AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **95(1)** (2026), Pages 1–32

Zero forcing irredundant sets

Bryan A. Curtis

Department of Mathematics, Iowa State University
Ames, IA 50011, U.S.A.
dr.bryan.curtis.math@gmail.com

Leslie Hogben*

American Institute of Mathematics, Caltech 8-32
1200 E California Blvd., Pasadena, 91125 U.S.A.
hogben@aimath.org

Adriana Roux†

Department of Mathematical Sciences
Stellenbosch University, Stellenbosch, 7600
South Africa
rianaroux@sun.ac.za

Abstract

Irredundance has been studied in the context of dominating sets, via the concept of a private neighbor. Here irredundance of zero forcing sets is introduced via the concept of a private fort and the upper and lower zero forcing irrdedundance numbers $\mathrm{ZIR}(G)$ and $\mathrm{zir}(G)$ are defined. Bounds on $\mathrm{ZIR}(G)$ and $\mathrm{zir}(G)$ are established and graphs having extreme values of $\mathrm{ZIR}(G)$ and $\mathrm{zir}(G)$ are characterized. The effect of the join and corona operations is studied. As the concept of a zero forcing irrdedundant set is new, there are many questions for future research.

1 Introduction

In this paper we introduce an analog of irredundant sets for zero forcing, called zero forcing irredundant sets, with a private fort playing the role of a private neighbor. We also investigate the relationships between the zero forcing and zero forcing irredundance numbers in analogy with the Domination Chain (see (1.1)). We begin this introduction with a brief discussion of domination and irredundance to motivate

* Also at: Department of Mathematics, Iowa State University, Ames, IA 50011, U.S.A.
† Corresponding author.

ISSN: 2202-3518

our work on zero forcing irredundant sets, followed by definitions of zero forcing and forts. We then introduce the definition of zero forcing irredundance. Finally we outline the structure of the rest of the paper and summarize notation we will use.

1.1 Motivation

A set S is a *dominating set* of a graph G if every vertex of G is in S or a neighbor of a vertex in S. A set S of vertices in a graph G is *irredundant* if each vertex in S dominates a vertex of G that is not dominated by any other vertex in S (this may be a neighbor or itself), called a *private neighbor*. The study of irredundance was introduced in 1978 by Cockayne, Hedetniemi and Miller [13] in relation to the study of minimal dominating sets. Irredundance has been studied extensively since then (see [19] and the references therein).

The *domination number* $\gamma(G)$ and the *upper domination number* $\Gamma(G)$ of G are the minimum and maximum cardinalities of a minimal dominating set of G. The *upper irredundance number* $\mathrm{IR}(G)$ and the *lower irredundance number* $\mathrm{ir}(G)$ of G are the maximum and minimum cardinalities of a maximal irredundant set of G. A set that is both dominating and irredundant is a minimal dominating and maximal irredundant set. In fact, minimal dominating sets have been characterized as exactly those sets that are both dominating and irredundant [13].

The relationships between the upper and lower parameters for domination and irredundance are expressed in the *Domination Chain* introduced in [13]:

$$\mathrm{ir}\, G \leq \gamma(G) \leq \Gamma(G) \leq \mathrm{IR}(G). \tag{1.1}$$

Various aspects of the Domination Chain have been studied, including complexity of computing parameters in the Domination Chain (see, for example, [4] and the references therein). The conditions under which these parameters are equal have been studied extensively (see [19] for a survey).

1.2 Zero forcing

Zero forcing was introduced as a bound for the maximum nullity among real symmetric matrices whose pattern of nonzero off-diagonal entries is described by the edges of a graph in [1] and independently in other applications.

Zero forcing is a propagation process on a graph G. Starting with an initial set of blue vertices, the process colors vertices blue by repeated applications of the *color change rule*: A blue vertex u can change the color of a white vertex w to blue if w is the only white neighbor of u. A *zero forcing set* of G is a subset of vertices B such that if B is the initial set of blue vertices, then the repeated application of the color change rule will cause the whole graph to turn blue. The *zero forcing number* of G is the minimum cardinality of a zero forcing set. The *upper zero forcing number* $\overline{\mathrm{Z}}(G)$ of a graph G is the maximum cardinality of a minimal zero forcing set of G.

1.3 Private forts

Let G be a graph. A nonempty set $F \subseteq V(G)$ is a *fort* if for all $v \in V(G) \setminus F$, $|F \cap N(v)| \neq 1$ where $N(v)$ denotes the set of neighbors of v. Note that $V(G)$ is a fort (vacuously) and in a connected graph of order at least two, a fort must contain at least two vertices. Forts present obstructions to zero forcing: Given a set B of blue vertices, some vertex in B can perform a force in G if and only if $V(G) \setminus B$ is not a fort. This equivalence, stated in the next theorem, was first noted by Brimkov, Fast and Hicks in [7].

Theorem 1.1. [7] *Let G be a graph. Then $B \subseteq V(G)$ is a zero forcing set if and only if B intersects every fort.*

A *minimal fort* is a fort that is not properly contained in any other fort.

Corollary 1.2. *Let G be a graph. Then $B \subseteq V(G)$ is a zero forcing set if and only if B intersects every minimal fort.*

Remark 1.3. Let G be a graph and let B be a minimal zero forcing set. Let $B' = \{u \in B : u \text{ is in some minimal fort of } G\}$. For every fort F, F contains a minimal fort F'. Since B is a zero forcing set, $\emptyset \neq B \cap F' = B' \cap F' \subseteq B' \cap F$, so B' is a zero forcing set. Since B is minimal, $B = B'$. So each element of B is an element of some minimal fort of G.

Remark 1.4. In general, a superset of a fort F need not be a fort, because if vertex v is added to F there may be a vertex u such that u has no neighbors in F but is adjacent to v. However, there are conditions under which all or certain supersets of a fort are a fort. If F is a fort and $N(v) \subseteq F$, then $F \cup \{v\}$ is a fort. If F is a fort such that $N(v) \subseteq F$ for every $v \notin F$, then every superset of F is a fort. A union of forts is a fort: Suppose F_1 and F_2 are forts of G. If $u \notin F_1 \cup F_2$, then u has zero or at least two neighbors in each of F_i, so u has zero or at least two neighbors in $F_1 \cup F_2$

A private fort is a natural analog of a private neighbor and is used to define zero forcing irredundance.

Definition 1.5. Let G be a graph and let $S \subseteq V(G)$. For $x \in S$, a fort F of G is a *private fort* of x (relative to S), if $S \cap F = \{x\}$. A *minimal private fort* of x is a private fort of x that does not properly contain another private fort of x.

1.4 Zero forcing irredundance

Analogously to a zero forcing set that intersects every fort, we have that $D \subseteq V(G)$ is a dominating set if and only if D intersects every closed neighborhood. With irredundance being a natural extension of domination, we introduce the concept of zero forcing irredundance.

Definition 1.6. Let G be a graph and let $S \subseteq V(G)$. Then S is a Z-*irredundant set* or *ZIr-set* if every element of S has a private fort. The *lower ZIr number* is

$$\operatorname{zir}(G) = \min\{|S| : S \text{ is a maximal ZIr-set}\}.$$

The *upper ZIr number* is

$$\mathrm{ZIR}(G) = \max\{|S| : S \text{ is a maximal ZIr-set}\}.$$

We use the terms *upper ZIR set* and *lower zir set* to refer to maximal ZIr-sets S and S' such that $|S| = \mathrm{ZIR}(G)$ and $|S'| = \mathrm{zir}(G)$, respectively.

In light of Corollary 1.2, it is reasonable to consider replacing private forts in Definition 1.6 with private *minimal* forts. However, this modification does not align with the definition of irredundance: v is a private neighbor of u if $S \cap N[v] = \{u\}$ (where $N[v] = N(v) \cup \{v\}$), and this definition does not require that $S \cap N[x] = \{u\}$ implies that $N[v] \subseteq N[x]$. It also has an undesired consequence: For every graph G and every vertex $v \in V(G)$, the set $\{v\}$ is an irredundant set. The same statement for Z-irredundance does not necessarily hold when restricting to private minimal forts.

1.5 Structure of the paper

Basic results about the relationship between $\mathrm{Z}, \overline{\mathrm{Z}}$, ZIR, and zir are presented in Section 2, with many parallels to γ, Γ, IR, and ir. Examples of upper ZIR and lower zir numbers for specific graph families are given in Section 3. Section 4 presents bounds on upper ZIR and lower zir numbers and graphs having extreme upper ZIR and lower zir numbers are characterized in Section 5. The effect of the join and corona operations is studied in Section 6 and it is shown in Section 7 that the lower zir number is noncomparable to several lower bounds for the zero forcing number. Section 8 provides information for future study of computational complexity and discusses the software used here for computations. Finally, Section 9 presents a summary table of upper ZIR and lower zir numbers, zero forcing numbers and other related parameters for various graph families and discusses directions for future research.

1.6 Notation

We conclude this introduction with some additional basic graph theory notation and terminology that we will use. A graph $G = (V(G), E(G))$ is simple, undirected, finite, and $V(G) \neq \emptyset$. Standard symbols are used for well-known graph families: K_n denotes a complete graph of order n; this graph has an edge between every pair of vertices. $\overline{K_n}$ denotes an empty graph of order n, which has no edges. P_n and C_n denote the path graph and cycle graph of order n; the vertices of P_n can be numbered $v_1, \ldots, v_n$ so that the edges are $v_i v_{i+1}$ for $i = 1, \ldots, n-1$, and similarly for the cycle with the addition of edge $v_n v_1$. A disjoint union is denoted by $\sqcup$ and is used for both sets and graphs (meaning the vertex sets are disjoint). The join $G \vee H$ of two disjoint graphs G and H has $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \hat{E}$ where $\hat{E} = \{uw : u \in V(G) \text{ and } w \in V(H)\}$. If G and H are disjoint and G has order n_G, the *corona* of G with H, denoted by $G \circ H$, is the graph obtained from the disjoint union of G and n_G copies of H by joining the ith vertex of G and vertices of the ith copy of H for each $i = 1, \ldots, n_G$.

Vertices u and w of a graph G are *twins* if $N(u) = N(w)$ (*independent twins*) or $N[u] = N[w]$ (*adjacent twins*). A set of vertices $\{w_1, \ldots, w_r\}$ is a *set of twins* if every two vertices in the set are independent twins or every two vertices in the set are adjacent twins. The next result is well known and useful in the study of zero forcing.

Proposition 1.7. [18, Proposition 9.15] *If a graph has a set of twins* $\{w_1, \dots, w_r\}$, *then each zero forcing set must contain at least* $r - 1$ *of these twins.*

2 Preliminary results

Many of the results in this section are not difficult to prove but provide intuition and establish a collection of tools we shall frequently use. Since zero forcing is one of the primary inspirations for this work, we begin by investigating its relationship with ZIr-sets. We have an immediate parallel with dominating and irredundant sets.

Remark 2.1. Let $S \subseteq V(G)$ be both a zero forcing set and ZIr-set. Then we can see that S is a minimal zero forcing set and maximal ZIr-set: Since every element $x \in S$ has a private fort F_x, $(S \setminus \{x\}) \cap F_x = \emptyset$ and $S \setminus \{x\}$ is not a zero forcing set. Since a superset of a zero forcing set is a zero forcing set, and no proper superset of S is miminal, S is a maximal ZIr-set.

Proposition 2.2. *Let* G *be a graph and* $S \subseteq V(G)$. *Then* S *is a minimal zero forcing set if and only if* S *is a maximal ZIr-set of* G *and* $S \cap F \neq \emptyset$ *for every fort* F *of* G.

Proof. Suppose that S is a minimal zero forcing set. Let $x \in S$. Since S is minimal, $S \setminus \{x\}$ is not a zero forcing set. By Theorem 1.1 there exists a fort F_x such that $(S \setminus \{x\}) \cap F_x = \emptyset$ and $S \cap F_x \neq \emptyset$. Thus F_x is a private fort of x relative to S. This argument holds for every vertex in S and so S is a ZIr-set. Theorem 1.1 guarantees $S \cap F \neq \emptyset$ for every fort F of G and hence S is a maximal ZIr-set.

Now suppose that S is a maximal ZIr-set of G and $S \cap F \neq \emptyset$ for every fort F of G. Then Theorem 1.1 implies that S is a zero forcing set, so S is a minimal zero forcing set by Remark 2.1. □

Note that since $V(G)$ is a fort for any graph G, $1 \leq \mathrm{zir}(G)$.

Corollary 2.3. *Let* G *be a graph. Then* $1 \leq \mathrm{zir}(G) \leq \mathrm{Z}(G) \leq \overline{\mathrm{Z}}(G) \leq \mathrm{ZIR}(G)$.

Not every maximal ZIr-set is a zero forcing set, as the next example shows.

Example 2.4. Consider the cycle C_5. The minimal forts are sets of three vertices that do not contain three consecutive vertices (consecutive in the order around the cycle). It follows that every set of two vertices is a maximal ZIr-set, but not every set of two vertices is a zero forcing set since a zero forcing set of C_5 must contain two adjacent vertices.

Next we present elementary results about ZIr sets. As the next remark illustrates, it often suffices to restrict to connected graphs when studying ZIr-sets.

Remark 2.5. If $G = G_1 \sqcup \cdots \sqcup G_k$, then S is a ZIr-set of G if and only if $S \cap G_i$ is a ZIr-set of G_i for $i = 1, \ldots, k$. Thus $\operatorname{zir}(G_1 \sqcup \cdots \sqcup G_k) = \operatorname{zir}(G_1) + \cdots + \operatorname{zir}(G_k)$ and $\operatorname{ZIR}(G_1 \sqcup \cdots \sqcup G_k) = \operatorname{ZIR}(G_1) + \cdots + \operatorname{ZIR}(G_k)$

The next example is an immediate consequence.

Example 2.6. Let $n \geq 1$. Since $\operatorname{zir}(K_1) = 1$, $\operatorname{zir}(\overline{K_n}) = Z(\overline{K_n}) = \overline{Z}(\overline{K_n}) = \operatorname{ZIR}(\overline{K_n}) = n$.

Remark 2.7. Suppose G is a graph of order $n \geq 2$ that has an edge. Every fort in a connected component containing an edge must contain at least two vertices. Adding an additional private fort to a union of private forts must add a new vertex, so the union of all k private forts associated with a maximal ZIr-set of k vertices must contain at least $k + 1$ vertices. Thus $\operatorname{ZIR}(G) \leq n - 1$.

Remark 2.8. Let G be a graph with no isolated vertices. Then every minimal zero forcing set is a maximal ZIr-set and no vertex is in every minimal zero forcing set [2]. Thus there does not exist a vertex v that is contained in every maximal ZIr-set of G.

Lemma 2.9. *Let G be a graph with no isolated vertices and let $S \subseteq V(G)$. If $N[v] \subseteq S$ for some vertex v, then S is not a ZIr-set, or equivalently, if S is a ZIr-set of G, then $V(G) \setminus S$ is a dominating set of G.*

Proof. Suppose there exists a vertex $v \in S$ such that $N(v) \subseteq S$. Having assumed that G has no isolated vertices, there exists a vertex $u \in N(v)$. Let $F \subseteq V(G)$ such that $F \cap S = \{u\}$. Since $N(v) \subseteq S$, v is adjacent to exactly one element of F. Thus there does not exist a private fort of u relative to S, and so S is not a ZIr-set of G. □

It is well known (and easy to see) that $\delta(G) \leq \mathrm{Z}(G)$. The next two results strengthen this bound.

