A note on regular graphs whose second largest eigenvalue does not exceed 1

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Abstract

In this paper, we determine all r-regular graphs whose second largest eigenvalue λ_2 satisfies $\lambda_2 \leq 1$, under the condition $n \geq 2r$, where n denotes the number of vertices. As an application, we provide a complete classification of all r-regular graphs with $\lambda_2 \leq 1$ for $9 \leq r \leq 10$. This result extends the previous classifications given by Stanić [*Lin. Multilin. Alg.* 58 (2010), 545–554] and by Koledin and Stanić, [*Novi Sad J. Math.* 43 (2013), 145–153], which fully resolved the cases $r \leq 8$.

1 Introduction

Let G be a simple graph with n vertices and adjacency matrix $A(G)$. Denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ the eigenvalues of $A(G)$, which are also referred to as the *eigenvalues* of G.

Our notation is standard, and for any undefined terms the reader is directed to [1, 9]. In particular, for specific graphs not defined here (such as the Schläfli graph and related examples), these monographs provide the necessary details.

However, to make the paper more self-contained, we include the following notation and definitions. A complete graph and a cycle with n vertices are denoted by K_n and C_n, respectively. An *r-regular graph* is a graph in which every vertex has degree r. A complete bipartite graph with partition sets of sizes p and q is denoted by $K_{p,q}$. The *cocktail party graph* $CP(k)$ is the unique regular graph on $2k$ vertices of degree $2(k-1)$; it can be obtained from K_{2k} by removing a perfect matching. The complement of a graph G is denoted by $\overline{G}$, while $\cup$ denotes the disjoint union of two graphs. A graph consisting of k disjoint copies of a graph G is denoted by kG. For

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two vertex-disjoint graphs G and H, the *join* of G and H, denoted by $G \vee H$, is the graph obtained from G and H by joining each vertex of G to every vertex of H.

The *line graph* $L(G)$ of a graph G is the graph whose vertices correspond to the edges of G, where two vertices in $L(G)$ are adjacent whenever the corresponding edges in G share exactly one endpoint. In this case, G is called the *root graph* of $L(G)$. A graph is called (r_1, r_2)-semiregular bipartite if it is bipartite (i.e., 2-colourable), and all vertices in the first colour class have degree r_1, while all vertices in the second colour class have degree r_2. It is known that a line graph is regular if and only if its root graph is either regular or semiregular bipartite [2].

The problem of determining those graphs whose second largest eigenvalue does not exceed 1 was initiated in [8], and has since attracted increasing attention (see [5, 6] and the references therein). Regular graphs with a relatively small second largest eigenvalue have a more "round" structure, i.e., a smaller diameter and higher connectivity. In addition, graphs that are not necessarily regular but are sparse and have strong connectivity properties are referred to as expanders (for more details, see [8]). There are also some research results on regular graphs with a bounded second largest eigenvalue (where the bound may be greater than 1). For example, bipartite regular graphs whose second largest eigenvalue is less than or equal to 2 are well-known and are identified as reflexive bipartite regular graphs [4]; such graphs correspond to the Lorentzian counterparts of spherical and Euclidean graphs arising in the theory of reflection groups [7]. For results concerning regular graphs with the same spectral property, we refer the reader to [3, 8, 9]. Here we quote some developments significant for this paper.

Every r-regular graph G with $\lambda_2 \leq 1$ must be connected whenever $r \geq 2$. In addition, the least eigenvalue of the complementary graph $\overline{G}$ is greater than or equal to -2 [9]. It is known that every graph whose least eigenvalue satisfies the previous property is either a line graph or the so-called *exceptional graph* for the same property; i.e., a graph which is not a line graph, but its least eigenvalue is not less than -2 [1]. The next result is well known.

Theorem 1.1. [8, Theorem 3.1] *Each connected regular graph satisfying $\lambda_2 \leq 1$ is a complement of a (not necessarily connected) regular graph, each component of which is either a connected regular line graph, or a cocktail party graph or one of 187 connected regular exceptional graphs given in [1], pp. 213–227.*

Each of the 187 connected regular exceptional graphs has between 8 and 28 vertices, while its degree is between 3 and 16. In [1], these graphs are divided into three layers, and if N is the number of vertices, and r_H the degree of an exceptional graph, then the graphs of the first (respectively, the second or the third) layer satisfy

$$N = 2(r_H + 2) \leq 28 \quad \text{(respectively, } N = \tfrac{3}{2}(r_H + 2) \leq 27 \text{ or } N = \tfrac{4}{3}(r_H + 2) \leq 16). \tag{1}$$

