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Outdegrees of generalized recursive trees grown from random batches of children

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Abstract

We study generalized recursive trees, that is, recursive trees where the nodes could recruit batches of children. We assume the size of the batch at each step follows the distribution of a generic random variable $X \in \{x_1, x_2, x_3, \ldots\} \subseteq \mathbb{N}$, with $\mathbb{P}(X = x_j) = p_j > 0$, for $j = 1, 2, 3, \ldots$. In this notation $x_1 < x_2 < x_3 < \cdots$. The support of X imposes certain admissible node outdegrees. We look at the joint distribution of the first three admissible outdegrees (the zero is included because a tree always has leaves). The values of x_1 and x_2 fall in three regimes in the first quadrant of the two-dimensional space. Namely, the regimes are $2x_1 < x_2$, $2x_1 = x_2$, and $2x_1 > x_2$. In all three regimes, we have asymptotic trivariate normal distributions for the counts of the first three outdegrees, with marked differences in the asymptotics of the vector of means and the covariance matrix. Pólya urns with random replacement matrices are our chief tool. We conclude the paper with an example from the middle regime, where the additions follow a distribution as one plus a Poisson random variable with parameter 1.

1 Introduction

A batched recursive tree grows according to the following algorithm. The tree starts out with a single root node labeled with 1. At time $n \geq 1$, an existing node in the tree is selected uniformly at random as a *parent* to a batch of new nodes of random

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size. The random size of the batch added follows the distribution of X, a random variable defined on the positive integers. The X nodes (considered as *children*) are adjoined to the chosen parent and all receive the label $n+1$; the process is then repeated. The operation of choosing a parent for children is often referred to as *recruiting*, and the selected parent is therefore a *recruiter* (see for example [1, 19]). At time n, the structure is thought of as a tree of *age* n. The number of children of a node is the node's *outdegree*.

Technically speaking, the batches generated at the various steps form an independent identically distributed (i.i.d.) sequence of random variables. The common distribution is to be called the *generating distribution*. In other words, at step n, we add a batch of size X_n, distributed like a generic X, and the sequence X_n is i.i.d. To the best of our knowledge, this model has not been studied before; in particular, it introduces random addition of nodes generated by a generalized distribution.

An instance of the distribution of the generating sequence is $X \equiv k$ (a constant), a degenerate distribution, as in the case of the well-studied recursive tree [5, 7, 9, 21], where $X = k = 1$. Or, X might be distributed over the set $\{3, 5, 17\}$, with respective probabilities $1/7, 5/7$, and $1/7$. We take this instance as a running example in this manuscript. In the latter instance, at time 1, the variable $X_1 \overset{\mathcal{L}}{=} X$ is generated (the notation $\overset{\mathcal{L}}{=}$ means equality in distribution). Say the realized value is 3; the root recruits three nodes labeled with 2. At time 2, the variable $X_2 \overset{\mathcal{L}}{=} X$ is generated. Say the realized value is 5; with equal probability either the root or one of its three children recruits a batch of five nodes labeled 3. Figure 1 shows the evolution of one such batched recursive tree, where $X_1 = 3$ and $X_2 = 5$. From left to right, the trees have probabilities 1, $1/7$, $1/7 \times 5/7 \times 1/4 = 5/196$.

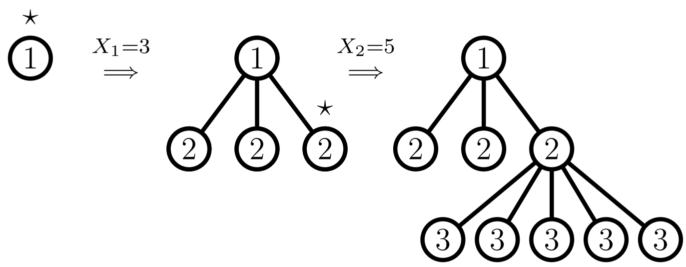

Figure 1: A recursive tree grown under the generating distribution of $X \in \{3, 5, 17\}$ with respective probabilities $1/7, 5/7$ and $1/7$. The recruiting nodes are flagged with asterisks.