Proposition 2.10. *Let G be a graph and $d \geq 1$. If $v_1, \ldots, v_d \in V(G)$ all have degree at least d, then $S = \{v_1, \ldots, v_d\}$ is a ZIr set.*

Proof. Assume $v_1, \ldots, v_d \in V(G)$ all have degree at least d. It suffices to show that the set $F_i = (V(G) \setminus S) \cup \{v_i\}$ is a private fort of v_i relative to S for every $i = 1, \ldots, d$. This is certainly the case if $d = 1$ since $V(G)$ is a fort, so suppose $d \geq 2$. Let $v \in V(G) \setminus F_i$ for some $i = 1, \ldots, d$. If v has two or more neighbors in $V(G) \setminus S$, then $|N(v) \cap F_i| \geq 2$. So, assume v has at most one neighbor in $V(G) \setminus S$. Since $v \in S$ and $|N(v)| \geq d$, v is adjacent to exactly one vertex in $V(G) \setminus S$ and to v_i because $S \setminus \{v\} \subseteq N(v)$. Thus $|F_i \cap N(v)| \geq 2$ as required. □

Corollary 2.11. *If G is a graph and $\delta(G) \geq 1$, then every set of size $\delta(G)$ is a ZIr-set. For every graph G, $\delta(G) \leq \operatorname{zir}(G)$.*

The next example shows the bound in Corollary 2.11 is sharp.

Example 2.12. Let $n \geq 2$. Then $\mathrm{zir}(K_n) = n - 1 = \mathrm{ZIR}(K_n)$ by Corollary 2.11 (since $\delta(K_n) = n - 1$) and Remark 2.7.

Example 2.12 implies that $\mathrm{Z}(K_n) = \overline{Z}(K_n) = n - 1$, but this is already known [1, 5]. In the next section we establish values of the parameters zir, Z, $\overline{\mathrm{Z}}$, and ZIR for additional graph families. These results show that both $\mathrm{zir}(G) = \mathrm{Z}(G) = \overline{\mathrm{Z}}(G) = \mathrm{ZIR}(G)$ and $\mathrm{zir}(G) < \mathrm{Z}(G) < \overline{\mathrm{Z}}(G) < \mathrm{ZIR}(G)$ are possible (Example 2.12 and Proposition 3.6).

Observation 2.13. *Let S be a ZIr-set of a graph G, of order n, let $x_i \in S$ for $i = 1, \ldots, k$, and let F_i be a private fort of x_i. Then $|S| \leq n - |\cup_{i=1}^{k} F_i| + k$.*

3 Determining ZIR and zir for graph families

In this section we establish the values of the upper and lower ZIr numbers for various families of graphs. In doing so, we highlight various techniques that will be used throughout the paper. We also include known zero forcing numbers and upper zero forcing numbers in each of the following results as the connections between the parameters and the underlying graph is rather interesting. See Table 9.1 for a summary of the values of zir, Z, $\overline{\mathrm{Z}}$, and ZIR determined for various graphs in this and other sections.

Let G be a graph on n vertices. The next example illustrates that the gap between $\mathrm{zir}(G)$ and $\mathrm{Z}(G)$ (and hence also $\mathrm{ZIR}(G)$) can be as large as $n - 3$. It is known that $\mathrm{Z}(K_{q,p}) = q + p - 2$ and $\overline{\mathrm{Z}}(K_{q,p}) = q + p - 2$ [1, 5].

Example 3.1. Suppose $1 \leq q \leq p$ and let $U = \{u_1, \ldots, u_q\}$ and $W = \{w_1, \ldots, w_p\}$ be the partite sets of $K_{q,p}$. Any set S that omits at least one vertex u_k from U and at least one vertex w_ℓ from W is a ZIr-set, since then $\{u_i, u_k\}$ is a private fort of u_i and $\{w_j, w_\ell\}$ is a private fort of w_j (if $q = 1$, the forts $\{u_i, u_k\}$ do not exist). If S is a ZIr-set containing U, then $S = U$ because a private fort of u_i necessarily contains W and similarly for $W \subseteq S$ (note that when $q = 1$, the private fort of u_1 is $V(K_{1,p})$). Thus $\mathrm{Z}(K_{q,p}) = \overline{\mathrm{Z}}(K_{q,p}) = \mathrm{ZIR}(K_{q,p}) = q + p - 2$ and $\mathrm{zir}(K_{q,p}) = q$.

Proposition 3.2. *Let $n \geq 4$. Then $\mathrm{zir}(C_n) = \mathrm{Z}(C_n) = \overline{\mathrm{Z}}(C_n) = 2$ and $\mathrm{ZIR}(C_n) = \lfloor \frac{n}{2} \rfloor$.*

Proof. It is known that $Z(C_n) = \overline{Z}(C_n) = 2$ [1, 5]. Together with Corollary 2.11, this implies $\mathrm{zir}(C_n) = 2$. Let $v_1, \ldots, v_n$ denote the vertices of C_n, where $v_n v_1 \in E(C_n)$ and $v_i v_{i+1} \in E(C_n)$ for $i = 1, \ldots, n - 1$. Let $S = \{v_{2i-1} : i = 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and $F_i = (V(C_n) \setminus S) \cup \{v_i\}$. By construction, F_{2i-1} is a private fort of v_{2i-1} for $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$. Thus S is a ZIr-set. Since $|S| = \lfloor \frac{n}{2} \rfloor$, $\mathrm{ZIR}(C_n) \geq \lfloor \frac{n}{2} \rfloor$.

Since every subset of $V(C_n)$ that does not contain a pair of consecutive vertices has cardinality at most $\lfloor \frac{n}{2} \rfloor$, it suffices to show that if S is a ZIr-set of C_n that contains a pair of adjacent vertices, then $|S| = 2$.

Assume that S is a ZIr-set of C_n that contains a pair of adjacent vertices. Without loss of generality let $\{v_1, v_2\} \subseteq S$. Let F be a fort of C_n. Suppose, to obtain a contradiction, that $F \cap \{v_1, v_2\} = \emptyset$. Then $v_3 \notin F$ since otherwise $|N(v_2) \cap F| = 1$. Recursively applying this argument implies $v_k \notin F$ for $k = 4, \ldots, n$ and hence $F = \emptyset$, a contradiction. Thus every fort of C_n intersects $\{v_1, v_2\}$ non-trivially. Since every element of S must have a private fort, it follows that $|S| = 2$. □

Observe that $C_3 = K_3$, so the parameters are determined by Example 2.12 (and $\mathrm{ZIR}(K_3) = 2 > \lfloor \frac{3}{2} \rfloor$).

Proposition 3.3. *For $n \geq 1$, $\mathrm{zir}(P_n) = \mathrm{Z}(P_n) = 1$. For $n \geq 4$, $\overline{\mathrm{Z}}(P_n) = 2$. For $n \geq 5$, $\mathrm{ZIR}(P_n) = \lfloor \frac{n-1}{2} \rfloor$.*

Proof. It is well known that $Z(P_n) = 1$ for $n \geq 1$, which implies $\mathrm{zir}(P_n) = 1$. For $n \geq 4$, $\overline{Z}(P_n) = 2$ [5]. Assume $n \geq 5$. Let $v_1, \ldots, v_n$ denote the vertices of P_n, where $v_i v_{i+1} \in E(P_n)$ for $i = 1, \ldots, n-1$. Just as in the proof of Proposition 3.2, if S is a ZIr-set of P_n that contains a pair of adjacent vertices, then $|S| \leq 2$. Observe that every fort in P_n contains v_1 and v_n.

Let $S = \{v_{2i} : i = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ and $F_i = (V(P_n) \setminus S) \cup \{v_i\}$. Note that $|S| = \lfloor \frac{n-1}{2} \rfloor$. By construction, F_{2i} is a private fort of v_{2i} for $i = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$. Thus S is a ZIr-set. Moreover, S is maximal since $|S| \geq 2$ and the addition of any vertex to S would introduce a pair of consecutive vertices or a vertex of degree 1. Since any set that does not contain a pair of consecutive vertices and does not contain v_1 or v_n has cardinality at most $\lfloor \frac{n-1}{2} \rfloor$ we conclude that $\mathrm{ZIR}(P_n) = \lfloor \frac{n-1}{2} \rfloor$. □

Observation 3.4. *Observe that $P_2 = K_2$ and $P_3 = K_{1,2}$, so $\mathrm{ZIR}(P_2) = 1$ and $\mathrm{ZIR}(P_3) = 1$ from Examples 2.12 and 3.1. It can be shown computationally [15] that $\mathrm{ZIR}(P_4) = 2$.*

For $k \geq 2$, the kth-*friendship graph* $\mathrm{Fr}(k)$ is the graph with $2k+1$ vertices and $3k$ edges constructed as the join of one vertex to k disjoint copies of K_2; $\mathrm{Fr}(3)$ is shown in Figure 3.1. It is known that $\mathrm{Z}(\mathrm{Fr}(k)) = k + 1$ [18, Theorem 9.5].

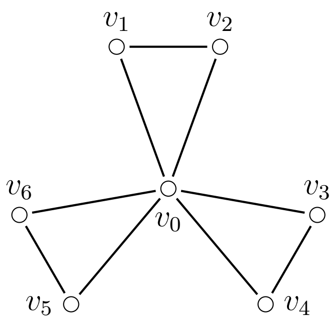

Figure 3.1: The friendship graph Fr(3)

Proposition 3.5. *For $k \geq 2$, $\mathrm{zir}(\mathrm{Fr}(k)) = \mathrm{Z}(\mathrm{Fr}(k)) = \overline{\mathrm{Z}}(\mathrm{Fr}(k)) = \mathrm{ZIR}(\mathrm{Fr}(k)) = k + 1$.*

Proof. Let $k \geq 2$ and let $V(\mathrm{Fr}(k)) = \{v_0, \ldots, v_{2k}\}$, where v_0 is the unique vertex of degree $2k$ and v_j is adjacent to v_{j+1} for $j \equiv 1 \mod 2$ and $1 \leq j < 2k$. We begin by examining the forts of $\mathrm{Fr}(k)$. For $i = 1, \ldots, k$, let $F_i = \{v_{2i-1}, v_{2i}\}$ and observe that F_i is a (minimal) fort. A union of forts of the form F_i is a fort by Remark 1.4. There are 2^k (minimal) forts that contain v_0 and exactly one element of each F_i for $i = 1, \ldots, k$; a superset of such a fort is a fort by Remark 1.4, and is called a *center-fort*. These are the only forts of $\mathrm{Fr}(k)$ because a fort that does not contain v_0 is a union of forts of the form F_i and a fort that does contain v_0 is a center-fort.

There are only k forts of the form F_i so if S is a ZIr-set and $|S| \geq k+1$, then one of the private forts relative to S must be a center-fort F', which contains at least $k+1$ vertices. Thus $|V(\mathrm{Fr}(k)) \setminus F'| \leq k$ and $|S| \leq k+1$. This implies $\mathrm{ZIR}(\mathrm{Fr}(k)) \leq k+1$.

Let S be a ZIr-set. We show that S is contained in a ZIr-set with cardinality $k+1$, which implies every maximal ZIr-set has cardinality $k+1$. Suppose that $S \cap F_\ell = \emptyset$ for some $\ell \in \{1, \ldots, k\}$ and choose a set of private forts relative to S for the vertices of S. Without loss of generality, we may assume that every chosen private fort that is not a center-fort is of the form F_i for some $i \in \{1, \ldots, k\}$. We show that $S' = S \cup \{v_{2\ell-1}\}$ is also a ZIr-set with private fort F_ℓ for $v_{2\ell-1}$ and appropriate modifications of private forts of the vertices in S. Suppose that the chosen private fort F for some $v_j \in S$ contains $v_{2\ell-1}$. Note that F is a center-fort. Since $v_j \notin F_\ell$, $F' = (F \cup \{v_{2\ell}\}) \setminus \{v_{2\ell-1}\}$ is a private fort relative to S' for v_j. Thus S is not maximal, and there is a ZIr-set $\widehat{S}$ such that $S \subseteq \widehat{S}$ and $\widehat{S} \cap F_i \neq \emptyset$ for $i = 1, \ldots, k$. If $F_i \subseteq \widehat{S}$ for some i, then $|\widehat{S}| \geq k+1$. So assume $|\widehat{S} \cap F_i| = 1$ for $i = 1, \ldots, k$. If $v_0 \in \widehat{S}$, then $|\widehat{S}| \geq k+1$. Suppose $v_0 \notin \widehat{S}$. For $\widehat{S} \cup \{v_0\}$, we have private forts F_i for elements of $\widehat{S}$ and the center-fort that does not contain any element of $\widehat{S}$ for v_0. Thus $\widehat{S} \cup \{v_0\}$ is a ZIr-set and every maximal ZIr-set has cardinality at least $k+1$. □

For $r \geq 3$, denote the two vertices in the smaller partite set of $K_{2,r}$ by u and u' and the set of vertices in the larger partite set by $W = \{w_1, \ldots, w_r\}$. For odd $s \geq 5$, denote the set of vertices of P_s by $Y = \{y_1, \ldots y_s\}$ in path order. Let $H(r,s)$ be the graph of order $r+s+1$ constructed from $K_{2,r}$ and P_s by identifying u' with y_s, retaining the label y_s (see Figure 3.2). The next example shows that all the parameters $\mathrm{zir}, \mathrm{Z}, \overline{\mathrm{Z}}$, and ZIR can be different (for example, $\mathrm{zir}(H(3,5)) = 2$, $\mathrm{Z}(H(3,5)) = 3$, $\overline{\mathrm{Z}}(H(3,5)) = 4$ and $\mathrm{ZIR}(H(3,5)) = 5$ [15]).

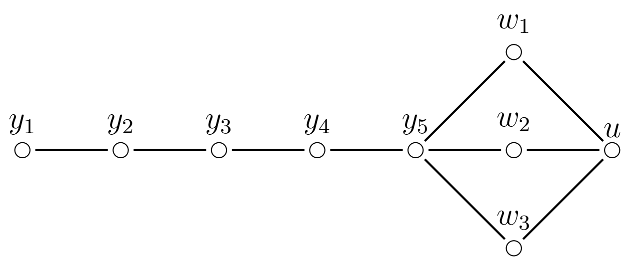

Figure 3.2: The graph $H(3,5)$, which has different values for $\mathrm{zir}, \mathrm{Z}, \overline{\mathrm{Z}}$, and ZIR

Proposition 3.6. *For $r \geq 2$ and odd $s \geq 3$, $\mathrm{zir}(H(r,s)) = 2$, $\mathrm{Z}(H(r,s)) = r$, and $\mathrm{ZIR}(H(r,s)) = r + \frac{s-1}{2}$. For $r \geq 2$ and odd $s \geq 5$, $\overline{\mathrm{Z}}(H(r,s)) = r+1$. For $r \geq 2$, $\overline{\mathrm{Z}}(H(r,3)) = r$.*

Proof. Let $r \geq 2$ and let s be odd with $s \geq 3$. By Proposition 1.7, any zero forcing set must contain at least $r-1$ of the vertices $\{w_1, \ldots, w_r\}$ since this is a set of twins. The only minimal zero forcing sets are $r-1$ of these vertices together with exactly one of the following sets: $\{u\}$, $\{y_1\}$, $\{y_s\}$, or (if $s \geq 5$) $\{y_i, y_{i+1}\}$ for $i = 2, \ldots, s-2$. Thus $\mathrm{Z}(H(r,s)) = r$; $\overline{\mathrm{Z}}(H(r,s)) = r+1$ if $s \geq 5$ and $\overline{\mathrm{Z}}(H(r,3)) = r$.

If F is a fort such that $F \cap Y \neq \emptyset$, then $y_1 \in F$ and at least every other vertex of Y is in F. For the ZIr-set $S_1 = \{u, y_1\}$, the only private fort F_u of u is $\{u, w_1, \ldots, w_r\}$, so S_1 is maximal and $\mathrm{zir}(H(r,s)) \leq 2$. Since S_1 is a ZIr-set, $\{y_1\}$ is not maximal. Since every other vertex has degree at least two, every other vertex can also be in a ZIr-set of two vertices by Proposition 2.10. Thus $\mathrm{zir}(H(r,s)) = 2$.