Note that any regular exceptional graph belongs to exactly one layer. This classification will be frequently used in the proof section. In the subsequent proof, when

we refer to a certain branch as a graph belonging to a fixed layer, we mean any graph with the corresponding number of vertices in that layer. The numbers of graphs in these three layers are 163, 20, and 3, respectively. In the first layer, the numbers of regular exceptional graphs with 10, 12, 14, 16, 18, 20, 22, 24, and 28 vertices are 5, 8, 21, 35, 38, 27, 18, 8, and 3, respectively. In the second layer, the numbers of regular exceptional graphs with 9, 12, 15, 18, 21, and 24 vertices are 2, 5, 6, 4, 2, and 1, respectively. In particular, the graph with 27 vertices is the Schläfli graph. In the third layer, there is one graph each for 8, 12, and 16 vertices. Further details can be found in [1], pp. 213–227.

Based on Theorem 1.1, Stanić determined all r-regular graphs with $\lambda_2 \leq 1$ for $2 \leq r \leq 4$ [8]. Subsequently, Koledin and Stanić determined all such r-regular graphs with $\lambda_2 \leq 1$ for $n > 2r$ and $r \geq 17$, as well as for $5 \leq r \leq 8$ [3]. In this paper, we determine all r-regular graphs with the same spectral property for $n \geq 2r$, and in particular, we completely classify those with $9 \leq r \leq 10$.

Hereafter, let $H_1, H_2, \ldots, H_5$ and $B_1, B_2, \ldots, B_{19}$ be the 24 graphs as shown in Figures 1 and 2, respectively.

Our main results read as follows.

Theorem 1.2. *Let G be an r-regular graph on n vertices satisfying $\lambda_2 \leq 1$. If $n \geq 2r$ and $r \geq 2$, then either $\overline{G}$ is one of the 187 connected regular exceptional graphs or $\overline{G} \in \{C_5, C_6, C_7, L(K_5), L(K_6), L(K_7), L(\overline{C_6}), L(K_{3,4}), L(CP(3)), L(\overline{C_7}), L(\overline{2C_3}), L(\overline{C_4 \cup C_3})\}$, or $\overline{G} \in \{L(K_{2,r+1}), 2K_r\}$ with $r \geq 2$.*

The previous theorem is used in proving the following two theorems.
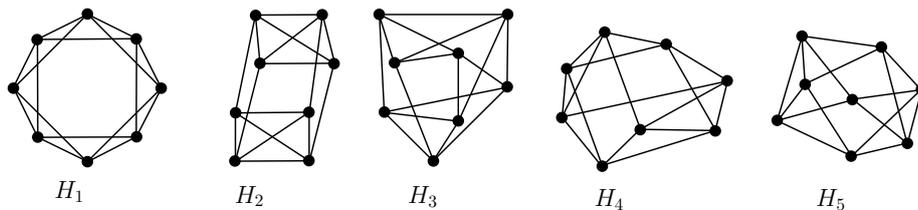

Figure 1: The 4-regular graphs H_1, H_2, H_3, H_4 and H_5 on eight vertices.