Recursive trees have been extensively studied and analyzed. See [4–6, 15] and the references therein for the myriad theoretical and practical aspects of recursive trees. They offer flexibility and adaptability, making them a powerful tool in various applications. Allowing $X \geq 1$ to be random widens the scope of the application of recursive trees. For example, in the context of chain letters [8], the m-ary option allows a letter holder to sell or distribute multiple copies of the letter simultaneously to a single or multiple new owners. Realistically, at the point of a transaction, the letter holder cannot predict the number of copies of the letter in the sale. This condition benefits from the proposed extension.

The outdegrees in the batched tree can be approached via Pólya urns. Before we present results on the outdegrees, we say a word about these urns to give a perspective.

2 Pólya urns

A Pólya urn is a pot containing colored balls. The urn starts nonempty. The urn evolves in discrete time by the drawing of a ball or balls from it, following certain replacement rules.

Suppose the total number of possible ball colors in the urn is c. For integer $r \in \mathbb{N}$, we use the notation $[r]$ to denote the set $\{1, 2, \ldots, r\}$. At time $n \geq 1$, a ball is drawn uniformly at random from the urn. The color of the sampled ball is observed and the ball is placed back in the urn. If the ball withdrawn is of color $i \in [c]$, we add $a_{i,j}$ balls of color $j \in [c]$ to the urn.

The dynamics of the urn are captured in a $c \times c$ *replacement matrix:*

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,c} \\ a_{2,1} & a_{2,2} & \dots & a_{2,c} \\ \vdots & \vdots & \ddots & \vdots \\ a_{c,1} & a_{c,2} & \dots & a_{c,c} \end{pmatrix}.$$

In the general case, the elements of the replacement matrix can take negative values and can be random. For a wider discussion and applications, see [14, 16, 17].

There are general theories for the evolution of urns [2, 12, 20]. For easy reference, we give here convergence theorems for the classes of urns that arise in the batched trees. We need the version where some of the replacement matrix entries are random.

Let $Y_{n,i}$ be the number of balls of color $i \in [c]$ after n draws, and let $\mathbf{Y}_n = (Y_{n,1}, \ldots, Y_{n,c})^T$ be the vector with these ball counts as components. When $\mathbb{E}[\mathbf{A}^T]$ is irreducible and balanced (with constant positive row sum), and all the off-diagonal elements are nonnegative, allowing the diagonal elements to be at least -1, we have some classic convergence results [2,12,20]. These convergence theorems come in terms of the eigenvalues and eigenvectors of $\mathbb{E}[\mathbf{A}^T]$. Under the conditions mentioned on the replacement matrix, there is one real simple eigenvalue λ_1, which is larger than the real part of any other eigenvalue, which is a consequence of the Perron–Frobenius theorem [10]. In this context, λ_1 is called the principal eigenvalue and the corresponding eigenvector, $\mathbf{v}_1$, is called the principal eigenvector.[1] Namely, the strong convergence theorem for Pólya urns states that

$$\frac{1}{n}\mathbf{Y}_n \xrightarrow{a.s.} \lambda_1 \mathbf{v}_1; \tag{1}$$

[1]Of the many possible scales for the principal eigenvector we choose the one that renders its norm-1 equal to 1, so that the components of the principal eigenvector are proportions of the number of balls of each color, and these proportions add up to 1, as they should.

the convergence is component-wise.

The multivariate central limit theorem for Pólya urns states that, if $\Re\,\lambda_2 < \frac{1}{2}\lambda_1$, then we have

$$\frac{1}{\sqrt{n}}\,(\mathbf{Y}_n - \lambda_1 \mathbf{v}_1) \xrightarrow{\mathcal{L}} \mathcal{N}_c(\mathbf{0}, \mathbf{\Sigma}), \tag{2}$$

where $\mathcal{N}_c(\mathbf{0}, \mathbf{\Sigma})$ stands for a c-component centered random vector (with a mean vector of c zeros) and a $c \times c$ covariance matrix $\mathbf{\Sigma}$. The covariance matrix for the class of urns discussed here can be computed using the approach in [11, 18] or [13].