Let $D = \{y_1, y_3, \ldots, y_{s-2}, y_s, u\}$ and $S_2 = V(H(r,s)) \setminus D$. Every vertex $v \in S_2$ is adjacent to at least two vertices in D, so $\{v\} \cup D$ is a private fort of v and S_2 is a ZIr-set. This shows that $\mathrm{ZIR}(H(r,s) \geq r + \frac{s-1}{2}$.

Now let S be a ZIr-set. There are $r+1$ vertices not in Y, and they cannot all be in S because if $w_j \in S$ then a private fort of w_j must contain u or some w_k with $k \neq j$. So $|S \cap \{u, w_1, \ldots, w_r\}| \leq r$. If S does not contain two consecutive vertices in Y, then $|S \cap Y| \leq \frac{s-1}{2}$. So assume that S contains two consecutive vertices y_{i-1}, y_i. Then $S \cap Y \subseteq \{y_{i-1}, y_i\}$ because a fort F such that $F \cap Y \neq \emptyset$ cannot omit two consecutive vertices of Y. Thus $|S \cap Y| \leq 2 \leq \frac{s-1}{2}$. In all cases, $|S| \leq r + \frac{s-1}{2}$ □

The value of ZIR is established for additional families of graphs in Theorems 4.4, 4.8 and 5.11 and Proposition 6.4 below.

4 Bounds

In this section we use known results about 2-domination to establish bounds on $\mathrm{ZIR}(G)$ and other methods to establish additional bounds on $\mathrm{ZIR}(G)$. A 2-*dominating set* of a graph G is a subset $D \subseteq V(G)$ such that every vertex not in D is adjacent to at least two vertices in D. The minimum cardinality of a 2-dominating set is the 2-*domination number* of the graph, indicated by $\gamma_2(G)$. The idea used in the next proof to show that $n - \gamma_2(G) \leq \mathrm{ZIR}(G)$ was already used informally in Proposition 3.6 and other results.

Proposition 4.1. *Let G be a graph of order $n \geq 2$ with no isolated vertices. Then $n - \gamma_2(G) \leq \mathrm{ZIR}(G) \leq n - \gamma(G)$. Both bounds are sharp.*

Proof. Let S be ZIR-set of G such that $|S| = \mathrm{ZIR}(G)$. By Lemma 2.9, $V(G) \setminus S$ is a dominating set and therefore $\gamma(G) \leq |V(G)| - |S| = n - \mathrm{ZIR}(G)$.

Let D be a γ_2-set of G and let $S = V(G) \setminus D$. For $v \in S$ set $F_v = D \cup \{v\}$ is a private fort of v relative to S. Therefore S is a ZIr-set and $\mathrm{ZIR}(G) \geq |S|$.

Example 3.1 shows $K_{q,p}$ with $q \geq 2$ realizes equality for the upper bound and Proposition 3.2 shows C_n realizes equality for the lower bound. □

Note that the previous lower bound does not need the restriction that G has no isolated vertices.

We use known results of Caro and Roddity [11] and Cockayne et al. [12] bounding $\gamma_2(G)$ for graphs with specified minimum degree to bound $\mathrm{ZIR}(G)$ for such graphs; note that the results in [11] and [12] are more general and imply Theorems 4.2 and 4.6.

Theorem 4.2. [11] *Let G be a graph of order n. If $\delta(G) \geq 3$, then $\gamma_2(G) \leq \frac{n}{2}$.*

Corollary 4.3. *Let G be a graph of order $n \geq 2$ with $\delta(G) \geq 3$. Then $\mathrm{ZIR}(G) \geq \frac{n}{2}$.*

This bound is sharp, as is shown in the next result. A *diamond* is a K_4 with one edge deleted. Let $D_1, D_2, \ldots, D_n$ be n diamonds each with vertex set $V_i = \{a_i, b_i, c_i, d_i\}$ where the edge a_ic_i is missing. Define the graph consisting of the k diamonds together with the edges c_ia_{i+1} for $1 \leq i \leq k-1$ and c_ka_1 to be the kth *necklace*, denoted by N_k. The 3rd necklace N_3 is illustrated in Figure 4.1.

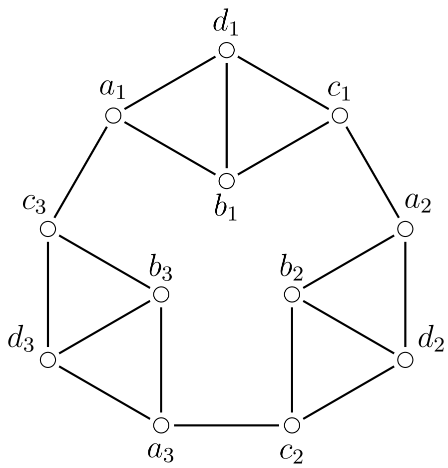

Figure 4.1: The graph N_3

Theorem 4.4. *Let $k \geq 2$. Then $\mathrm{ZIR}(N_k) = 2k = \frac{1}{2}|V(N_k)|$.*

Proof. By Corollary 4.3, $\mathrm{ZIR}(N_k) \geq 2k$. Let S be a ZIr-set of N_k. If $|V_i \cap S| \leq 2$ for all i, then $|S| \leq 2k$. Note that $|S \cap V_i| = 4$ is impossible because $V_i = N[b_i] \not\subseteq S$ by Lemma 2.9. We show that if $|S \cap V_i| = 3$, then $|S \cap V_{i-1}|, |S \cap V_{i+1}| \leq 1$. Once this is established, we conclude that $|S| \leq 2k$.

First, we assume that c_{j-1} and a_j cannot be in a private fort of the vertices in $V_j \cap S$, and show $|V_j \cap S| \leq 1$. Let $x \in \{b_j, c_j, d_j\} \cap S$ and let F_x be a private fort of x. If $F_x \cap \{b_j, d_j\} \neq \varnothing$, then a_j must be adjacent to two vertices in the fort and $c_{j-1} \notin F_x$, so $\{b_j, d_j\} \subseteq F_x$. If $x = c_j$, then $b_j \in F_x$ or b_j needs two neighbors in F_x so $d_j \in F_x$, so $\{b_j, c_j, d_j\} \subseteq F_x$. Therefore $|V_j \cap S| \leq 1$. By symmetry, if c_{j-1} and a_j cannot be in a private fort of the vertices in $V_{j-1} \cap S$, then $|S \cap V_{j-1}| \leq 1$. Thus

$$c_{j-1}, a_j \text{ cannot be in a private fort of any } v \in V_j \cap S \implies |V_j \cap S| \leq 1; \quad (4.1)$$
$$c_{j-1}, a_j \text{ cannot be in a private fort of any } v \in V_{j-1} \cap S \implies |V_{j-1} \cap S| \leq 1. \quad (4.2)$$

Now suppose that $|V_i \cap S| = 3$. First assume that $a_i, b_i, c_i \in S$. Note that d_i cannot belong to a private fort of a vertex not in V_i, since then b_i would be adjacent to only one vertex in the fort. Because d_i (and b_i) cannot be in a private fort of a vertex not in V_i, c_{i-1} cannot belong to any private fort of a vertex in outside V_i (or a_i would have only one neighbor in this fort). Thus a_i and c_{i-1} cannot belong to any private fort of a vertex in V_{i-1}, and so $|S \cap V_{i-1}| \leq 1$. By symmetry, $|S \cap V_{i+1}| \leq 1$.

Now assume that $b_i, c_i, d_i \in S$. Note that a_i cannot belong to a private fort of any vertex outside V_i, since then b_i and d_i would be adjacent to only one vertex in that fort. Furthermore, if c_{i-1} is in a private fort of a vertex not in V_i, a_i would only be adjacent to one vertex in that fort. Similarly, a_{i+1} cannot belong to a private fort of a vertex not in V_i, since then c_i would be adjacent to only one vertex in that fort. By (4.1) and (4.2), $|S \cap V_{i-1}| \leq 1$ and $|S \cap V_{i+1}| \leq 1$. □

Remark 4.5. It is known that $\mathrm{Z}(N_k) = k + 2$ [18, Theorem 9.5]. Suppose S is a minimal zero forcing set. Since b_i and d_i are twins, at least one of b_i and d_i must be in S by Proposition 1.7. In order for the forcing to start, there must be a blue vertex with two blue neighbors so there must be at least two other vertices in S, and with these vertices we have a zero forcing set. Thus $\overline{\mathrm{Z}}(N_k) = k + 2$.

Theorem 4.6. [12] *Let G be a graph of order n. If $\delta(G) = 2$, then $\gamma_2(G) \leq \frac{2n}{3}$.*

Corollary 4.7. *Let G be a graph of order $n \geq 3$ and $\delta(G) = 2$. Then $\mathrm{ZIR}(G) > \frac{n}{3}$.*

Proof. From Proposition 4.1 and Theorem 4.6 it follows that $\mathrm{ZIR}(G) \geq \frac{n}{3}$. Graphs with 2-domination number equal to $\frac{2n}{3}$ were characterized in [16] as $G = H \circ K_2$ for any graph H. We show that $\mathrm{ZIR}(G) > \frac{n}{3}$.

Label the vertices of H as $v_1, v_2, \ldots, v_{n_H}$ and the vertices of K_2 joined to vertex v_i as a_i and b_i. The set $S = \{v_1\} \cup \{a_i : i = 1, \ldots, n_H\}$ is a ZIr-set of G where the private fort of each a_i is $\{a_i, b_i\}$ and the private fort of v_1 is $V(G) \setminus \{a_i : i = 1, \ldots, n_H\}$. □

Theorem 5.11 provides a family of graphs G satisfying $\mathrm{ZIR}(G) = \frac{1}{3}|V(G)|$, but these graphs have $\delta(G) = 1$, so we do not know how close to the bound $\frac{1}{3}|V(G)|$ we can get for graphs G with $\delta(G) = 2$.

The next family of graphs realizes $\mathrm{ZIR}(G) = n - \gamma_2(G)$ and $\delta(G) = 2$. Let H_k be the graph consisting of k 5-cycles, where the vertices of the i-th cycle is labeled $v_{i,1}, v_{i,2}, \ldots, v_{i,5}$, together with the edges $v_{i,3}v_{i+1,1}$ for $i \in \{1, \ldots, k-1\}$ and the edge $v_{k,3}v_{1,1}$; see Figure 4.2.

Theorem 4.8. *The graph H_k of order $n = 5k$ has $\delta(H_k) = 2$, $\gamma_2(H_k) = \frac{3n}{5}$, and $\mathrm{ZIR}(H_k) = \frac{2n}{5} = n - \gamma_2(H_k)$. Furthermore, $\mathrm{Z}(H_k) = k + 2$.*

Proof. Let B_i denote the set of vertices of the i-th cycle of H_k. First we show that $\gamma_2(H_k) = \frac{3n}{5}$. Let D be a 2-dominating set of H_k. If $v_{i,2} \in D$, then to 2-dominate $v_{i,4}$ and $v_{i,5}$ $|D \cap B_i \setminus \{v_{i,2}\}| \geq 2$. It therefore follows that $|D \cap B_i| \geq 3$. On the other hand, if $v_{i,2} \notin D$, then certainly $v_{i,1}, v_{i,3} \in D$. Since $v_{i,4}$ and $v_{i,5}$ are not 2-dominated yet, it follows that $|D \cap \{v_{i,4}, v_{i,5}\}| \geq 1$ and so $|D \cap B_i| \geq 3$. Hence $\gamma_2(H_k) \geq \frac{3n}{5}$. Since $\{v_{i,2}, v_{i,4}, v_{i,5} : i = 1, \ldots, k\}$ is a 2-dominating set, equality follows.

Since $\mathrm{ZIR}(H_k) \geq \frac{2n}{5}$, it suffices to show that $\mathrm{ZIR}(H_k) \leq \frac{2n}{5}$. Let S be a ZIr-set of H_k. If $|S \cap B_i| \leq 2$ for every $i \in \{1, \ldots, k\}$ then the result follows. Assume therefore

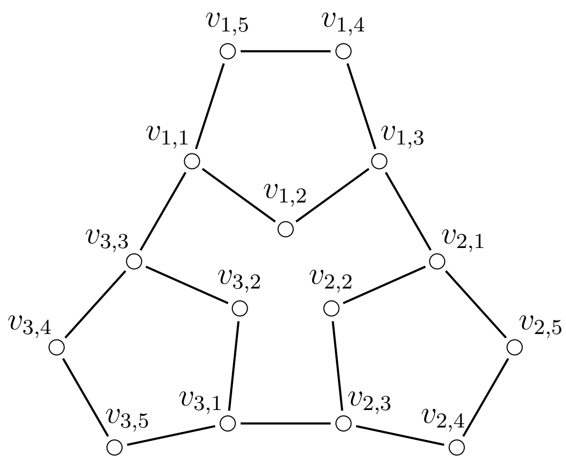

Figure 4.2: The graph H_3

that $|S \cap B_i| \geq 3$ for some $i \in \{1, \ldots, k\}$. If $|S \cap B_i| > 3$, there exists a degree-2 vertex v in B_i such that $N[v] \subseteq S$, which is impossible according to Lemma 2.9. Therefore $|S \cap B_i| = 3$. To complete the proof, we show that $|S \cap B_{i-1}| \leq 1$ and $|S \cap B_{i+1}| \leq 1$. Let $F_{i,j}$ denote a private fort of $v_{i,j}$ (relative to S).

First we examine $B_i \cap S$ further, showing $v_{i,1}$ and $v_{i,3}$ cannot both be in S. To obtain a contradiction, assume $\{v_{i,1}, v_{i,3}\} \subseteq S$. Since $N(v_{i,2}) = \{v_{i,1}, v_{i,3}\}$, $v_{i,2} \in F_{i,1}$. If $v_{i,5} \in F_{i,1}$, then $v_{i,4}$ would necessarily also be in $F_{i,1}$, contradicting $|S \cap B_i| = 3$. So $v_{i,5} \notin F_{i,1}$, and it follows that $v_{i,4} \in F_{i,1}$ to ensure that $v_{i,5}$ is adjacent to at least two vertices in $F_{i,1}$. A similar argument for a private fort $F_{i,3}$ shows that $v_{i,5} \in F_{i,3}$. In this case, $B_i \subseteq F_{i,1} \cup F_{i,3}$. Thus $|B_i \cap S| = 2$, which is a contradiction. Hence $|\{v_{i,1}, v_{i,3}\} \cap S| \leq 1$.

For the two cases $|\{v_{i,1}, v_{i,3}\} \cap S| = 0$ and $|\{v_{i,1}, v_{i,3}\} \cap S| = 1$, we prove both the following statements:

(a) There are j and j' such that $v_{i+1,1} \in F_{i,j}$, and $v_{i-1,3} \in F_{i,j'}$.

(b) $v_{i+1,1} \notin F_{i+1,j}$ for any j and $v_{i-1,3} \notin F_{i-1,j}$ for any j.

Case $|\{v_{i,1}, v_{i,3}\} \cap S| = 0$:

First, we prove (a) for both $v_{i+1,1}$ and $v_{i-1,3}$. Since $|B_i \cap S| = 3$, $B_i \cap S = \{v_{i,2}, v_{i,4}, v_{i,5}\}$. Then $v_{i,3}$ is the only neighbor of $v_{i,4}$ not in S and therefore $v_{i,3} \notin F_{i,2}$. To ensure that $v_{i,3}$ is adjacent to two vertices of $F_{i,2}$, $v_{i+1,1} \in F_{i,2}$. Similarly, $v_{i-1,3} \in F_{i,2}$.