Theorem 1.3. *Let G be a 9-regular graph on n vertices satisfying $\lambda_2(G) \leq 1$. Then, either $\overline{G} \in \{\overline{K_{10}}, 2K_9, 2CP(4), L(K_{2,10}), C_{12}, C_3 \cup C_9, C_4 \cup C_8, C_5 \cup C_7, 2C_6, 2C_3 \cup C_6, C_3 \cup C_4 \cup C_5, 3C_4, 4C_3, K_5 \cup L(\overline{C_6}), K_5 \cup L(\overline{2C_3}), CP(3) \cup L(K_{2,4}), L(H_1), L(H_2), \ldots, L(H_5), L(K_{4,4})\}$, or $\overline{G} = K_5 \cup F_1$, where F_1 is a 4-regular exceptional graph with 9 vertices in the second layer, or $\overline{G} = CP(3) \cup F_2$, where F_2 is a 4-regular exceptional graph with 8 vertices in the third layer, or $\overline{G}$ is a 6-regular exceptional graph with 16 vertices in the first layer, or $\overline{G}$ is a 14-regular exceptional graph on 24 vertices in the second layer.*
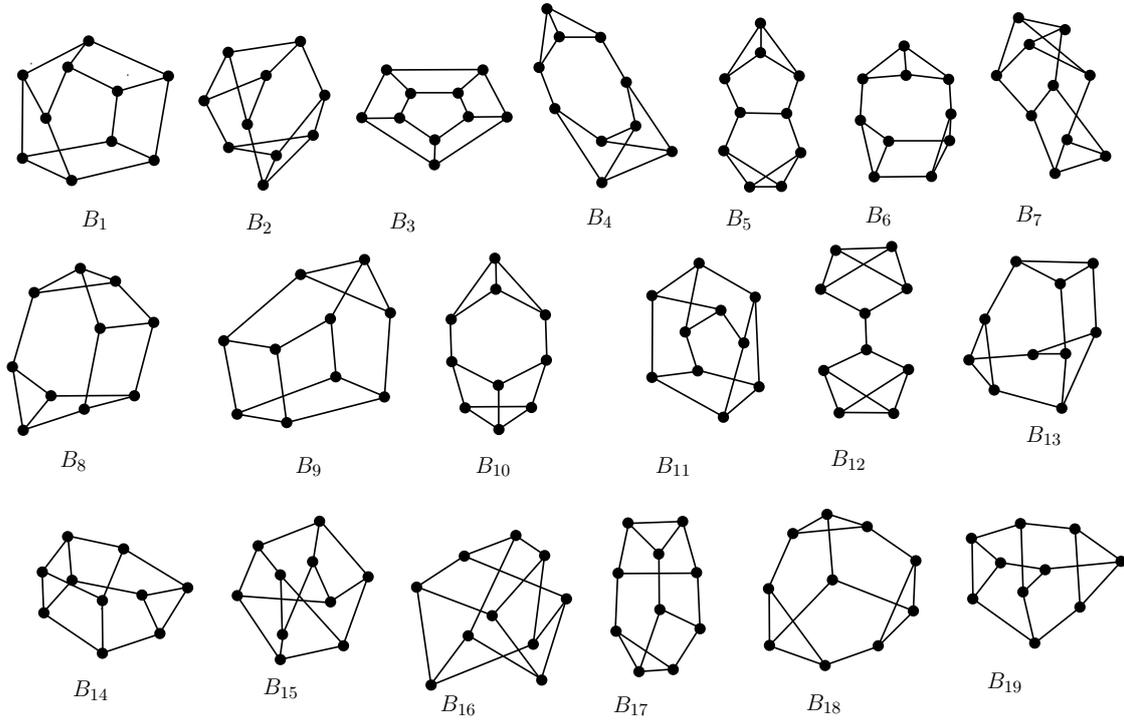
Figure 2: The 3-regular graphs $B_1, B_2, \ldots, B_{19}$ on ten vertices.

Theorem 1.4. *Let G be a 10-regular graph on n vertices satisfying $\lambda_2(G) \leq 1$. Then, either $\overline{G} \in \{\overline{K_{11}},\ \overline{CP(6)},\ 2K_{10},\ L(K_7),\ L(K_{2,11}),\ 3K_5,\ K_6 \cup L(K_{2,5}),\ K_7 \cup L(K_5), L(K_{3,6}),\ K_9 \cup CP(5),$ the Schläfli graph, $L(K_{2,3}) \cup 2K_4,\ CP(3) \cup L(\overline{C_6}),\ CP(3) \cup L(\overline{2C_3}),\ C_{13},\ C_3 \cup C_{10},\ C_4 \cup C_9,\ C_5 \cup C_8,\ C_6\ \cup C_7,\ 2C_3 \cup C_7,\ C_3 \cup 2C_5,\ 2C_4 \cup C_5, C_3 \cup C_4 \cup C_6,\ 3C_3 \cup C_4,\ L(B_1),\ L(B_2), \ldots, L(B_{19})\}$ or $\overline{G} = K_4 \cup F_1$, where F_1 is a 3-regular exceptional graph with 10 vertices in the first layer, or $\overline{G} = CP(3) \cup F_2$, where F_2 is a 4-regular exceptional graph with 9 vertices in the second layer.*

Note that all the graphs referred to in the previous theorems satisfy $\lambda_2(G) \leq 1$. Proofs are separated in the next section.

2   Proofs of Theorems 1.2–1.4

Recall that the *edge-degree* of an arbitrary edge e is the number of edges adjacent to e. The following lemma comes from [3].

