}, \frac{1}{r-1}\right)$ and their sizes.

We did not make a serious attempt to extend our results to higher dimensions and leave this direction for future work.

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