To prove (b), suppose $v_{i+1,1} \in F_{i+1,j}$ for some $j \in \{1, \ldots, 5\}$. Since both of the neighbors of $v_{i,3}$ in B_i are in S, $v_{i,3}$ is necessarily in $F_{i+1,j}$ but then $v_{i,4}$ is not adjacent to two vertices of $F_{i+1,j}$. Therefore $v_{i+1,1} \notin F_{i+1,j}$ for any j and similarly $v_{i-1,3} \notin F_{i-1,j}$ for any j.

Case $|\{v_{i,1}, v_{i,3}\} \cap S| = 1$:

Assume that $\{v_{i,1}, v_{i,3}\} \cap S = \{v_{i,3}\}$. It follows from Lemma 2.9 that $N[v_{i,4}] \not\subseteq S$ and thus at most one of $v_{i,4}$ or $v_{i,5}$ is in S. Since $|B_i \cap S| = 3$, $v_{i,2} \in S$ and exactly one of $v_{i,4}$ or $v_{i,5}$ is in S.

First, we prove (a) for $v_{i+1,1}$. Either $v_{i,4} \in S$ or $v_{i,4} \in F_{i,5}$. In both cases $v_{i,3}$ is adjacent to a vertex in $F_{i,j}$ where $j = 4$ or 5, and therefore $v_{i+1,1} \in F_{i,j}$.

Next, we prove (a) for $v_{i-1,3}$. Let $j' = 4$ or 5, according to which of $v_{i,4}$ or $v_{i,5}$ is in S. If $v_{i,1} \in F_{i,j'}$, then $v_{i,2}$ would have only one neighbor in $F_{i,j'}$. Thus $v_{i,1} \notin F_{i,j'}$, and $v_{i-1,3} \in F_{i,j'}$ to ensure that $v_{i,1}$ is adjacent to two vertices of $F_{i,j'}$.

To prove (b), suppose $v_{i+1,1} \in F_{i+1,j}$ for some $j \in \{1, \ldots, 5\}$. Since $v_{i,2}, v_{i,3} \in S$, $v_{i,4}$ is necessarily in $F_{i+1,j}$ and therefore $v_{i,5} \in S$. To ensure that $v_{i,5}$ is adjacent to two vertices of $F_{i+1,j}$, $v_{i,1} \in F_{i+1,j}$ but then $v_{i,2}$ is only adjacent to one vertex of $F_{i+1,j}$. Therefore $v_{i+1,1} \notin F_{i+1,j}$ for any j. Now suppose that $v_{i-1,3} \in F_{i-1,j}$ for some $j \in \{1, \ldots, 5\}$. Since $v_{i,2}, v_{i,3} \in S$, $v_{i,1} \notin F_{i-1,j}$ and therefore $v_{i,5} \in F_{i-1,j}$. But then $v_{i,4} \in S$ and only adjacent to one vertex of $F_{i-1,j}$. Hence $v_{i-1,3} \notin F_{i-1,j}$ for any j.

To complete the proof we show that $|S \cap B_{i+1}| \leq 1$ and $|S \cap B_{i-1}| \leq 1$.

From (a), $v_{i+1,1} \in F_{i,\ell}$ for some specific $\ell \in \{1, \ldots, 5\}$ and it therefore follows that $|\{v_{i+1,2}, v_{i+1,3}\} \cap F_{i,\ell}| \geq 1$ and $|\{v_{i+1,4}, v_{i+1,5}\} \cap F_{i,\ell}| \geq 1$.

Now consider the vertices in $S \cap B_{i+1}$. From (b) it follows that $v_{i+1,1} \notin F_{i+1,j}$ for any $j \in \{1, \ldots, 5\}$, which implies $v_{i+1,1} \notin S$. Also we can see that $v_{i,3} \notin F_{i+1,j}$ either: If $v_{i,3} \in S$ then clearly $v_{i,3} \notin F_{i+1,j}$. So assume $v_{i,3} \notin S$ and $v_{i,3} \in F_{i+1,j}$. If $v_{i,1} \notin S$, then $v_{i,2}, v_{i,4}, v_{i,5} \in S$ and $v_{i,4}$ is adjacent to only one vertex in $F_{i+1,j}$. If $v_{i,1} \in S$, then also $v_{i,2} \in S$ and $v_{i,2}$ is adjacent to only one vertex in $F_{i+1,j}$. Thus $v_{i,3} \notin F_{i+1,j}$.

Since $v_{i,3} \notin F_{i+1,j}$ for any j, $\{v_{i+1,2}, v_{i+1,5}\} \not\subseteq S$, otherwise $v_{i+1,1}$ is adjacent to only one vertex in $F_{i+1,2}$ and $F_{i+1,5}$.

Suppose first that $\{v_{i+1,2}, v_{i+1,5}\} \cap S = \{v_{i+1,2}\}$. Then $v_{i+1,5} \in F_{i+1,2}$ to ensure that $v_{i+1,1}$ is adjacent to two vertices of $F_{i+1,2}$. Recall $v_{i+1,1} \in F_{i,\ell}$, so $v_{i+1,3} \in F_{i,\ell}$ to ensure that $v_{i+1,2}$ is adjacent to two vertices of $F_{i,\ell}$. If also $v_{i+1,4} \in S$, then $v_{i+1,5} \in F_{i+1,4}$. This is impossible since $v_{i+1,1}$ is then adjacent to only one vertex of $F_{i+1,4}$. Thus $\{v_{i+1,2}, v_{i+1,5}\} \cap S = \{v_{i+1,2}\}$ implies $|S \cap B_{i+1}| \leq 1$. If $\{v_{i+1,2}, v_{i+1,5}\} \cap S = \{v_{i+1,5}\}$ it follows similarly that $|S \cap B_{i+1}| \leq 1$.

Now, if $\{v_{i+1,2}, v_{i+1,5}\} \cap S = \emptyset$, then $|S \cap B_{i+1}| \leq 2$. Assume $|S \cap B_{i+1}| = 2$ with $v_{i+1,3}, v_{i+1,4} \in S$. Then $v_{i+1,2}, v_{i+1,5} \in F_{i,\ell}$ since $v_{i+1,1} \in F_{i,\ell}$ and $v_{i+1,3}, v_{i+1,4} \notin F_{i,\ell}$. But then $v_{i+1,4}$ is adjacent to only one vertex in $F_{i,\ell}$, a contradiction. It follows that $|S \cap B_{i+1}| \leq 1$. It follows similarly that $|S \cap B_{i-1}| \leq 1$ and hence $\mathrm{ZIR}(H_k) \geq \frac{2}{5}n$ as required.

For the zero forcing number, observe that $\{v_{1,3}, v_{1.4}\} \cup \{v_{i,2} : i = 1, \ldots, k\}$ is a zero forcing set of size $k + 2$. Since $\{v_{i,2}, v_{i,4}, v_{i,5}\}$ is a fort, a zero forcing set must contain at least one vertex of each cycle. Since $\deg(v_{i,1}) = \deg(v_{i,3}) = 3$, forcing cannot move from one cycle to the next unless some cycle has three vertices or two adjacent cycles have two each. Thus $\mathrm{Z}(H_k) = k + 2$. □

Finally, we present bounds that do not involve $\gamma_2(G)$.

Remark 4.9. Note that $\mathrm{Z}(G) \leq \mathrm{ZIR}(G)$, and it is known that $\mathrm{Z}(G(n,p)) = n - o(n)$ where $G(n,p)$ is the Erdős-Rényi random graph of order n with edge probability p [18, Section 9.11].

Proposition 4.10. *Let G be a connected graph of order $n \geq 2$. Then* $\mathrm{ZIR}(G) \leq \frac{\Delta(G)}{\Delta(G)+1} n$ *and this bound is sharp.*

Proof. Let S be a maximal ZIr-set of G. Let X denote the set of edges incident to a vertex in S and a vertex in $V \setminus S$. Since G is connected, a vertex $x \in S$ has a neighbor w. If $w \in S$, then x has another neighbor $u \in F_w$ (the private fort of w), and $u \notin S$. Thus every vertex in S is adjacent to at least one vertex not in S and therefore $|X| \geq |S|$. Furthermore, every vertex in $V \setminus S$ is adjacent to at most $\Delta(G)$ vertices of S and therefore $\Delta(G)|V \setminus S| \geq |X|$. It follows that

$$\Delta(G)(n - |S|) = \Delta(G)|V \setminus S| \geq |X| \geq |S|.$$

Since S is any maximal ZIr-set, $\mathrm{ZIR}(G) \leq \frac{\Delta(G)}{\Delta(G)+1}n$. By Example 2.12, $\mathrm{ZIR}(K_n) = n - 1 = \frac{n-1}{(n-1)+1}n$. □

Corollary 4.11. *If G is a connected cubic graph of order $n \geq 4$, then $\frac{1}{2}n \leq \mathrm{ZIR}(G) \leq \frac{3}{4}n$ and the lower bound is sharp.*

Proof. The inequalities are immediate from Corollary 4.3 and Proposition 4.10. The lower bound is sharp by Theorem 4.4. □

The only example we know where the upper bound is equality is $\mathrm{ZIR}(K_4) = 3$.

5 Extreme values of zir and ZIR

In this section we characterize graphs of order n having the extreme high values n and $n - 1$ for zir and ZIR and characterize graphs having the extreme high value $n - 2$ for zir. We also characterize graphs having the extreme low value 1 for zir and present a family of graphs with the lowest known ZIR value for arbitrarily large n.

Remark 5.1. Let G be a graph of order n. It is well known that $\mathrm{Z}(G) = n \Leftrightarrow G \cong \overline{K_n} \Leftrightarrow \overline{\mathrm{Z}}(G) = n$, and this extends to zir and ZIR: Since $\mathrm{zir}(G) = n$ implies $\mathrm{Z}(G) = n$ and $\mathrm{ZIR}(G) = n$ implies $G \cong \overline{K_n}$ by Remark 2.7, we have $G \cong \overline{K_n} \Leftrightarrow \mathrm{zir}(G) = n \Leftrightarrow \mathrm{ZIR}(G) = n$.

For a graph on n vertices, it is well known that $\mathrm{Z}(G) = n - 1$ if and only if $G \cong K_{n-r} \sqcup rK_1$ with $n - r \geq 2$. It was shown recently by Brimkov and Carlson in [6] that $\overline{\mathrm{Z}}(G) = n - 1$ if and only if $G \cong K_{n-r} \sqcup rK_1$. We extend this to zir and ZIR.

Proposition 5.2. *Let G be a graph of order $n \geq 2$. The following are equivalent:*

1. $G \cong K_{n-r} \sqcup rK_1$ *with* $n - r \geq 2$.
2. $\mathrm{zir}(G) = n - 1$.
3. $\mathrm{Z}(G) = n - 1$.
4. $\overline{\mathrm{Z}}(G) = n - 1$.
5. $\mathrm{ZIR}(G) = n - 1$.

Proof. If G does not have an edge, then $G \cong \overline{K_n}$ and $\operatorname{zir}(\overline{K_n}) = \operatorname{Z}(\overline{K_n}) = \overline{\operatorname{Z}}(\overline{K_n}) = \operatorname{ZIR}(\overline{K_n}) = n$. So assume G has an edge, so that $\operatorname{ZIR}(G) \leq n-1$ by Remark 2.7. Then with Example 2.12 and Corollary 2.3, we have that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). Since $\overline{\operatorname{Z}}(H) = |V(H)| - 1$ implies $H \cong K_{n-r} \sqcup rK_1$ with $n - r \geq 2$ for any graph H of order n [6], (4) $\Rightarrow$ (1). Thus (1) – (4) are equivalent.

So assume $\operatorname{ZIR}(G) = n-1$. Then there is a ZIr-set S such that $|S| = n-1$. Since every set of $n-1$ vertices is a zero forcing set of G, S is a minimal zero forcing set by Remark 2.1. Thus $\overline{\operatorname{Z}}(G) = n-1$. □

The next result provides many examples of graphs with $\operatorname{ZIR}(G) = |V(G)| - 2$.

Corollary 5.3. *If $G = H \vee 2K_1$ or $G = H \vee K_2$ with $H \not\cong K_{|V(H)|}$, then $\operatorname{ZIR}(G) = |V(H)| = |V(G)| - 2$. If $G = H \vee K_2$ and H has no isolated vertices, then $\operatorname{Z}(G) = \operatorname{Z}(H) + 2$.*

Proof. The statement about ZIR is immediate from the previous proposition and Proposition 4.1 (because $V(G) \setminus V(H)$ is a 2-dominating set). It is known that $\operatorname{Z}(H \vee K_1) = \operatorname{Z}(H) + 1$ for a graph H with no isolated vertices [18, Proposition 9.16]. Since $H \vee K_2 \cong (H \vee K_1) \vee K_1$, this implies $\operatorname{Z}(H \vee K_2) = \operatorname{Z}(H) + 2$ if H has no isolated vertices. □

The previous corollary does not list $\operatorname{Z}(H \vee 2K_1)$ because $\operatorname{Z}(H)+1 \leq \operatorname{Z}(H \vee 2K_1) \leq Z(H) + 2$ and both bounds are sharp (consider $H \cong K_r$ for the lower bound and $H \cong P_r$ for the upper bound). It is not true that all graphs G of order n with $\operatorname{ZIR}(G) = n-2$ have $\gamma_2(G) = 2$, as seen in the next example.

Example 5.4. Denote the vertices of the partite sets of $K_{3,4}$ by $U = \{u_1, u_2, u_3\}$ and $\{x_1, x_2, y_1, y_2\}$. Construct G from $K_{3,4}$ by adding the edges x_1y_1 and x_2y_2. See Figure 5.1. Note that no set of two vertices 2-dominates G since the degree of every vertex is four. Since U is a 2-dominating set, $\gamma_2(G) = 3$. We see that $S = \{u_1, u_2, u_3, y_1, y_2\}$ is a ZIr-set by noting that $F_{u_i} = \{u_i, x_1, x_2\}$ is a private fort of u_i for $i = 1, 2, 3$ and $F_{y_j} = \{y_j, x_j\}$ is a private fort of y_j for $j = 1, 2$. Thus $\operatorname{ZIR}(G) = 5 = |V(G)| - 2$.

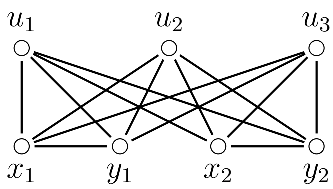

Figure 5.1: A graph G with $\operatorname{ZIR}(G) = |V(G)| - 2$ and $\gamma_2(G) = 3$.

There is a known characterization of $\operatorname{Z}(G) \geq n-2$ for graphs G of order $n \geq 3$. In [1] it is shown that $\operatorname{Z}(G) \geq n-2$ if and only if G does not contain P_4, $P_3 \sqcup K_2$, dart, $\ltimes$, or $3K_2$ as an induced subgraph. In [3] it is shown that G does not contain P_4, $P_3 \sqcup K_2$, dart, $\ltimes$, or $3K_2$ as an induced subgraph if and only if

$$G \cong \overline{(K_{s_1} \sqcup \cdots \sqcup K_{s_t} \sqcup K_{q_1,p_1} \sqcup K_{q_2,p_2} \sqcup \cdots \sqcup K_{q_k,p_k}) \vee K_r}$$

with $s_i \geq 1$, $p_i \geq 1$, $q_i \geq 0$, $t, k, r \geq 0$, and $t+k+r \geq 1$. Observe that $K_1 \cong K_{0,1}$ and $K_2 \cong K_{1,1}$, so we may assume $s_i \geq 3$. Without loss of generality we may also assume $p_i \geq q_i$ (which implies $p_i \geq 1$) and $q_i \geq q_{i+1}$. Furthermore, $K_{0,a_1} \sqcup \cdots \sqcup K_{0,a_\ell} \cong K_{0,a_1+\cdots+a_\ell}$, so we may assume $q_{k-1} \geq 1$. These results can be combined as in the next theorem.