Lemma 2.1. [3] *Let G be a connected r-regular graph on n vertices. If $\overline{G} = L(H)$ for some regular graph H, then $n - r$ is odd, $n - r + 1$ divides $4n$ and H is r_H-regular with $N = \frac{2n}{r_H}$ vertices, where $r_H = \frac{n-r+1}{2}$.*

Now we are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2: Since G is an r-regular graph, we have $\overline{G}$ is $(n - r - 1)$-regular. If $\overline{G}$ is disconnected, then it has at least two components, each having at least $n - r$ vertices. This implies that $2(n - r) \leq n$. Note that $n \geq 2r$. Thus, $n = 2r$ and $\overline{G}$ contains exactly two components; each has exactly r vertices. In this case, $\overline{G}$ is $(r - 1)$-regular and so each of the two components of $\overline{G}$ is a complete graph K_r, giving that $\overline{G} = 2K_r$. Assume now that $\overline{G}$ is connected. Since $n \geq 2r$, it cannot be a cocktail party graph. Next, assume that $\overline{G}$ is not one of 187 connected regular exceptional graphs. By Theorem 1.1, we suppose that

$$\overline{G} = L(H), \text{ where } H \text{ is regular or semiregular bipartite.}$$

Case 1. H is a regular graph. Suppose that H is r_H-regular with N vertices. By Lemma 2.1, we have $N r_H = 2n$. Since $n = 2r_H + r - 1$ it follows that $N = 4 + \frac{2(r-1)}{r_H}$. Since $n \geq 2r$, we have $r_H \geq \frac{r+1}{2}$, and so $\frac{2(r-1)}{r_H} \leq 4 - \frac{8}{r+1} < 4$. It immediately follows that

$$N = 4 + \frac{2(r - 1)}{r_H} \in \{5, 6, 7\}. \tag{2}$$

Since $\overline{G} = L(H)$ is connected, H is connected. Next, we suppose that $H \notin \{K_5, K_6, K_7, C_5, C_6, C_7\}$. Then $3 \leq r_H \leq N - 2$.

If $N = 5$, then $3 \leq r_H = 2(r - 1) \leq N - 2$ and $n = 2r_H + r - 1 = 5(r - 1)$ by (2). Thus, $r_H = 2(r - 1) = 3$, a contradiction.

If $N = 6$, then $3 \leq r_H = r - 1 \leq N - 2 = 4$ and $n = 2r_H + r - 1 = 3(r - 1)$ by (2). Since H is a connected $(r - 1)$-regular graph with 6 vertices and $r \geq 2$, we have $4 \leq r \leq 5$. For $r = 4$, H is a connected 3-regular graph with 6 vertices, and so $\overline{H} = C_6$ or $\overline{H} = 2C_3$. For $r = 5$, H is a connected 4-regular graph with 6 vertices, and so $H = CP(3)$.

If $N = 7$, then $r_H = \frac{2}{3}(r - 1) \leq N - 2 = 5$ and $n = 2r_H + r - 1 = \frac{7}{3}(r - 1)$ by (2). We have $3 \leq \frac{2}{3}(r - 1) \leq 5$. Note that 3 must divide $r - 1$. Thus $r = 7$. This implies that H is a connected 4-regular graph with 7 vertices and so $\overline{H} = C_7$ or $\overline{H} = C_4 \cup C_3$.

Case 2. H is an (r_1, r_2)-semiregular bipartite graph. Suppose that H has n_1 and n_2 vertices in the corresponding colour classes. Since G is an (r_1, r_2)-semiregular bipartite graph with n edges, we have

$$r_1 r_2 \leq r_1 n_1 = r_2 n_2 = n, \quad \text{and} \quad r_1 + r_2 = n - r + 1. \tag{3}$$

Without loss of generality we can suppose that $r_1 \geq r_2$. If r_2 is equal to 1, then $H = K_{1,n}$, because H is connected with exactly n edges. This implies that $\overline{G} = L(K_{1,n}) = K_n$ and so G is disconnected, a contradiction. Thus, we have $2 \leq r_2 \leq r_1 = n - r + 1 - r_2 \leq n - r - 1$, implying that $2(n - r - 1) \leq r_1 r_2 \leq n$. Since $n \geq 2r$, we get $2r \leq n \leq 2(r + 1)$.

Let $n = 2r$. Then $r_1 + r_2 = r + 1$ and both r_1 and r_2 must divide $2r$. Since $r_1 \geq r_2 \geq 2$, we have $n_1 \leq n_2 \leq r$ by (3). From (3) we also have

$$\frac{2r}{n_1} + \frac{2r}{n_2} = r + 1,$$

which implies that $n_1 \geq 3$. If $n_1 \geq 4$, then $\frac{2r}{n_1} + \frac{2r}{n_2} \leq r$, which is impossible. Thus $n_1 = 3$ and so $n_2 = \frac{6r}{r+3}$. This together with (3) implies that $r_1 = \frac{2r}{3}$ and $r_2 = \frac{r}{3} + 1$. Since $(\frac{r}{3} + 1)n_2 = 2r$, we have $(6 - n_2)(r + 3) = 18$, giving that $r + 3$ must divide 18. Combining this with $r_1 r_2 \leq n = 2r$ and $r \geq 2$, we have $r = 3$ or $r = 6$. For $r = 3$, we have $n_2 = 3 = n_1$ and $r_1 = r_2 = 2$ by (3). This implies that $H = C_6$. Otherwise, $r = 6$. By (3), H is a semiregular bipartite graph with $n_1 = 3 = r_2$ and $n_2 = r_1 = 4$, which gives rise to the graph $\overline{G} = L(K_{3,4})$.