3 Strong laws for the smallest three outdegrees

We carry out all the calculations with n referring to the highest label in a tree of age $n - 1$. In the general setup, we have

$$X \in \{x_1, x_2, x_3, \dots\} \subseteq \mathbb{N}.$$

Note that this setup covers finite distributions, taking only a finite portion of $\mathbb{N}$ to have positive probabilities and assigning probability 0 to the rest of the positive integers.

Remark 3.1 Some standard recursive tree shapes may not arise at all in a certain batched recursive tree. For instance, a tree with a root and two children is not possible to come by in the running example.

Remark 3.2 In a batched recursive tree, the labels on a root-to-leaf path form an increasing sequence of numbers. Authors in the past dubbed such trees "increasing," see [3]. The batched model we propose widens the class and associated results of increasing trees.

We take $\mathbb{P}(X = x_j) = p_j > 0$. Depending on the support of X, there is a set of admissible outdegrees. In the running example with $X \in \{3, 5, 17\}$, the only admissible outdegrees are

$$0, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, \dots\ .$$

Note that 0 (the outdegree of a leaf) is always an admissible outdegree, as any tree has leaves.

We are going to look at nodes of the first three admissible outdegrees. The values x_1 and x_2 in the generating distribution fall into regimes in the first quadrant of the two-dimensional space. We detect three different regimes, according as $2x_1 < x_2$, $2x_1 = x_2$, and $2x_1 > x_2$. We call these regimes A, B and C, respectively. See the transition diagram in Figure 2.

For any regime, the dynamics of the outdegrees of the first three admissible outdegrees can be tracked by four colors. Color 1 corresponds to the leaves (outdegree 0); Color 2 corresponds to nodes of the second admissible outdegree; Color 3 corresponds to nodes of the third admissible outdegree; Color 4 corresponds to nodes of outdegrees higher than the third admissible one.

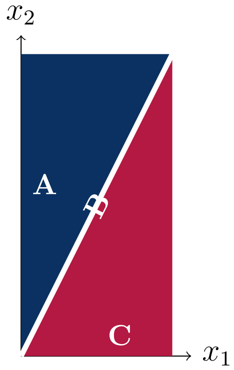

Figure 2: The three regimes of the first three outdegrees.

Let $\Delta_{n,i}$ be the number of nodes of the ith admissible outdegree in a batched recursive tree with highest label n. We look at the joint distribution of $\Delta_{n,1}, \Delta_{n,2}, \Delta_{n,2}$. The rightmost structure in Figure 1 depicts a batched tree with $\Delta_{3,1} = 7$ leaves, $\Delta_{3,2} = 1$, $\Delta_{3,3} = 1$, and $\Delta_{3,i} = 0$, for $i \geq 4$.

Theorem 3.1. *Let $\Delta_{n,i}$ be the number of nodes of the ith admissible outdegree in a batched recursive tree, with highest label n, growing under the generating distribution of $\mathbf{X} = \{x_1, x_2, \ldots\}$, in which $\mathbb{P}(\mathbf{X} = x_j) = p_j > 0$, for $j = 1, 2, 3, \ldots$ and $\mathbb{E}[X] = \mu$. Let $\mathbf{\Delta_n}$ be the vector $(\Delta_{n,1}, \Delta_{n,2}, \Delta_{n,3})^T$. We have the strong convergence:*

Regime A, $2x_1 < x_2$:

$$\frac{1}{n}\mathbf{\Delta}_n \xrightarrow{a.s.} \frac{\mu^2}{\mu+1}\begin{pmatrix} 1 \\ \frac{p_1}{\mu+1} \\ \frac{p_1^2}{(\mu+1)^2} \end{pmatrix}.$$

Regime B, $2x_1 = x_2$:

$$\frac{1}{n}\mathbf{\Delta}_n \xrightarrow{a.s.} \frac{\mu^2}{\mu+1}\begin{pmatrix} 1 \\ \frac{p_1}{\mu+1} \\ \frac{\mu p_2 + p_1^2 + p_2}{(\mu+1)^2} \end{pmatrix}.$$

Regime C, $2x_1 > x_2$:

$$\frac{1}{n}\mathbf{\Delta}_n \xrightarrow{a.s.} \frac{\mu^2}{\mu+1}\begin{pmatrix} 1 \\ \frac{p_1}{\mu+1} \\ \frac{p_2}{\mu+1} \end{pmatrix}.$$

Proof. We handle *Regime A* in some detail and leave out similar details in *Regimes B* and *C*.