Theorem 5.5. [1, 3] *For a graph G of order $n \geq 3$, $\mathrm{Z}(G) \geq n-2$ if and only if*

$$\overline{G} \cong (K_{s_1} \sqcup \ldots K_{s_t} \sqcup K_{q_1,p_1} \sqcup K_{q_2,p_2} \sqcup \cdots \sqcup K_{q_k,p_k}) \vee K_r$$

with $s_i \geq 3$, $p_i \geq q_i$, $q_i \geq q_{i+1}$, $p_i \geq 1$, $q_{k-1} \geq 1$ and $q_k \geq 0$, $t, k, r \geq 0$, and $t+k+r \geq 1$.

Note that $\mathrm{Z}(G) \geq |V(G)| - 2$ implies $\mathrm{ZIR}(G) \geq |V(G)| - 2$. For example, the graph G in Example 5.4 is covered by Theorem 5.5 since $\overline{G} = K_3 \sqcup K_{2,2}$. However, $\mathrm{ZIR}(G) \geq |V(G)|-2$ does not imply $\mathrm{Z}(G) \geq |V(G)|-2$, as seen in the next example.

Example 5.6. Consider the graph $G = rK_2 \vee 2K_1$ with $r \geq 3$. Corollary 5.3 establishes $\mathrm{ZIR}(G) = 2r = |V(G)| - 2$. To see that $\mathrm{Z}(G) \leq r+2$, note that a set consisting of one vertex from each K_2 and the vertices u and u' of $2K_1$ is a zero forcing set of cardinality $r+2$, so $\mathrm{Z}(G) \leq r+2 < 2r = |V(G)| - 2$. Furthermore, $\mathrm{Z}(G) = r+2$, because any zero forcing set must contain at least one vertex of each K_2 and at least one of u, u', and one more vertex (either contain both u and u' or contain both vertices of some K_2).

We use Theorem 5.5 to characterize graphs G having $\mathrm{zir}(G) = |V(G)| - 2$.

Theorem 5.7. *Let G be a graph of order $n \geq 3$. Then $\mathrm{zir}(G) = n-2$ if and only if*

$$\overline{G} \cong (K_{q_1,p_1} \sqcup K_{q_2,p_2} \sqcup \cdots \sqcup K_{q_k,p_k}) \vee K_r$$

with $p_i \geq q_i$, $q_i \geq q_{i+1}$, $p_i \geq 1$, $q_{k-1} \geq 1$ and $q_k \geq 0$, $k \geq 1$, $r \geq 0$, and

$$q_1 \geq 2 \quad \textit{or} \quad (q_1 = 1 \textit{ and } k \geq 2).$$

Proof. Note that $\mathrm{zir}(G) = n-2$ implies $\mathrm{Z}(G) \geq n-2$. Thus it suffices to assume G has the form in Theorem 5.5, and determine which of these forms have $\mathrm{zir}(G) = n-2$. Recall $\mathrm{zir}(G) = n$ if and only if $G \cong nK_1$, and $\mathrm{zir}(G) = n-1$ if and only if $G \cong K_{n-r} \sqcup rK_1$ with $n - r \geq 2$. This eliminates several cases: $t = 1, k = 0$ means $G \cong (s_1 + r)K_1$; $t = 0, k = 1, q_1 = 0$ means $G \cong K_{p_1} \sqcup rK_1$; $t = 0, k = 1, q_1 = 1$ means $G \cong K_{p_1} \sqcup (r+1)K_1$. Thus $t = 1, k = 0$ and $t = 0, k = 1, q_1 = 0, 1$ imply $\mathrm{zir}(G) > n-2$. This means G has one of the following forms:

(a) $t \geq 2$.

(b) $t = 1$ and $k \geq 1$.

(c) $t = 0$ and $q_1 \geq 2$.

(d) $t = 0$, $k \geq 2$, and $q_1 = 1$.

Observe that the presence or absence of r isolated vertices in G ($\vee K_r$ in $\overline{G}$) has no effect on whether or not $\text{zir}(G) = n - 2$, because each isolated vertex is its own private fort. Thus we assume $r = 0$.

We first assume we have one of cases (a) and (b) and show that $\text{zir}(G) < n - 2$. In both these cases, $t \geq 1$ and $s_1 \geq 3$. Let U denote the vertices of $\overline{K_{s_1}}$. Since $t \geq 2$ or $k \geq 1$, G is connected. Let $S = V(G) \setminus U$; note $S \neq \emptyset$. Since $G[U]$ is the empty graph on s_1 vertices and every vertex in U is adjacent to every vertex not in U, U is in every private fort of a vertex in S. Therefore S is a maximal ZIr-set, where $\{v\} \cup U$ is a private fort of $v \in S$ with $|S| = n - |U| < n - 2$.

Case (c) and (d) satisfy the description of G in the statement of the theorem, so it remains to show that every graph G of this form satisfies $\text{zir}(G) = n-2$. Denote the partite sets as $Z_1, \ldots, Z_{2k}$, or Z_{2k-1} if $q_k = 0$, where $|Z_{2j-1}| = p_j$ and $|Z_{2j}| = q_j$ for $j = 1, \ldots, k$. Let S be a maximal ZIr-set. We show that $|S| \geq n - 2$, which implies $\text{zir}(G) = n-2$. The only case in which a partite set is a dominating set is if $q_k = 0$; in this case each vertex of Z_{2k-1} dominates G. Suppose first that Z_{2k-1} is a dominating set and $(V(G) \setminus S) \subseteq Z_{2k-1}$. If $|V(G) \setminus S| \leq 2$, then $|S| \geq n - 2$. If $|V(G) \setminus S| \geq 3$ then S would not be maximal, because for any distinct $u, w \in Z_{2k-1}$, $V \setminus \{u, w\}$ is a ZIr-set since $\{v, u, w\}$ is a private fort of v. So assume $(V(G) \setminus S) \not\subseteq Z_{2k-1}$ or Z_{2k-1} is not a dominating set. For all i, if Z_i is not a dominating set of G, then $(V(G) \setminus S) \not\subseteq Z_i$ by Lemma 2.9. This means $(V(G) \setminus S) \not\subseteq Z_i$ for all i, so there exist distinct indices j and ℓ and vertices $z_j \in (V(G) \setminus S) \cap Z_j$ and $z_\ell \in (V(G) \setminus S) \cap Z_\ell$. We show that $V \setminus \{z_j, z_\ell\}$ is a ZIr-set, so the maximality of S implies $V \setminus \{z_j, z_\ell\} \subseteq S$ and $|S| \geq n - 2$. Let $v \neq z_j, z_\ell$ and $v \in Z_i$. If $i \neq j, \ell$, the $\{v, z_j, z_\ell\}$ is a private fort of v. If $i = j$, then $\{v, z_j\}$ is a private fort of v, and the case $i = \ell$ is similar. □

Remark 5.8. Suppose $\text{zir}(G) = 1$ and $\{x\}$ is a maximal ZIr-set. If F_x is a fort that contains x, then $F \subseteq F_x$ for every fort F that does not contain x, because if $F \not\subseteq F_x$, then there exists $y \in F \setminus\ F_x$ and $\{x, y\}$ is a ZIr-set.

Theorem 5.9. *Let G be a graph of order $n \geq 1$. Then* $\text{zir}(G) = 1$ *if and only if $G = P_n$ or $G = K_{1,n-1}$.*

Proof. If $G = P_n$ or $G = K_{1,n-1}$, then $\text{zir}(G) = 1$ by Proposition 3.3 or Example 3.1. Assume $\text{zir}(G) = 1$. If G had more than one connected component, then $\text{zir}(G) \geq 2$ by Remark 2.7, so G is connected. If $n = 1$ then $G = P_1$. So assume $n \geq 2$ and $G \neq K_{1,p}$ with $p \geq 1$. Then G has at least two vertices of degree at least two.

Since $\text{zir}(G) = 1$, there exists a maximal ZIr-set $\{x\}$. By Proposition 2.10, $\deg x \not\geq 2$ since $\{x\}$ is maximal and there are two vertices of degree two or more. Thus $\deg x = 1$.

Now suppose that G has a vertex of degree at least three. Let c be a vertex such that $\deg c \geq 3$ and $\deg w \geq 3$ implies $\text{dist}(x, w) \geq \text{dist}(x, c)$. Label the vertices on the unique path from c to x by $v_1, \ldots, v_k$ with $\text{dist}(v_i, c) = i$. Since c is a cut-vertex and $\deg c \geq 3$, $F_1 = V(G) \setminus \{c, v_1, \ldots, v_k, x\}$ is a fort. Since $x \notin F_1$ and $\{x\}$ is a maximal ZIr-set, it follows from Remark 5.8 that every fort that contains x must also contain F_1. For any vertex u, $\deg u \geq 2$ implies $F_2 = V(G) \setminus \{u\}$ is a fort that contains x. If $u \neq v_i$, then $F_1 \not\subseteq F_2$. Thus $\deg u \geq 2$ implies $u = v_i$

for some i. Since $\deg c \geq 3$, there are two vertices $w_1, w_2 \in N(c)$ with $w_1, w_2 \neq v_1$. Thus $\deg w_1 = \deg w_2 = 1$. Thus $F_3 = V(G) \setminus \{c, w_1\}$ is a fort that contains x. But $F_1 \not\subseteq F_3$, which is a contradiction. Thus G cannot have a vertex of degree greater than two. Since $\deg x = 1$, G is a path. □

Note that $\mathrm{ZIR}(G) = 1$ implies $\mathrm{Z}(G) = 1$ and thus $G \cong P_n$. But $\mathrm{ZIR}(P_n) = 1$ only when $n \leq 3$ (see Proposition 3.3 and Observation 3.4).

We do not know what the lowest value of $\mathrm{ZIR}(G)$ is over all graphs of order n as $n \to \infty$. A computer search of graphs of order $n \leq 9$ found several graphs G with order 9 and $\mathrm{ZIR}(G) = 3$, including $C_3 \circ 2K_1$. This led to Theorem 5.11 below that $\mathrm{ZIR}(C_r \circ 2K_1) = \frac{1}{3}|V(C_r \circ 2K_1)|$, the lowest known value of $\mathrm{ZIR}(G)$ among graphs G of order n as n becomes arbitrarily large. Before stating Theorem 5.11, we introduce some notation and prove a lemma. We adopt the following labeling for vertices of $H \circ tK_1$ where r is the order of H: Label the vertices of H as $u_0, \ldots, u_{r-1}$ and let the set of leaves adjacent to u_i be denoted by $L_i = \{v_{i,j} : j = 1, \ldots, t\}$ for $i = 0, \ldots, r-1$. Furthermore, let $W_i = \{u_i\} \cup L_i$ for $i = 0, \ldots, r-1$.

Lemma 5.10. *Let H be a connected graph of order $r \geq 3$, let $t \geq 2$, let $G = H \circ tK_1$, and let S be a ZIr-set of G. Then G has a ZIr-set S' such that $|S| = |S'|$ and every vertex of S' is a leaf.*

Proof. Let S be a ZIr-set and let $S_i = S \cap W_i$. Note that if F is a fort and $u_i \in F$, then $W_i \subseteq F$. So $u_i \in S$ implies $v_{i,j} \notin S$ for $j = 1, \ldots, t$ because none of these vertices could have a private fort relative to S. If $u_i \in S$, we wish to replace u_i by an element of L_i in S to obtain a ZIr-set of the same cardinality with one fewer vertex of H. Repeating this process as needed will establish the result. To replace u_i by $v_{i,1}$, we need to verify that we can adjust other private forts if needed so that $v_{i,1}$ is not in the private fort of any element of S except u_i. Let $x \in S_k$ with $k \neq i$. Choose a minimal private fort F_x. We show that minimality implies $|F_x \cap L_i| \leq 1$: Note that $u_i \notin F_x$ since $u_i \in S$. The only reason for any $v_{i,j}$ to be in F_x is that u_i has exactly one neighbor in F_x that is not in L_i. Thus including one leaf in L_i is sufficient. If $F_x \cap L_i \neq \emptyset$, without loss of generality we modify F_x so that $F_x \cap L_i = \{v_{i,t}\}$. Define $S' = S \setminus \{u_i\} \cup \{v_{i,1}\}$. Then S' is a ZIr-set with private forts $\{v_{i,1}, v_{i,t}\}$ for $v_{i,1}$ and F_x for $x \notin W_i$. □

Note that maximality need not be preserved in moving to an all-leaf ZIr-set of the same cardinality.

Theorem 5.11. *Let $r \geq 3$. Then*

$$\mathrm{Z}(C_r \circ 2K_1) = \overline{\mathrm{Z}}(C_r \circ 2K_1) = \mathrm{ZIR}(C_r \circ 2K_1) = r = \frac{1}{3}|V(K_r \circ 2K_1)|.$$

Proof. In $G = C_r \circ 2K_1$, we modify the labeling convention to label the two leaf neighbors of u_i by x_i and y_i. Index arithmetic is done in $\mathbb{Z}_r$, i.e., modulo r. Observe that x_i and y_i are twins for $i = 0, \ldots, r-1$, so at least one of them must be a zero forcing set. Thus $r \leq \mathrm{Z}(G) \leq \overline{\mathrm{Z}}(G) \leq \mathrm{ZIR}(G)$.

Now we show that $\mathrm{ZIR}(G) \leq r$. Let $W_i = \{u_i, x_i, y_i\}$. Let S be a ZIr-set and let $S_i = S \cap W_i$. By Lemma 5.10, we may replace any occurrence of u_i in S by x_i and assume S consists entirely of leaves. Thus the possible values of $s = |S_i|$ are 0, 1, or 2. For $s = 0, 1, 2$ define $T_s = \{i : |S_i| = s\}$ and observe that $r = |T_2| + |T_1| + |T_0|$ and $|S| = 2|T_2| + |T_1|$. We show that $|T_2| \leq |T_0|$, which implies that $|S| \leq r$. For each $z \in S$ we choose a minimal private fort F_z. If $z \in S_i$, $u_j \in F_z$, and $j \neq i$, then $j \in T_0$ because $u_j \in F_z$ implies $W_j \subseteq F_z$.

If $T_2 = \emptyset$, then there is nothing to prove, so without loss of generality (by renumbering if necessary) assume $0 \in T_2$, so $S_0 = \{x_0, y_0\}$. Note that $y_0 \notin F_{x_0}$, which implies $u_0 \notin F_{x_0}$. Thus u_0 needs at least two neighbors in F_{x_0}, so one of u_{r-1} and u_1 must be in F_{x_0}. Without loss of generality, assume that $u_1 \in F_{x_0}$, which implies $1 \in T_0$. We proceed in increasing order around the cycle. If there are two consecutive cycle vertices $u_j, u_{j+1} \notin F_{x_0}$, then we stop as soon as this is encountered and set $j_0 = j$. If we do not stop because there is no j such that $u_j, u_{j+1} \notin F_{x_0}$, then at least half of the cycle vertices u_j are in F_{x_0}. Thus $|T_0| \geq \frac{r}{2} \geq |T_2|$.

For the remainder of the proof we assume F_z omits two consecutive cycle vertices for each $z \in S$. The assumption that $u_1 \in F_{x_0}$ and we have stopped with $u_{j_0}, u_{j_0+1} \notin F_{x_0}$ and $u_{j_0-1} \in F_{x_0}$ implies at least one of x_{j_0} or y_{j_0} is in F_{x_0}. Define $I_0 = \{0, 1, \ldots, j_0 - 1\}$. Observe that $|I_0 \cap T_0| \geq |I_0 \cap T_2|$ and $j_0 \notin T_2$.

We start with the interval $I = I_0$ and expand as needed through the process described next, such that the interval $I = [0, b]$ retains the following properties (which are true for I_0):

1. $|I \cap T_0| \geq |I \cap T_2|$.