Next let $n = 2r + 1$. Then both r_1 and r_2 must be odd (since they divide n), and r must be even (since $r_1 + r_2 = r + 2$). This implies that $r_1 \geq r_2 \geq 3$ and it then follows that $2r + 1 = n \geq r_1 r_2 \geq 3(r - 1)$, so for $r \leq 4$. Combining this with $r_1 + r_2 = r + 2 \geq 6$, we have $r_1 = r_2 = 3 < 4 = r$ and $n = 9$. By (3), we also have $n_1 = n_2 = 3$ and thus $H = K_{3,3}$.

Now let $n = 2r + 2$. We have $r_1 r_2 \geq 2(r + 1) = n$, and since $r_1 r_2 \leq n$ it immediately follows that $r_1 = 2$ and $r_2 = r + 1$. Therefore, $H = K_{2,r+1}$, i.e., $\overline{G} = L(K_{2,r+1})$. $\quad\square$

From Theorem 1.2, we can roughly characterize the r-regular graphs on n vertices for $n \geq 2r$. For the case $r = 9$, we only need to consider $n \in \{10, 12, 14, 16\}$, since nr must be even.

Proof of Theorem 1.3: Suppose that G contains exactly e edges. Then $2e = nr = 9n$ and so n must be even. If $n \geq 18$, then by Theorem 1.2, either G is a complement of any of the regular exceptional graphs on 24 vertices from the second layer, or $\overline{G} = L(K_{2,10})$ or $\overline{G} = 2K_9$. Since G is 9-regular with n vertices, we have $n \geq 10$ with equality if and only if $G = K_{10}$. If $n = 12$, then $\overline{G}$ is a 2-regular graph. Thus $\overline{G}$ must be a cycle, or a disconnected graph all of whose components are cycles. Now, we can conclude that $\overline{G}$ must be one of C_{12}, $C_3 \cup C_9$, $C_4 \cup C_8$, $C_5 \cup C_7$, $2C_6$, $2C_3 \cup C_6$, $C_3 \cup C_4 \cup C_5$, $3C_4$, $4C_3$. Next, we consider the cases that $n \in \{14, 16\}$. Since $r = 9$, $\overline{G}$ is not the cocktail party graph.

Case 1. $n = 16$. Now $\overline{G}$ is a 6-regular graph. We start with the case that $\overline{G}$ is connected. By (1), the regular exceptional graphs with degree 6 in the complement of G can be found in the first layer. If $\overline{G} = L(H)$, where H is a regular graph, then by Lemma 2.1, H is a 4-regular graph with $N = 8$ vertices. There are exactly six 4-regular graphs with $N = 8$ vertices; they are $K_{4,4}, H_1, H_2, \ldots, H_5$, as shown in Figure 1. Thus we have $\overline{G} = L(H_i)$, where $1 \leq i \leq 6$. If $\overline{G}$ is the line graph of a semiregular bipartite graph, then, by (3), we have $r_1 + r_2 = 8$ and both r_1 and r_2 divide 16, so we have $n_1 = n_2 = r_1 = r_2 = 4$, the only solution, implying that $\overline{G} = L(K_{4,4})$.

Otherwise $\overline{G}$ is disconnected. Since $\overline{G}$ is 6-regular, each component of $\overline{G}$ has at least 7 vertices, which implies that $\overline{G}$ contains exactly two components. If $\overline{G}$ contains one component having 7 vertices, then this component must be K_7 and the other component (say H) has exactly 9 vertices. By (1), H is not one of these 187 regular exceptional graphs. Since H is 6-regular, it is not a cocktail party graph. By Lemma 2.1, H is not a line graph of any regular graph. This implies that H is the line graph of an (r_1, r_2)-semiregular bipartite graph. By (3), $r_1 + r_2 = r_{\overline{G}} + 2 = 8$ and both r_1 and r_2 divide 9, but this is impossible.

Next, we suppose that each component of $\overline{G}$ contains at least 8 vertices. This implies that $\overline{G}$ contains exactly two components, each having 8 vertices. Let H be one component of $\overline{G}$. Since $\overline{G}$ is 6-regular and each component of $\overline{G}$ contains 8 vertices, $\overline{H}$ must be a 1-regular graph and so $H = CP(4)$. Thus, $\overline{G} = 2CP(4)$.