Regime A:

In this regime, we have $2x_1 < x_2$. The second admissible value is $2x_1$. So, in the color code, the four colors in the palette are:

- Color 1: encodes the leaves (outdegree 0),
- Color 2: encodes nodes of outdegree x_1,
- Color 3: encodes nodes of outdegree $2x_1$,
- Color 4: encodes nodes of outdegree higher than $2x_1$.

We discuss only the rationale of the top row of the ball replacement matrix. Other rows can be argued similarly.

When a leaf recruits, a copy of X is generated and a batch of this size is adjoined to the recruiting leaf. The recruiting leaf now becomes a node of outdegree X, a net gain of $X - 1$ leaves (balls of color 1 in the urn). If $X = x_1$, the status of the recruiting leaf changes to become of outdegree x_1 and one ball of color 2 is added to the urn. Otherwise, X is of value greater than x_1, the recruiting parent's outdegree is upgraded to be higher than $2x_1$ — we add one ball of color 4 to the urn. This argument gives us the top row of the replacement matrix below. The reader can check the rest of the entries in the matrix:

$$\mathbf{A} = \begin{pmatrix} X-1 & \mathbb{I}_{\{X=x_1\}} & 0 & \mathbb{I}_{\{X\ge x_2\}} \\ X & -1 & \mathbb{I}_{\{X=x_1\}} & \mathbb{I}_{\{X\ge x_2\}} \\ X & 0 & -1 & 1 \\ X & 0 & 0 & 0 \end{pmatrix}.$$

The expected value of this matrix is

$$\mathbb{E}[\mathbf{A}] = \begin{pmatrix} \mu-1 & p_1 & 0 & 1-p_1 \\ \mu & -1 & p_1 & 1-p_1 \\ \mu & 0 & -1 & 1 \\ \mu & 0 & 0 & 0 \end{pmatrix}.$$

After a calculation for $\mathbb{E}[\mathbf{A}^T]$, we get the eigenvalues $\lambda_1 = \mu$, and $\lambda_2 = \lambda_3 = \lambda_4 = -1$. The principal eigenvalue is $\lambda_1 = \mu$, and the corresponding principal eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} \frac{\mu}{\mu+1} \\ \frac{\mu p_1}{(\mu+1)^2} \\ \frac{\mu p_1^2}{(\mu+1)^3} \\ \frac{1+(1-p_1)\mu^2+(2-p_1-p_1^2)\mu}{(\mu+1)^3} \end{pmatrix}.$$

We get the result in the theorem via the strong law for Pólya urns (cf. (1)).

Regime B:

This regime is for distributions in which $x_2 = 2x_1$. The replacement matrix can be argued in a similar manner to that in *Regime A*. One finds

$$\mathbf{A} = \begin{pmatrix} X-1 & \mathbb{I}_{\{X=x_1\}} & \mathbb{I}_{\{X=x_2\}} & \mathbb{I}_{\{X>x_2\}} \\ X & -1 & \mathbb{I}_{\{X=x_1\}} & \mathbb{I}_{\{X\ge x_2\}} \\ X & 0 & -1 & 1 \\ X & 0 & 0 & 0 \end{pmatrix}.$$

The expected value of this matrix is

$$\mathbb{E}[\mathbf{A}] = \begin{pmatrix} \mu - 1 & p_1 & p_2 & 1 - p_1 - p_2 \\ \mu & -1 & p_1 & 1 - p_1 \\ \mu & 0 & -1 & 1 \\ \mu & 0 & 0 & 0 \end{pmatrix}.$$