2. If $b \in T_2$, then $b - 1 \in I \cap T_0$. If $b \in T_0$, then $b + 1 \notin T_2$.

If $T_2 \subset I$, then the proof is complete by condition (1). If not, let p be the first element of T_2 that is not in I starting at b and proceeding in increasing order. We examine a private fort F_{x_p}. As in the construction of I_0, one of u_{p-1} or u_{p+1} is in F_{x_p}. If $u_{p-1} \in F_{x_p}$, then update the interval to $I = [0, \ldots, p]$ and observe that the conditions (1) and (2) remain true because $p-1 \in T_0$ and $j \notin T_2$ for $j = b+1, \ldots, p-2$ by the way p was chosen. So suppose $u_{p-1} \notin F_{x_p}$, which implies $u_{p+1} \in F_{x_p}$. We proceed in increasing order around the cycle. If for all $j = p+2, \ldots, r-2$, $u_j \notin F_{x_p}$ implies $u_{j+1} \in F_{x_p}$, then $I = \{0, \ldots, r-1\}$ satisfies (1) and the proof is complete. So let j be the first index (in increasing order) such that $u_j, u_{j+1} \notin F_{x_p}$. Then, update the interval to $I = [0, \ldots, j-1]$ and observe that the conditions (1)–(2) remain true. This completes the proof. □

6 Effect of graph operations on upper ZIr number

In this section we examine the effect of graph operations in ZIr-sets and ZIr numbers, primarily the upper ZIr number. We first consider the effect of removing a cut-vertex.

Proposition 6.1. *Let G be a connected graph such that c is a cut-vertex of G and $G - c$ has $\ell \geq 3$ components, denoted by $H_1, \dots, H_\ell$ and ordered so that $\mathrm{ZIR}(H_1) \geq \cdots \geq \mathrm{ZIR}(H_\ell)$. For $i = 1, \dots, \ell$, let S_i be a ZIr-set for H_i. Then $\bigcup_{i \neq k} S_i$ is a ZIr-set for $k = 1, \dots, \ell$ and $\mathrm{ZIR}(G) \geq \mathrm{ZIR}(H_1) + \cdots + \mathrm{ZIR}(H_{\ell-1})$. This bound is sharp.*

Proof. Fix $k \in \{1, \dots, \ell\}$ and let $S = \bigcup_{i \neq k} S_i$. For each $x \in S$, there is an $i \neq k$ such that $x \in S_i$. Then x has a private fort F in H_i. If $|F \cap N(c)| = 0$ or $|F \cap N(c)| \geq 2$, then F is a private fort of x in G relative to S. If $|F \cap N(c)| = 1$, then $F' = F \cup V(H_k)$ is a private fort of x in G relative to S. Since every $x \in S$ has a private fort of x in G relative to S, S is a ZIr-set, and by choosing $k = \ell$ we have $\mathrm{ZIR}(G) \geq \mathrm{ZIR}(H_1) + \cdots + \mathrm{ZIR}(H_{\ell-1})$.

To see that the bound is sharp, consider the graph $G = C_r \circ 2K_1$. Each cycle vertex is a cut-vertex with components $H_1 \cong P_{r-1} \circ 2K_1$, $H_2 \cong K_1$, and $H_3 \cong K_1$. By Theorem 5.11, $\mathrm{ZIR}(C_r \circ 2K_1) = r$. Furthermore, $\mathrm{Z}(P_{r-1} \circ 2K_1) = r - 1$ since $P_{r-1} \circ 2K_1$ is a tree. Thus $\mathrm{ZIR}(C_r \circ 2K_1) = r \geq \mathrm{ZIR}(H_1) + \mathrm{ZIR}(H_2) = \mathrm{ZIR}(P_{r-1} \circ 2K_1) + \mathrm{ZIR}(K_1) \geq \mathrm{Z}(P_{r-1} \circ 2K_1) + \mathrm{ZIR}(K_1) = (r - 1) + 1 = r$. □

We now consider the join of two graphs $G \vee H$. Proposition 6.2 handles the more general case and Proposition 6.3 is needed for the special case of $H = K_1$.

Proposition 6.2. *Let G and H be graphs on $n_G \geq 2$ and $n_H \geq 2$ vertices, respectively. Then $n_G + n_H - 4 \leq \mathrm{ZIR}(G \vee H) \leq n_G + n_H - 1$. Both bounds are sharp. In particular, $\mathrm{ZIR}(P_{n_G} \vee P_{n_H}) = n_G + n_H - 4$ for $n_G, n_H \geq 7$.*

Proof. Let $D \subseteq V(G \vee H)$ such that $|D \cap V(G)| = 2$ and $|D \cap V(H)| = 2$. Then D is a 2-dominating set. By Proposition 4.1, $\mathrm{ZIR}(G \vee H) \geq n_G + n_H - 4$. Since $G \vee H$ has an edge, Remark 2.7 implies $\mathrm{ZIR}(G \vee H) \leq n_G + n_H - 1$.

The upper bound is realized by $G \cong K_{n_G}$ and $H \cong K_{n_H}$. We now show that the lower bound is realized by $G \cong P_{n_G}$ and $H \cong P_{n_H}$ for $n_G \geq 7$ and $n_H \geq 7$. Let $v_1, \dots, v_{n_G}$ and $u_1, \dots, u_{n_H}$ denote the vertices of G and H, respectively, where v_i (respectively u_i) is adjacent to v_j (respectively u_j) if and only if $i = j \pm 1$. Assume, by way of contradiction, that $\mathrm{ZIR}(G \vee H) \geq n_G + n_H - 3$. Then $G \vee H$ has a ZIr-set S of size $n_G + n_H - 3$.

First, suppose $V(G) \subseteq S$. Observe that every private fort of every vertex in $S \cap V(H)$ relative to S is a fort of H, and therefore contains u_1 and u_{n_H}. Thus, $u_1, u_{n_H} \notin S$ and so there exists some $j \in \{2, n_G - 1\}$ such that $u_j \in S$. This is a contradiction since every fort in H containing u_j has size at least 5, which implies $|S| \leq n_G + n_H - 5 + 1$. Thus $V(G)$ is not a subset of S. Similarly, $V(H)$ is not a subset of S.

Without loss of generality, assume $|V(G) \setminus S| = 1$ and $|V(H) \setminus S| = 2$. Since $n_H \geq 7$, there exists some $u_j \in V(H)$ such that $N[u_j] \cap V(H) \subseteq S$, and there exists some $u_k \in S \cap V(H)$ such that u_k is not adjacent to u_j and $j \neq k$. Let F_{u_k} be a private fort of u_k relative to S. Since every fort of H is a dominating set of H, F_{u_k} is not a fort of H. Thus, $F_{u_k} \cap V(G) = \{v_\ell\}$ where $V(G) \setminus S = \{v_\ell\}$. This is a contradiction since $N[u_j] \cap F_{u_k} = \{v_\ell\}$. Thus, $\mathrm{ZIR}(G \vee H) = n_G + n_H - 4$. □

The lower bound in the previous result can fail when $n_G = 1$ and $n_H = 2$, as illustrated by the friendship graph $\mathrm{Fr}(k) = K_1 \vee kK_2$ since $|V(\mathrm{Fr}(k))| = 2k+1$ and $\mathrm{ZIR}(\mathrm{Fr}(k)) = k+1$ by Proposition 3.5.

Proposition 6.3. *Let G be a graph of order n_G with no isolated vertices. Then*

$$n_G - \gamma(G) \le \mathrm{ZIR}(G \vee K_1) \le n_G - \gamma(G) + 1. \tag{6.1}$$

If $\mathrm{ZIR}(G) = n_G - \gamma(G)$, *then* $\mathrm{ZIR}(G \vee K_1) = n_G - \gamma(G) + 1 = \mathrm{ZIR}(G) + 1$. *Both bounds in* (6.1) *are sharp.*

Proof. Let u denote the vertex of K_1, let S be a maximal ZIr-set of G, and let D be a dominating set of G. Then $D \cup \{u\}$ is a 2-dominating set of $G \vee K_1$, so $\gamma_2(G \vee K_1) \le \gamma(G) + 1$ and $\mathrm{ZIR}(G \vee K_1) \ge (n_G + 1) - (\gamma(G) + 1) = n_G - \gamma(G)$ by Proposition 4.1.

Let $\hat{S}$ be a ZIR-set of $G \vee K_1$. If u is in a private fort F of some vertex in $\hat{S}$, then F contains at least $\gamma(G)$ vertices of G since u is adjacent to every vertex in G. Therefore $|\hat{S}| \le (|V(G)|+1) - |F| + 1 \le n_G + 1 - (\gamma(G)+1) + 1 = n_G - \gamma(G) + 1$. If u is not in any private fort of a vertex in $\hat{S}$, then $u \notin \hat{S}$ and every private fort relative to $\hat{S}$ in $G \vee K_1$ is also a private fort relative to $\hat{S}$ in G. Therefore $\hat{S}$ is a ZIr-set of G, and $\mathrm{ZIR}(G \vee K_1) = |\hat{S}| \le \mathrm{ZIR}(G) \le n_G - \gamma(G)$ by Proposition 4.1. Thus in either case $\mathrm{ZIR}(G \vee K_1) \le n_G - \gamma(G) + 1$.

By Lemma 2.9, $V(G) \setminus S$ is a dominating set of G (recall S is a maximal ZIr-set of G). Thus, $\{u\} \cup (V(G) \setminus S)$ is a private fort of u in $G \vee K_1$ relative to $S \cup \{u\}$. So $\mathrm{ZIR}(G \vee K_1) \ge \mathrm{ZIR}(G) + 1$. Thus $\mathrm{ZIR}(G) = n_G - \gamma(G)$ implies $\mathrm{ZIR}(G \vee K_1) \ge n_G - \gamma(G) + 1$.

The complete graph $G = K_n$ shows the upper bound is sharp since $n_G = n$, $\gamma(K_n) = 1$, $K_n \vee K_1 \cong K_{n+1}$, and $\mathrm{ZIR}(K_{n+1}) = n$. By Corollary 5.3, the graph $C_r \vee K_2 \cong (C_r \vee K_1) \vee K_1$ has $\mathrm{ZIR}(C_r \vee K_2) = r = (r+1) - 1 = |V(C_r \vee K_1)| - \gamma(C_r \vee K_1)$. □

The *wheel* graph of order $r+1$ is $W_{r+1} = C_r \vee K_1$.

Proposition 6.4. *Let $r \ge 5$. Then* $\mathrm{ZIR}(W_{r+1}) = r - \gamma(C_r) = r - \left\lceil \frac{r}{3} \right\rceil$ *and* $\mathrm{zir}(W_{r+1}) = \mathrm{Z}(W_{r+1}) = \overline{\mathrm{Z}}(W_{r+1}) = 3$.

Proof. To establish $\mathrm{ZIR}(W_{r+1}) = r - \gamma(C_r)$, it suffices to show that $\mathrm{ZIR}(W_{r+1}) \le r - \gamma(C_r)$ by Proposition 6.3. Let u denote the vertex of K_1, let $\hat{S}$ be a maximal ZIr-set of W_{r+1}, and for $x \in \hat{S}$, let F_x denote private fort of x with respect to $\hat{S}$ in W_{r+1}.

Suppose first that $u \notin F_y$ for some $y \in \hat{S} \cap V(C_r)$. Then F_y is a private fort of y in C_r relative to $\hat{S} \setminus \{u\}$. Note that every fort of C_r contains at least $\left\lceil \frac{r}{2} \right\rceil$ vertices. If $u \notin \hat{S}$, then $|\hat{S}| \le r - \left\lceil \frac{r}{2} \right\rceil + 1 = \left\lfloor \frac{r}{2} \right\rfloor + 1$. If $u \in \hat{S}$, then $u \notin F_y$ for every $y \in \hat{S} \cap C_r$. Thus $\hat{S} \setminus \{u\}$ is a ZIr-set of C_r and $|\hat{S}| \le \mathrm{ZIR}(C_r) + 1 = \left\lfloor \frac{r}{2} \right\rfloor + 1$. Thus in both cases $u \notin \hat{S}$ and $u \in \hat{S}$, $|\hat{S}| \le \left\lfloor \frac{r}{2} \right\rfloor + 1 \le r - \left\lceil \frac{r}{3} \right\rceil$ for $r \ge 5$, where the last inequality is verified algebraically.

Let $\hat{S} \cap V(C_r) = \{y_1, \ldots, y_\ell\}$ and suppose that $u \in F_{y_i}$ for $i = 1, \ldots, \ell$ (which implies $u \notin \hat{S}$). It suffices to show that $\ell \leq r - \gamma(C_r)$. Since $u \in F_{y_i}$, F_{y_i} must contain a dominating set for C_r for every y_i. Recall that $|\hat{S}| \leq r+1-|\cup_{i=1}^k F_{y_i}|+k = r - |\left(\cup_{i=1}^k F_{y_i}\right) \cap V(C_r)| + k$ for $k = 1, \ldots, \ell$ (Observation 2.13). In order to have $|\hat{S}| = r - |\left(\cup_{i=1}^k F_{y_i}\right) \cap V(C_r)| + k$ there must be a set $\hat{D}$ such that $\hat{D} \cup \{y_i\}$ is a minimum dominating set for $i = 1, \ldots, \ell$. Since $r \geq 5$, there does not exist such a set and therefore $|\hat{S}| < r - |\left(\cup_{i=1}^k F_{y_i}\right) \cap V(C_r)| + k \leq r - \gamma(C_r) + 1$.

It is well known that $\mathrm{Z}(W_{r+1}) = 3$ and easy to see that $\overline{\mathrm{Z}}(W_{r+1}) = 3$ since any zero forcing set must contain two consecutive cycle vertices and u, or three consecutive cycle vertices, and any set of this type is minimal. By Corollary 2.11, $3 = \delta(W_{r+1}) = 3 \leq \mathrm{zir}(W_{r+1})$. □

We now consider the corona product of two graphs. The following notational conventions will be used throughout the rest of this section. Let G and H be graphs on n_G and n_H vertices, respectively. Let $v_1, \ldots, v_{n_G}$ denote the vertices of G. For $i = 1, \ldots, n_G$ let H_i denote the copy of H in $G \circ H$ whose vertices are adjacent to v_i, and let H_i' denote the subgraph of $G \circ H$ induced by $V(H_i) \cup \{v_i\}$, which is isomorphic to $H \vee K_1$. The next result states basic facts about forts and ZIr-sets of coronas $G \circ H$ when H has no isolated vertices.

Proposition 6.5. *Let G be a graph, let H be a graph with no isolated vertices, and let F be a fort of $G \circ H$. Then $F_i' = F \cap V(H_i')$ is the empty set or a fort of H_i'. If S is a ZIr-set of $G \circ H$, then $S_i' = S \cap V(H_i')$ is a ZIr-set of H_i' and $S = \cup_{i=1}^{n_G} S_i'$. Thus* $\mathrm{ZIR}(G \circ H) \leq n_G \,\mathrm{ZIR}(H \vee K_1)$.

Proof. Since H has no isolated vertices, every fort of H has at least two elements. Thus a fort of H_i is also a fort of H_i'. Assume $F_i' \neq \emptyset$. If $v_i \in F_i'$, then F_i' is a fort of H_i'. If $v_i \notin F_i'$, then F_i' is a fort of H_i, so is also a fort of H_i'. The remaining statements follow from this property of forts. □

Before proving our first result on the corona product of two graphs we introduce some new terminology. Let S_0 be a maximal ZIr-set of a graph G. An *abandoned fort* relative S_0 is a fort in G that contains no elements of S_0 and S_0 *abandons* a fort if there exists an abandoned fort relative to S_0. We say that G *abandons* a fort if there exists an upper ZIR set S that abandons a fort.