Case 2. $n = 14$. Then $\overline{G}$ is a 4-regular graph. Firstly, we consider the case when $\overline{G}$ is connected. By (1), $\overline{G}$ is not one of these 187 regular exceptional graphs. If $\overline{G} = L(H)$ is a regular line graph, then by (3), H is not a semiregular bipartite graph. Thus, H is a regular graph. But in view of Lemma 2.1, $n - r + 1 = 6$ divides $4n = 56$, a contradiction. So, we only consider the disconnected cases. Since $\overline{G}$ is 4-regular, each component of $\overline{G}$ contains at least 5 vertices. This implies that $\overline{G}$ has exactly two components.

Let $\overline{G} = G_1 \cup G_2$, where G_i is a 4-regular graph with N_i vertices. Then, we suppose that $5 \leq N_1 \leq N_2 \leq 9$.

If $N_1 = 5$, then $G_1 = K_5$. In this case, G_2 is a 4-regular graph with 9 vertices, and thus G_2 is not a cocktail party graph. When G_2 is a regular exceptional graph, then G_2 is a 4-regular exceptional graph with 9 vertices belonging to the second layer by (1). When $G_2 = L(H)$ for a regular graph H, then H is a 3-regular graph with 6 vertices. This implies that either $\overline{H} = C_6$ or $\overline{H} = 2C_3$. Thus $\overline{G} = K_5 \cup L(\overline{C_6})$ or $\overline{G} = K_5 \cup L(\overline{2C_3})$. Otherwise G_2 is the line graph of an (r_1, r_2)-semiregular bipartite graph. By (3), $r_1 + r_2 = r_{\overline{G}} + 2 = 6$ and both r_1 and r_2 divide 9. This implies that $\overline{G} = K_5 \cup L(K_{3,3})$ (equivalent to $\overline{G} = K_5 \cup L(\overline{2C_3})$).

If $N_1 = 7$, then $N_2 = 7$. Since every cocktail party graph contains an even number of vertices, and in view of (1), neither G_1 nor G_2 is a cocktail party graph or a regular exceptional graph. By Lemma 2.1, neither G_1 nor G_2 is a line graph of any regular graph. Thus, both G_1 and G_2 are the line graph of some semiregular bipartite graph. Next, suppose that G_1 is the line graph of an (r_1, r_2)-semiregular bipartite graph. By (3), $r_1 + r_2 = r_{\overline{G}} + 2 = 6$ and both r_1 and r_2 divide 7, which is impossible.

Finally, we consider the case of $N_1 = 6$ and $N_2 = 8$. Since G_2 is a 4-regular graph with eight vertices, it is not a cocktail party graph. If G_2 is a regular exceptional graph, then it is a 4-regular exceptional graph with 8 vertices belong to the third layer. If $G_2 = L(H)$ for a regular graph H, then H is a 3-regular graph with $\frac{16}{3}$ vertices by Lemma 2.1, a contradiction. Otherwise, $G_2 = L(H)$ for some (r_1, r_2)-semiregular bipartite graph. By (3), we have $r_1 + r_2 = r_{\overline{G}} + 2 = 6$ and both r_1 and r_2 divide 8. This implies that $r_1 = 2$ and $r_2 = 4$. By (3), we can conclude that $H = K_{2,4}$. Note that G_1 is a 4-regular graph with six vertices. Thus, $\overline{G_1}$ is 1-regular and so we have $G_1 = CP(3)$. Now, we have $\overline{G} = CP(3) \cup L(K_{2,4})$, or $\overline{G} = CP(3) \cup H$, where H is any of the 4-regular exceptional graphs with 8 vertices in the third layer. $\square$

Note that a 10-regular graph may have an odd number of vertices. Consequently, in contrast to the proof of Theorem 1.3, the proof of Theorem 1.4 becomes considerably more involved.

Proof of Theorem 1.4: If $n \geq 20$, then by Theorem 1.2, G is a complement of the Schläfli graph or $\overline{G} = L(K_7)$ or $\overline{G} = L(K_{2,11})$ or $\overline{G} = 2K_{10}$. Otherwise, $11 \leq n \leq 19$, where $n = 11$ implies that $G = K_{11}$. For $n = 12$, $\overline{G}$ is 1-regular and so $G = CP(6)$. For $n = 13$, $\overline{G}$ is 2-regular and so $\overline{G}$ is equal to C_{13}, $C_3 \cup C_{10}$, $C_4 \cup C_9$, $C_5 \cup C_8$, $C_6 \cup C_7$, $2C_3 \cup C_7$, $C_3 \cup C_4 \cup C_6$, $C_3 \cup 2C_5$, $2C_4 \cup C_5$, $3C_3 \cup C_4$. Next, we suppose that $14 \leq n \leq 19$. Since $r = 10$, $\overline{G}$ is not the cocktail party graph. Note that if $\overline{G}$ is r-regular and r is odd, then each component of $\overline{G}$ must contain an even number of vertices to satisfy the handshaking lemma.