After a calculation for $\mathbb{E}[\mathbf{A}^T]$, we get the eigenvalues $\lambda_1 = \mu$, and $\lambda_2 = \lambda_3 = \lambda_4 = -1$. The principal eigenvalue is $\lambda_1 = \mu$, and the corresponding principal eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} \frac{\mu}{\mu+1} \\ \frac{\mu p_1}{(\mu+1)^2} \\ \frac{\mu(\mu p_2 + p_1^2 + p_2)}{(\mu+1)^3} \\ \frac{1+(1-p_1-p_2)\mu^2+(2-p_1^2-p_1-p_2)\mu}{(\mu+1)^3} \end{pmatrix}.$$

We get the result via the strong law for Pólya urns (cf. (1)).

Regime C:

This regime is for distributions in which $2x_1 > x_2$. The replacement matrix can be argued in a similar manner to that in *Regime A*. One finds

$$\mathbf{A} = \begin{pmatrix} X - 1 & \mathbb{I}_{\{X=x_1\}} & \mathbb{I}_{\{X=x_2\}} & \mathbb{I}_{\{X>x_2\}} \\ X & -1 & 0 & 1 \\ X & 0 & -1 & 1 \\ X & 0 & 0 & 0 \end{pmatrix}.$$

The expected value of this matrix is

$$\mathbb{E}[\mathbf{A}] = \begin{pmatrix} \mu - 1 & p_1 & p_2 & 1 - p_1 - p_2 \\ \mu & -1 & 0 & 1 \\ \mu & 0 & -1 & 1 \\ \mu & 0 & 0 & 0 \end{pmatrix}.$$

After a calculation for $\mathbb{E}[\mathbf{A}^T]$, we get the eigenvalues $\lambda_1 = \mu$, and $\lambda_2 = \lambda_3 = \lambda_4 = -1$. The principal eigenvalue is $\lambda_1 = \mu$, and the corresponding principal eigenvector is

$$\mathbf{v}_1 = \begin{pmatrix} \frac{\mu}{\mu+1} \\ \frac{\mu p_1}{(\mu+1)^2} \\ \frac{\mu p_2}{(\mu+1)^2} \\ \frac{1+\mu(1-p_1-p_2)}{(\mu+1)^2} \end{pmatrix}.$$

We get the result via the strong law for Pólya urns (cf. (1)). □

4 Gaussian laws for the smallest three outdegrees

We remind the reader that we carry out all the calculations with n referring to the highest label in the tree.

Theorem 4.1. *Let $\Delta_{n,i}$ be the number of nodes of the ith admissible outdegree in a batched recursive tree, with highest label n, growing under the generating distribution of $X = \{x_1, x_2, \dots\}$, in which $\mathbb{P}(\mathbf{X} = x_j) = p_j > 0$, for $j = 1, 2, 3, \dots$. Let $\mathbf{\Delta}_n$ be the vector $(\Delta_{n,1}, \Delta_{n,2}, \Delta_{n,3})^T$. We have Gaussian laws:*

Regime A, $2x_1 < x_2$:

$$\frac{1}{\sqrt{n}}\left(\mathbf{\Delta}_n - \frac{\mu^2}{\mu+1}\begin{pmatrix}1\\ \frac{p_1}{\mu+1}\\ \frac{p_1^2}{(\mu+1)^2}\end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}_3\left(\mathbf{0}, \mathbf{\Sigma}_A\right),$$

Regime B, $2x_1 = x_2$:

$$\frac{1}{\sqrt{n}}\left(\mathbf{\Delta}_n - \frac{\mu^2}{\mu+1}\begin{pmatrix}1\\ \frac{p_1}{\mu+1}\\ \frac{\mu p_2+p_1^2+p_2}{(\mu+1)^2}\end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}_3\left(\mathbf{0}, \mathbf{\Sigma}_B\right),$$

Regime C, $2x_1 > x_2$:

$$\frac{1}{\sqrt{n}}\left(\mathbf{\Delta}_n - \frac{\mu^2}{\mu+1}\begin{pmatrix}1\\ \frac{p_1}{\mu+1}\\ \frac{p_2}{\mu+1}\end{pmatrix}\right) \xrightarrow{\mathcal{L}} \mathcal{N}_3\left(\mathbf{0}, \mathbf{\Sigma}_C\right).$$

The limiting covariance matrices $\mathbf{\Sigma}_A, \mathbf{\Sigma}_B$ and $\mathbf{\Sigma}_C$ are effectively computable and depend on the regime.