Example 6.6. Many abandoned forts naturally occur as a result of 2-dominating sets. This is not surprising since a fort is a 2-dominating set of its closed neighborhood. Examples we have already considered include $P_n, n \geq 3$ and $C_n, n \geq 4$. The two vertices not in H are an abandoned fort relative to $S = V(H)$ for $H \vee 2K_1$, or $H \vee K_2$ when $H \ncong K_r$ (cf. Corollary 5.3). Figure 6.1 provides an example of a connected graph that abandons a fort that is not a 2-dominating set: the upper ZIR set $\{v_1, v_4, v_5\}$ abandons the fort $\{v_2, v_3\}$, which is not 2-dominating.

A single vertex forms a fort if and only if that vertex is isolated. Thus, we have the following observation.

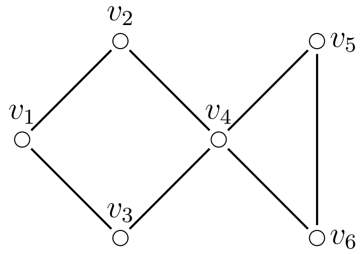

Figure 6.1: A graph with an upper ZIR-set that abandons a fort that is not 2-dominating.

Observation 6.7. *If a maximal ZIr-set abandons a fort F, then $|F| \geq 2$.*

Theorem 6.8. *Let G and H be graphs on n_G and n_H vertices, respectively. Assume H has no isolated vertices and $H \vee K_1$ abandons a fort. Then* $\mathrm{ZIR}(G \circ H) = n_G \, \mathrm{ZIR}(H \vee K_1)$.

Proof. By Remark 6.5, $\mathrm{ZIR}(G \circ H) \leq n_G \, \mathrm{ZIR}(H \vee K_1)$. Equality is obtained by constructing an appropriately sized ZIr-set.

Let $S = S_1 \cup \cdots \cup S_{n_G}$, where each S_i is an upper ZIR set of H_i' that abandons a fort F_i. Without loss of generality assume that either $v_i \in F_i$ for every $i = 1, \ldots, n_G$, or $v_i \notin F_i$ for every $i = 1, \ldots, n_G$. Let $u \in S_k$ for some $k \in \{1, \ldots, n_G\}$ and let F be a private fort of u relative to S_k in H_k'. It suffices to show u has a private fort relative to S in $G \circ H$.

First, suppose $v_k \in F$. If $v_i \in F_i$ for every $i = 1, \ldots, s$, then the union of F with every $F_i \neq F_k$ is a private fort of u relative to S in $G \circ H$. If $v_i \notin F_i$ for every $i = 1, \ldots, s$, then the union of F with each F_i such that $v_i \in N(v_k)$ is a private fort of u relative to S in $G \circ H$.

Now suppose $v_k \notin F$. Then every vertex in $V(G \circ H) \setminus V(H_k')$ is adjacent to no vertices in F. Thus, F is a private fort of u relative to S in $G \circ H$. Thus S is a ZIr-set of $G \circ H$ and so $\mathrm{ZIR}(G \circ H) \geq |S| = n_G \, \mathrm{ZIR}(H \vee K_1)$. □

Remark 6.9. Let G be a graph and let S be a maximal ZIr-set set of G. Then S abandons a fort if and only if S is not a zero forcing set, because S is a zero forcing set if and only if it intersects every fort by Theorem 1.1. Furthermore, if G does not abandon a fort, then every upper ZIR set is a zero forcing set, so $\mathrm{ZIR}(G) = \overline{\mathrm{Z}}(G)$. Thus Theorem 6.8 applies to any graph H with no isolated vertices such that $\overline{\mathrm{Z}}(H \vee K_1) < \mathrm{ZIR}(H \vee K_1)$.

The next two examples provide families of graphs H to which the previous theorem applies.

Example 6.10. If $\mathrm{ZIR}(H \vee K_1) = n_H - \gamma(H)$ and $H \ncong K_r$, then $H \vee K_1$ abandons a fort: Let D_H be a dominating set of H and let u be the vertex of K_1. Then the 2-dominating set $F = D_H \cup \{u\}$ is an abandoned fort of the upper ZIR set $S = V(H) \setminus D_H$ for $H \vee K_1$. In particular, $W_{r+1} = C_r \vee K_1$ and $\mathrm{ZIR}(W_{r+1}) = |V(C_r)| - \gamma(C_r)$ by Proposition 6.4. Thus $\mathrm{ZIR}(G \circ C_r) = n_G(r - \lceil \frac{r}{3} \rceil)$ for $r \geq 4$.

Example 6.11. Let $\hat{H}$ be a graph that is not complete. Define $H = \hat{H} \vee K_1$. Since $\hat{H}$ is not complete, neither is $H \vee K_1$, so $\mathrm{ZIR}(H \vee K_1) \leq |V(H \vee K_1)| - 2 = |V(\hat{H})|$. As noted in Example 6.6, $F = V(H \vee K_1) \setminus V(\hat{H})$ is an abandoned fort relative to the upper ZIR set $S = V(\hat{H})$.

Next, we present an example of a graph H such that H does not abandon a fort but $H \vee K_1$ does abandon a fort.

Example 6.12. Recall that $\mathrm{Fr}(k) = K_1 \vee kK_2$ denotes the kth friendship graph. It was shown in Proposition 3.5 that any upper ZIR set S of $\mathrm{Fr}(k)$ must contain v_0 (the vertex of degree $2k$. Any fort F_{v_0} of $\mathrm{Fr}(k)$ that contains v_0 also contains at least one vertex from each K_2. However, S must contain a vertex from every K_2 or it would not be maximal. Since every fort of $\mathrm{Fr}(k)$ contains v_0 or S contains either 0 or 2 for each K_2, $\mathrm{Fr}(k)$ does not abandon a fort. Since $\mathrm{Fr}(k) \vee K_1 \cong 2K_2 \vee K_2$, $\mathrm{Fr}(k) \vee K_1$ abandons a fort.

We now analyze $G \circ H$ when $H \vee K_1$ does not abandon a fort. Recall that when H has no isolated vertices, every fort of H contains at least two vertices, which implies the next observation.

Observation 6.13. *If H contains no isolated vertices, then every fort of H_i is a fort of $G \circ H$.*

Observation 6.14. *If $n_H \geq 2$, then $\mathrm{ZIR}(G \circ H) \geq n_G$ because $V(H_i)$ is a fort of $G \circ H$ for $i = 1, \ldots, n_G$. This bound is sharp since $\mathrm{ZIR}(C_r \circ 2K_1) = r$ by Theorem 5.11.*

Our next result is a lower bound that applies when assuming only that H has no isolated vertices (although Theorem 6.8 determines $\mathrm{ZIR}(G \circ H)$ when $H \vee K_1$ abandons a fort).

Proposition 6.15. *Let G and H be graphs on n_G and n_H vertices, respectively. If H contains no isolated vertices, then $\mathrm{ZIR}(G \circ H) \geq \mathrm{ZIR}(G)\,\mathrm{ZIR}(H \vee K_1) + (n_G - \mathrm{ZIR}(G))\,\mathrm{ZIR}(H)$. The bound is sharp.*

Proof. Let R be a ZIr-set of G such that $|R| = \mathrm{ZIR}(G)$. Without loss of generality $R = \{v_1, \ldots, v_k\}$, where $k = \mathrm{ZIR}(G)$. For $i = 1, \ldots, k$ let F_i denote a private fort of v_i relative to S in G, and let S_i be an upper ZIR set of H_i'. For $i = k+1, \ldots, n_G$ let S_i be an upper ZIR set of H_i, and let $T_i = V(H_i) \setminus S_i$. Let $S = S_1 \cup \cdots \cup S_{n_G}$ and let $u \in S$. Then $u \in S_j$ for some $j \in \{1, \ldots, n_G\}$.

First, assume $j \in \{k+1, \ldots, n_G\}$. Then u has a private fort F relative to S_j in H_j. By Observation 6.13, F is a private fort of u relative to S in $G \circ H$.

Now assume $j \in \{1, \ldots, k\}$. Then u has a private fort F relative to S_j in H_j'. If $v_j \notin F$, then F is a private fort of u relative to S in $G \circ H$. So, suppose $v_j \in F$ and let T be the union of each T_i such that $v_i \in F_j$. Then $F \cup F_j \cup T$ is a private fort of u relative to S in $G \circ H$. Thus, S is a ZIr-set and $\mathrm{ZIR}(G \circ H) \geq |S| = \mathrm{ZIR}(G)\,\mathrm{ZIR}(H \vee K_1) + (n_G - \mathrm{ZIR}(G))\,\mathrm{ZIR}(H)$.

When $G = K_1$, $G \circ H = H \vee K_1$ for any H. Thus $\mathrm{ZIR}(G)\,\mathrm{ZIR}(H \vee K_1) + (n_G - \mathrm{ZIR}(G))\,\mathrm{ZIR}(H) = (1)\,\mathrm{ZIR}(H \vee K_1) + (1-1)\,\mathrm{ZIR}(H) = \mathrm{ZIR}(H \vee K_1) = \mathrm{ZIR}(G \circ H)$. □

Although Proposition 6.15 excluded isolated vertices for H, it is easy to compute $\mathrm{ZIR}(G \circ K_1)$, as in the next example.

Example 6.16. Let G be a graph of order r. We can see that the set $S = V(G)$ is a ZIr-set of cardinality r: Let $u_i \in V(H_i)$ be the unique neighbor of v_i that is not in $V(G)$. Then $F_i = \{v_i\} \cup \{u_j : v_j \in N[v_i]\}$ is a private fort of v_i. By Proposition 4.1 $\mathrm{ZIR}(G \circ K_1) \le 2r - \gamma(G \circ K_1) = 2r - r = r$, so $\mathrm{ZIR}(G \circ K_1) = r$. It is known (and easy to see, e.g., by considering consider C_{4k} vs. K_r) that $\mathrm{Z}(G \circ K_1)$ depends on G.

The next result may be useful in the case where $H \vee K_1$ does not abandon a fort and H has one or more isolated vertices.

Proposition 6.17. *Let G be a graph on n_G vertices and let H be a graph. Then* $\mathrm{ZIR}(G \circ H) \ge \alpha(G)\,\mathrm{ZIR}(H \vee K_1) + (n_G - \alpha(G))(\mathrm{ZIR}(H) - 1)$ *and this bound is sharp.*

Proof. Let A be an independent set of G such that $|A| = \alpha(G)$. For each $v_i \in A$, let S'_i be an upper ZIR set of H'_i. For each $v_i \in V(G) \setminus A$, let R_i be an upper ZIR set of H_i. Let $u_i \in R_i$ with private fort F_i relative to R_i in H_i. Then $S_i = R_i \setminus \{u_i\}$ abandons the fort F_i in H_i.

Let $S = (\bigcup_{i:v_i \in A} S'_i) \cup (\bigcup_{i:v_i \notin A} S_i)$. We show that S is a ZIr-set of $G \circ H$. Let $u \in S$. Suppose first that $u \in S_j$ for some $j = 1, \ldots, n_G$. Then u has a private fort F relative to S_j in H_j and hence, $F \cup F_j \subseteq V(H_j)$ is a private fort of u relative to S_j in H'_j because $|F \cup F_j| \ge 2$. Now suppose $u \in S'_j$ for some $j = 1, \ldots, n_G$. If u has a private fort $F \subseteq V(H_j)$ relative to $S \cap V(H'_j)$ in H'_j, then F is a private fort of u relative to S in $G \circ H$. So, suppose this is not the case. Then u has a private fort F relative to $S \cap V(H'_j)$ that contains v_j. Then the union of F with every F_i such that $v_i \in N(v_j)$ is a private fort of u relative to S in $G \circ H$. Thus, S is a ZIr-set and $\mathrm{ZIR}(G \circ H) \ge \alpha(G)\,\mathrm{ZIR}(H \vee K_1) + (n_G - \alpha(G))(\mathrm{ZIR}(H) - 1)$.

For sharpness, consider any graph G of order n_G and $H = sK_1$ with $s > n_G$. Then $\mathrm{ZIR}(H) = s$ and $\mathrm{ZIR}(H \vee K_1) = \mathrm{ZIR}(K_{1,s}) = s - 1$. It follows that $\mathrm{ZIR}(G \circ H) \ge \alpha(G)(s-1) + (n_G - \alpha(G))(s-1) = n_G\,\mathrm{ZIR}(H \vee K_1) = n_G(s-1)$. Let S be an upper ZIR set of $G \circ H$. Note that if $v_i \in S$, then $S \cap V(H'_i) = \{v_i\}$ and therefore $|S \cap V(H'_i)| \le s$. If there exist an H'_i such that $|S \cap V(H'_i)| = s$, then $V(H_i) \subseteq S$ and v_i is in the private fort F_v of every $v \in S \cap H_i$. To ensure that v_i is adjacent to two vertices of F_v, there exist a neighbour of v_i in G that is in F_v, say v_k. Since $v_k \in F_v$ it follows that $V(H_k) \subseteq F_v$. Therefore $|S| \le (n_G - 1)s = sn_G - s$ and since $s > n_G$ it contradicts the lower bound on $\mathrm{ZIR}(G \circ H)$. It follows that $|S \cap V(H_i)| \le s - 1$ for each i and therefore $\mathrm{ZIR}(G \circ H) = n_G(s-1)$. □

7 Nonrelationships

In this section we present examples that show that some parameters that are lower bounds for zero forcing number are noncomparable to the lower ZIr number. We also show that the independence number is noncomparable to zir and ZIR; this is not surprising since it is well known that it is not comparable to Z, but worth noting since α appears in a lower bound for ZIR.

One of the origins of zero forcing of a graph G was the study of maximum nullity among the set of symmetric matrices that have graph G. Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$. The *graph* of a real symmetric $n \times n$ matrix A, denoted by $\mathcal{G}(A)$, has vertex set $\{v_1, \ldots, v_n\}$ and has edge set $\{v_iv_j : i \neq j \text{ and } a_{ij} \neq 0\}$. The *maximum nullity* of the graph G, denoted by $\mathrm{M}(G)$, is defined to be

$$\mathrm{M}(G) = \max\{\operatorname{null} A : \mathcal{G}(A) = G \text{ and } A^T = A\}$$

where A^T denotes the transpose of A. It is known that $\mathrm{M}(G) \leq \mathrm{Z}(G) \leq \mathrm{ZIR}(G)$ for all G, so it seems natural to ask whether there is a relationship between maximum nullity and lower ZIR number. The next examples show there is no relationship between M and zir.

Example 7.1. The star $K_{1,p}$ has $\mathrm{zir}(K_{1,p}) = 1 < p - 1 = \mathrm{M}(K_{1,p})$ by Example 3.1 (maximum nullity of the star is well known [18, Theorem 9.5]).

Example 7.2. Let G be the pentasun shown in Figure 7.1. Then it can be verified computationally that $\mathrm{zir}(G) = 3$ [15] and it is known that $\mathrm{M}(G) = 2$ [18, Example 9.118].

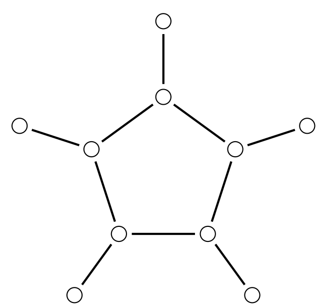

Figure 7.1: A graph G having $M(G) < \mathrm{zir}(G)$

For any positive integer k, a *k-tree* is K_{k+1} or a graph that is obtained from K_{k+1} by repeatedly adding a new vertex and joining it to an existing k-clique. For a graph G, the minimum k such that G is a subgraph of some k-tree is called the *tree-width* of G, denoted by $\mathrm{tw}(G)$. It is known that $\mathrm{tw}(G) \leq \mathrm{Z}(G)$ for all G [18], so it seems natural to ask whether there is a relationship between tree-width and lower ZIR number. The next examples show tree-width and lower ZIR number are not comparable.