Case 1. $n = 19$. Then $\overline{G}$ is an 8-regular graph. We start by considering the case that $\overline{G}$ is connected. By (1), $\overline{G}$ is not one of these 187 regular exceptional graphs. If $\overline{G} = L(H)$ for a regular graph H, then H is a 5-regular graph with $\frac{38}{5}$ vertices by Lemma 2.1, a contradiction. If $\overline{G}$ is the line graph of a semiregular bipartite graph, then by (3), we have $r_1 + r_2 = r_{\overline{G}} + 2 = 10$ and both r_1 and r_2 divide 19, but this is impossible.

Otherwise, $\overline{G}$ is disconnected. Since $r_{\overline{G}} = 8$, each component of $\overline{G}$ has at least 9 vertices, which implies that $\overline{G}$ contains exactly two components. If $\overline{G}$ contains one component having 9 vertices, then this component must be K_9 and the other component (say H) has exactly 10 vertices. Since H must be 8-regular, $\overline{H}$ must be 1-regular and so $H = CP(5)$. Now, we can conclude that $\overline{G} = K_9 \cup CP(5)$.

Case 2. $n = 18$. Then $r_{\overline{G}} = 18 - 10 - 1 = 7$. Firstly, we consider the case when $\overline{G}$ is connected. By (1), $\overline{G}$ is not one of these 187 regular exceptional graphs. If $\overline{G} = L(H)$, where H is a regular graph, then by Lemma 2.1, $n - r + 1 = 9$ divides $4n = 76$, a contradiction. If $\overline{G}$ is the line graph of a semiregular bipartite graph, then by (3), we have $r_1 + r_2 = 9$ and both r_1 and r_2 divide 18. The only solution implies that $\overline{G} = L(K_{3,6})$. Then we consider the disconnected cases. Since $r_{\overline{G}} = 7$, each component of $\overline{G}$ has at least 8 vertices, which implies that $\overline{G}$ contains exactly two components. Let $\overline{G} = G_1 \cup G_2$, where G_i is a 7-regular graph with N_i vertices. Then, we suppose that $8 \leq N_1 \leq N_2 \leq 10$. Since G_1 and G_2 are both 7-regular graphs, we must have $N_1 = 8$ (and so $G_1 = K_8$) and $N_2 = 10$. In this case, G_2 is a 7-regular graph with 10 vertices, and obviously G_2 is neither a cocktail party graph nor a regular exceptional graph by (1). By Lemma 2.1, G_2 is not a line graph of any regular graph. Thus, G_2 is the line graph of an (r_1, r_2)-semiregular bipartite graph. By (3), $r_1 + r_2 = r_{\overline{G}} + 2 = 9$ and both r_1 and r_2 divide 10, which is impossible.

Case 3. $n = 17$. Then $r_{\overline{G}} = 6$. Firstly, we consider the case when $\overline{G}$ is connected. By (1), $\overline{G}$ is not one of these 187 regular exceptional graphs. If $\overline{G} = L(H)$ for a regular graph H, then by Lemma 2.1, we have $n - r + 1 = 8$ divides $4n = 68$, a contradiction. If $\overline{G}$ is the line graph for a semiregular bipartite graph, then by (3), we have $r_1 + r_2 = 8$ and both r_1 and r_2 divide 17, which is not possible.

Next, we consider the disconnected cases. Since $r_{\overline{G}} = 6$, each component of $\overline{G}$ has at least 7 vertices, which implies that $\overline{G}$ contains exactly two components. Let $\overline{G} = G_1 \cup G_2$, where G_i is a 7-regular graph with N_i vertices. Then, we suppose that $7 \leq N_1 \leq N_2 \leq 10$.

If $N_1 = 7$, then $G_1 = K_7$. In this case, G_2 is a 6-regular graph with 10 vertices,

and obviously G_2 is neither a cocktail party graph nor a regular exceptional graph by (1). When $G_2 = L(H)$ for a regular graph H, then H is 4-regular with 5 vertices by Lemma 2.1, and so $H = K_5$. We can conclude that $\overline{G} = K_7 \cup L(K_5)$. Otherwise, G_2 is the line graph of an (r_1, r_2)-semiregular bipartite graph. By (3), $r_1 + r_2 = 8$ and both r_1 and r_2 divide 10, which is impossible.