Proof. The computations for the strong law in Theorem 3.1 show that in all three regimes we have $-1 = \Re\lambda_2 = \Re\lambda_3 = \Re\lambda_4 < 1/2\mu = 1/2\lambda_1$. So, the central limit theorem (cf. (2)) applies to all three regimes. It only remains to calculate the limiting covariance matrix in each case.

We can use a method developed in [18] or an integral formula given in [13] to determine the limiting covariance matrix $\mathbf{\Sigma}_A$. According to [18], the limiting covariance matrix $\mathbf{\Sigma}$ solves the matrix equation[2]

$$\mu\,\mathbf{Q} = \bar{\mathbf{A}}^T\mathbf{Q} + \mathbf{Q}\bar{\mathbf{A}} + \mu\,\bar{\mathbf{A}}^T\mathbf{diag}(\mathbf{v}_1)\bar{\mathbf{A}} - \mu^3\,\mathbf{v}_1\mathbf{v}_1^T.$$

□

5 An Example from *Regime B*

Suppose X is distributed like 1 plus a Poisson random variable with parameter 1. The first two values in this distribution are $x_1 = 1$, and $x_2 = 2$, with probabilities

$$p_1 = \mathbb{P}(X = 1) = e^{-1}, \qquad p_2 = \mathbb{P}(X = 2) = e^{-1}.$$

[2]The formulas in the covariance matrix $\mathbf{\Sigma}_A$ are so huge that they render the presentation impractical. However, when the parameters are given numerically, the computation is straightforward. This is illustrated by an example following the theorem.

Here, we have $2x_1 = x_2$, which puts the case in *Regime B*. For this generating distribution, the three components of the vector $\boldsymbol{\Delta}_n$ count the leaves, nodes of outdegree 1 and nodes of outdegree 2, respectively. According to Theorem 3.1, we have

$$\frac{1}{n}\begin{pmatrix} \Delta_{n,1} \\ \Delta_{n,2} \\ \Delta_{n,3} \end{pmatrix} \xrightarrow{a.s.} \frac{4}{27e^2}\begin{pmatrix} 9e^2 \\ 3e \\ 1+3e \end{pmatrix}.$$

We next solve the linear system of equations in the theorem, with $\mu = 2$ after calculating $\mathbf{v}_1$. We get

$$\mathbf{\Sigma_B} = \begin{pmatrix} \frac{1}{9} & -\frac{17e^{-1}}{108} & -\frac{17e^{-1}}{108} + \frac{13e^{-2}}{1296} \\ -\frac{17e^{-1}}{108} & \frac{e^{-1}}{9} + \frac{101e^{-2}}{648} & \frac{47e^{-2}}{648} + \frac{133e^{-3}}{15552} \\ -\frac{17e^{-1}}{108} + \frac{13e^{-2}}{1296} & \frac{47e^{-2}}{648} + \frac{133e^{-3}}{15552} & \frac{e^{-1}}{9} + \frac{125e^{-2}}{648} + \frac{673e^{-3}}{7776} - \frac{625e^{-4}}{93312} \end{pmatrix}.$$

So, the specific trivariate central limit theorem in this example is

$$\frac{1}{\sqrt{n}}\left(\begin{pmatrix} \Delta_{n,1} \\ \Delta_{n,2} \\ \Delta_{n,3} \end{pmatrix} - \frac{4}{27e^2}\begin{pmatrix} 9e^2 \\ 3e \\ 1+3e \end{pmatrix} n\right) \xrightarrow{\mathcal{L}} \mathcal{N}_3(\mathbf{0}, \mathbf{\Sigma_B}).$$

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