Example 7.3. As shown in Proposition 3.5, the friendship graph $\mathrm{Fr}(k)$ has $\mathrm{zir}(\mathrm{Fr}(k)) = k + 1$. To see that $\mathrm{tw}(\mathrm{Fr}(k)) = 2$, construct $\mathrm{Fr}(k)$ as a subgraph of a 2-tree G: First define a sequence of 2-trees L_i as follows: $L_1 = K_3$ with $V(L_1) = \{0, 1, 2\}$. For $i = 1, \ldots, 2k - 2$, construct L_{i+1} by adding vertex $i + 2$ adjacent to exactly vertices 0 and $i + 1$ of L_i. Observe that $V(L_i) = \{0, 1, \ldots, i + 1\}$ and L_i is a 2 tree. From L_{2k-1}, delete the edges $\{\{2i, 2i + 1\} : i = 1, \ldots, k - 1\}$ to obtain $\mathrm{Fr}(k)$.

Example 7.4. Let G be the graph shown in Figure 7.2. Then it has been verified computationally that $\mathrm{zir}(G) = 4$ and $\mathrm{tw}(G) = 5$. [15].

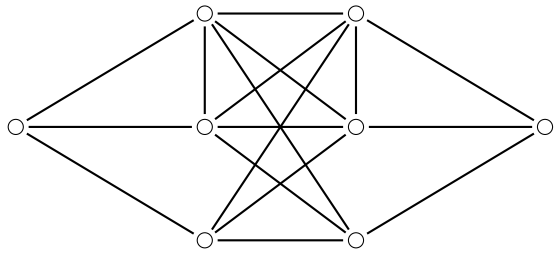

Figure 7.2: A graph G having $\mathrm{tw}(G) > \mathrm{zir}(G)$

Since $\gamma_P(G) \leq \mathrm{Z}(G)$, it is immediate that $\gamma_P(G) \leq \mathrm{ZIR}(G)$. Note that $C_r \circ 2K_1$ is an extremal example for power domination (least possible value of $\gamma_P(G)$ for graphs with $3r$ vertices) and may be an extremal example for $\mathrm{ZIR}(G)$ (may be least possible value of $\mathrm{ZIR}(G)$ for graphs with $3r$ vertices).

The next example shows that not every maximal ZIr-set is a power dominating set, but does not show that $\mathrm{zir}(G) < \gamma_P(G)$ is possible.

Example 7.5. The graph G in Figure 7.3 satisfies $\mathrm{zir}(G) = 2$ and $\gamma_P(G) = 2$. Observe that $S = \{v_3, v_4\}$ is a maximal ZIr-set of G. However, S is not a power dominating set.

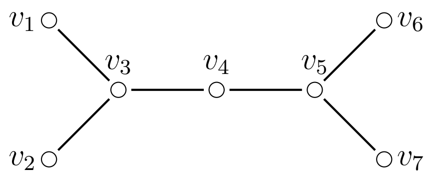

Figure 7.3: A graph with a lower zir set that is not power dominating.

Question 7.6. *Is $\gamma_P(G) \leq \mathrm{zir}(G)$ for all graphs G?*

Well-known examples that show α is noncomparable with Z also show that α is noncomparable with zir and ZIR.

Example 7.7. Consider K_n with $n \geq 3$: $\alpha(K_n) = 1 < n - 1 = \mathrm{zir}(K_n)$.

Example 7.8. Consider $C_r \circ 2K_1$ with $r \geq 3$: $\alpha(C_r \circ 2K_1) = 2r > r = \mathrm{ZIR}(C_r \circ 2K_1)$.

The domination number has appeared in several bounds for ZIR. It is known that γ is noncomparable with Z. Consider, for example, $K_n, n \geq 3$ ($\gamma(K_n) = 1$ and $\mathrm{Z}(K_n) = n - 1$) and $C_r \circ K_1$ where $r = 4\ell$ ($\gamma(C_r \circ K_1) = r$ and $\mathrm{Z}(C_r \circ K_1) = \frac{r}{2}$). What can be said γ and ZIR? From Corollary 4.3 a graph G with $\delta(G) \geq 3$ has $\mathrm{ZIR}(G) \geq \frac{n}{2}$. Since a graph G with no isolated vertices has $\gamma(G) \leq \frac{n}{2}$, it follows that $\gamma(G) \leq \mathrm{ZIR}(G)$ for graphs with minimum degree at least 3.

Question 7.9. *Is $\gamma(G) \leq \mathrm{ZIR}(G)$ for all graphs G?*

8 Computational considerations

We have not made a careful study of the computational complexity of computing zero forcing irredundance numbers, but hope that the information in this section may help in future studies of either the complexity of computing these parameters, or in computational experimentation.

Determining all minimal forts is computationally challenging because there are examples of graph families such that the numbers of minimal forts is exponential in the order [8]. For comparison, Genesen, Haas, and Hogben presented a family of trees G_n of order $3n$ such that G_n has 2^n minimum zero forcing sets and also showed that the set of minimum zero forcing sets of a graph of order n can be computed in $2^{O(n)}$ operations [17].

The software [15] can compute $\mathrm{ZIR}(G)$ and $\mathrm{zir}(G)$ in less than a minute for graphs up to order fifteen using *Sage* installed on a 2024 MacBook Pro with an M3 Max chip and 36GB of memory running Sequoia 15.5; well-known graph families such as paths, cycles, complete graphs, and complete bipartite graphs tend to run significantly faster than random graphs. We have not attempted to optimize the code or make it as efficient as possible. Rather our focus was on accurately (albeit slowly) determining $\mathrm{ZIR}(G)$ and $\mathrm{zir}(G)$ for a given small graph G to help in the initial investigation of examples.

Programs for computing zero forcing number by brute force have existed for twenty years, and over time a variety of methods for computing the zero forcing number have been developed that are significantly faster than brute force. Such methods include the wavefront algorithm documented in Butler et al. [9] and the use of integer programming described in [7]. Integer programs and their relaxations to linear programs have also been used recently in conjunction with forts in [10]. There are likely methods better than brute force for determining all forts and zero forcing irredundance numbers.

9 Concluding remarks

In this section we summarize the values of lower ZIr number, zero forcing number, upper zero forcing number, and upper ZIr number for various graph families in Table 9.1 and present some open questions for future work. In most cases listed in Table 9.1, the zero forcing number was already known and most can be found (with references) in [18, Theorem 9.5].

Next, we list several questions that were implicit in earlier discussions.

Question 9.1. *What is the least value of* $\mathrm{ZIR}(G)$ *over all graphs of order* n*?*

Question 9.2. *Theorem 5.11 provides a family of graphs* G *satisfying* $\mathrm{ZIR}(G) = \frac{1}{3}|V(G)|$*, but these graphs have* $\delta(G) = 1$*. What is an asymptotically tight lower bound on* $\mathrm{ZIR}(G)$ *for connected graphs* G *of order* n *with* $\delta(G) = 2$ *as* $n \to \infty$*?*

Question 9.3. *If* G *is a connected cubic graph of order* $n \geq 4$*, then* $\mathrm{ZIR}(G) \leq \frac{3}{4}n$ *by Corollary 4.11, and equality is realized by* K_4*. Do there exist connected cubic*

result #	graph G	order	zir(G)	Z(G)	$\overline{\mathrm{Z}}(G)$	ZIR(G)
2.6	$\overline{K_n}$	n	n	n	n	n
2.12	K_n	n	$n-1$	$n-1$	$n-1$	$n-1$
3.1	$K_{q,p}$, $1 \le q \le p$	$q+p$	q	$q+p-2$	$q+p-2$	$q+p-2$
3.3	P_n, $5 \le n$	n	1	1	2	$\lfloor \frac{n-1}{2} \rfloor$
3.2	C_n, $4 \le n$	n	2	2	2	$\lfloor \frac{n}{2} \rfloor$
3.5	Fr(k), $k \ge 2$	$2k+1$	$k+1$	$k+1$	$k+1$	$k+1$
3.6	$H(r,s)$, $r \ge 2, s \ge 5, s$ odd $H(r,3)$, $r \ge 2$	$r+s+1$ $r+s+1$	2 2	r r	$r+1$ r	$r+\frac{s-1}{2}$ $r+1$
4.4, 4.5	N_k	$4k$		$k+2$	$k+2$	$2k$
4.8	H_k	$5k$		$k+2$		$2k$
5.11	$C_r \circ 2K_1$	$3r$		r	r	r
5.6	$rK_2 \vee 2K_1$	$2r+2$		$r+2$		$2r$
6.2	$P_{n_1} \vee P_{n_2}$, $n_1, n_2 \ge 7$	n_1+n_2				n_1+n_2-4
6.4	$W_{r+1} = C_r \vee K_1$, $r \ge 5$	$r+1$	3	3	3	$r - \lceil \frac{r}{3} \rceil$
5.3	$H \vee 2K_1$	$\|V(H)\|+2$				$\|V(H)\|$
5.3	$H \vee K_2$, $H \ne K_{\|V(H)\|}$ for ZIR, $\delta(H) \ge 1$ for Z	$\|V(H)\|+2$		$\mathrm{Z}(H)+2$		$\|V(H)\|$
6.10	$H \circ C_r$, $r \ge 5$	$\|V(H)\|(r+1)$				$\|V(H)\| (r - \lceil \frac{r}{3} \rceil)$
6.11	$H \circ W_{r+1}$, $r \ge 5$	$\|V(H)\|(r+2)$				$\|V(H)\|r$
6.16	$H \circ K_1$	$2\|V(H)\|$				$\|V(H)\|$

Table 9.1: Summary of values of $\mathrm{zir}(G), \mathrm{Z}(G), \overline{\mathrm{Z}}(G)$, $\mathrm{ZIR}(G)$ for various graph families G

graphs G of arbitrarily large order n such that $\mathrm{ZIR}(G) = \frac{3}{4}n$? If not, what is an asymptotically tight upper bound on $\mathrm{ZIR}(G)$ for connected cubic graphs G of order n as $n \to \infty$?

Section 6 only scratches the surface of how graph operations affect ZIr numbers. There are many related questions remaining. For example, what is the effect of graph operations on the lower zir number?

Question 9.4. *If H abandons a fort, is it necessarily the case that $H \vee K_1$ abandons a fort?*

Section 7 contains two questions about whether the lower zir number is at least the power domination number and whether the upper ZIR number is at least the domination number. We can also look for relationships and non-relationships with additional parameters.

Cockayne and Mynhardt [14] determined conditions under which integers $1 \le k_1 \le k_2 \le k_3 \le k_4$ can be realized as $\mathrm{ir}(G) = k_1, \gamma(G) = k_2, \Gamma(G) = k_3$, and $\mathrm{IR}(G) = k_4$. One condition is that $k_1 = 1$ implies that $k_2 = 1$; this condition would however not be true for the ZIr-numbers and zero forcing numbers since $\mathrm{zir}(K_{1,r}) = 1$, but $\mathrm{Z}(K_{1,r}) = r-1$. We know from Proposition 3.6 that $k_1 = 2, k_2 \ge k_1$, $k_3 = k_2+1$, and $k_4 > k_3$ can be realized by $H(r,s)$.

Question 9.5. *For which integer values $1 \le k_1 \le k_2 \le k_3 \le k_4$ does there exist a graph such that $\mathrm{zir}(G) = k_1, Z(G) = k_2, \bar{Z}(G) = k_3$ and $\mathrm{ZIR}(G) = k_4$?*

Section 8 discusses questions about the computational complexity of zero forcing irredundance numbers and better methods for determining these parameters.

Question 9.6. *Determine whether the problems "Is* $\mathrm{ZIR}(G) \leq k$*?" and "Is* $\mathrm{zir}(G) \leq k$*?" are NP-complete.*

Question 9.7. *Develop more efficient methods for computing* ZIR *and* zir*.*

Acknowledgements

The research of B. Curtis is partially supported by NSF grant 1839918. The research of A. Roux is supported in part by the National Research Foundation of South Africa (Grant Number: 121931). The authors thank the referees for their helpful comments.

References

[1] AIM Minimum Rank — Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S.M. Cioaba, D. Cvetković, S.M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen and A. Wangsness Wehe), Zero forcing sets and the minimum rank of graphs, *Linear Algebra Appl.* 428 (2008), 1628–1648.

[2] F. Barioli, W. Barrett, S. Fallat, H.T. Hall, L. Hogben, B. Shader, P. van den Driessche and H. van der Holst, Zero forcing parameters and minimum rank problems, *Linear Algebra Appl.* 433 (2010), 401–411.

[3] W.W. Barrett, H. van der Holst and R. Loewy, Graphs whose minimal rank is two, *Electron. J. Linear Algebra* 11 (2004), 258–280.

[4] C. Bazgan, L. Brankovic, K. Casel and H. Fernau, Domination chain: characterisation, classical complexity, parameterised complexity and approximability, *Discrete Appl. Math.* 280 (2020), 23–42.

[5] N. H. Bong, J. Carlson, B. Curtis, R. Haas and L. Hogben, Isomorphisms and properties of TAR reconfiguration graphs for zero forcing and other X-set parameters, *Graphs Combin.* 39 (2023), Paper No. 86, 23 pp.

[6] B. Brimkov and J. Carlson, Minimal Zero Forcing Sets, *Australas. J. Combin.* 90 (2024), 363–377.

[7] B. Brimkov, C. C. Fast and I. V. Hicks, Computational approaches for zero forcing and related problems, *European J. Oper. Res.* 273 (2019), 889–903.

[8] B. Brimkov, D. Mikesell and I. V. Hicks, Improved computational approaches and heuristics for zero forcing, *INFORMS Journal on Computing* 33 (2021), 1384–1399.

[9] S. Butler, J. Grout and H. T. Hall, Using variants of zero forcing to bound the inertia set of a graph, *Electron. J. Linear Algebra* 30 (2015), 1–18.

[10] T. R. Cameron, L. Hogben, F. H. J. Kenter, S. A. Mojallal and H. Schuerger, Forts, (fractional) zero forcing, and Cartesian products of graphs, available at `https://arxiv.org/abs/2310.17904`.

[11] Y. Caro and Y. Roditty, A note on the k-domination number of a graph, *Internat. J. Math. Math. Sci.* 13 (1990), 205–206.

[12] E. J. Cockayne, B. Gamble and B. Shepherd, An upper bound for the k-domination number of a graph, *J. Graph Theory* 9 (1985), 533–534.

[13] E. J. Cockayne, S. T. Hedetniemi and D. J. Miller, Properties of hereditary hypergraphs and middle graphs, *Canad. Math. Bull.* 21 (1978), 461–468.

[14] E. J. Cockayne and C. M. Mynhardt, The sequence of upper and lower domination, independence and irredundance numbers of a graph, *Discrete Math.* 122 (1993), 89–102.

[15] B. A. Curtis and L. Hogben, *Sage* code for ZIr-sets, available as Jupyter notebook `https://aimath.org/~hogben/ZIR.ipynb` and as PDF `https://aimath.org/~hogben/ZIR.pdf`.

[16] O. Favaron, A. Hansberg and L. Volkmann, On k-Domination and Minimum Degree in Graphs, *J. Graph Theory* 57 (2008), 33–40.

[17] J. Geneson, R. Haas and L. Hogben, Reconfiguration graphs of zero forcing sets, *Discrete Appl. Math.* 329 (2023), 126–138.

[18] L. Hogben, J. C.-H. Lin and B. L. Shader, *Inverse Problems and Zero Forcing for Graphs*, Math. Surveys Monogr., 270, American Mathematical Society, Providence, RI, 2022.

[19] C. M. Mynhardt and A. Roux, Irredundance, in *Structures of domination in graphs* (Dev. Math., 66), (Eds.: T.W. Haynes, S.T. Hedetniemi and M.A. Henning), Springer, Cham, 2021, 135–181.

(Received 15 Apr 2024; revised 17 Mar 2026)