If $N_1 = 8$, then $N_2 = 9$. Since G_1 is 6-regular, then $\overline{G_1}$ must be 1-regular, we have $G_1 = CP(4)$. In this case, G_2 is a 6-regular graph with 9 vertices. By (1), G_2 is not one of these 187 regular exceptional graphs. Since G_2 is 6-regular, it is neither a cocktail party graph nor a line graph of a regular graph by Lemma 2.1. Thus, G_2 is the line graph of an (r_1, r_2)-semiregular bipartite graph. By (3), $r_1 + r_2 = r_{\overline{G}} + 2 = 8$ and both r_1 and r_2 divide 9, which is impossible.

Case 4. $n = 16$. Then $r_{\overline{G}} = 5$ is odd. First, we focus on the case of connected graphs. By (1), $\overline{G}$ is not one of these 187 regular exceptional graphs. If $\overline{G} = L(H)$ for a regular graph H, then by Lemma 2.1, $n - r + 1 = 7$ divides $4n = 64$, a contradiction. Otherwise, $\overline{G}$ is the line graph of a semiregular bipartite graph. By (3), we have $r_1 + r_2 = 7$ and both r_1 and r_2 divide 16, which is not possible.

Next, we consider the disconnected cases. Since $r_{\overline{G}} = 5$, each component of $\overline{G}$ has at least 6 vertices, which implies that $\overline{G}$ contains exactly two components. Let $\overline{G} = G_1 \cup G_2$, where G_i is a 5-regular graph with N_i vertices. Then, we suppose that $6 \leq N_1 \leq N_2 \leq 10$. Since G_1 and G_2 are both 5-regular graphs, both N_1 and N_2 are even. Thus, either $N_1 = 6 < 10 = N_2$ or $N_1 = N_2 = 8$.

If $N_1 = 6$, then $G_1 = K_6$. In this case, G_2 is a 5-regular graph with 10 vertices, and obviously G_2 is neither a cocktail party graph nor a regular exceptional graph by (1). If $G_2 = L(H)$ for a regular graph H, then by Lemma 2.1, $N_2 - 5 + 1 = 6$ divides $4N_2 = 40$, a contradiction. Otherwise, G_2 is the line graph of a semiregular bipartite graph H, then by (3), we have $r_1 + r_2 = 7$ and both r_1 and r_2 divide 10, so $H = K_{2,5}$. We can conclude that $\overline{G} = K_6 \cup L(K_{2,5})$.

If $N_1 = 8$, then $N_2 = 8$. By (1), G_1 is not one of these 187 regular exceptional graphs. Since G_1 is 5-regular, it is neither a cocktail party graph nor a line graph of any regular graph by Lemma 2.1. Thus G_1 is the line graph of an (r_1, r_2)-semiregular bipartite graph. By (3), $r_1 + r_2 = r_{\overline{G}} + 2 = 7$ and both r_1 and r_2 divide 8, which is impossible. Similarly, G_2 is the same as G_1.

Case 5. $n = 15$. Then $r_{\overline{G}} = 15 - 10 - 1 = 4$. Firstly, we consider the case when $\overline{G}$ is connected. By (1), $\overline{G}$ is not one of these 187 regular exceptional graphs. If $\overline{G} = L(H)$ for a regular graph H, then by Lemma 2.1, H is a 3-regular graph with $N = 10$ vertices. Note that $B_1, B_2, \ldots, B_{19}$ are all 3-regular graphs with $N = 10$ vertices. If $\overline{G}$ is the line graph of a semiregular bipartite graph, then by (3), we have $r_1 + r_2 = 6$ and both r_1 and r_2 divide 15. The only solution implies that $r_1 = r_2 = 3$ and $n_1 = n_2 = 5$. In this case, we have $\overline{G} \in \{L(B_1), L(B_2)\}$.

Next, we consider the disconnected cases. Since $r_{\overline{G}} = 4$, each component of $\overline{G}$ has at least 5 vertices, which implies that $\overline{G}$ contains exactly two or three components. Clearly, if $\overline{G}$ contains three components, then $\overline{G} = 3K_5$. Thus, we may suppose that $\overline{G} = G_1 \cup G_2$, where G_i is a 4-regular graph with N_i vertices. Then $5 \leq N_1 \leq N_2 \leq 10$.

If $N_1 = 5$, then $G_1 = K_5$. In this case, G_2 is a 4-regular graph with 10 vertices, and obviously G_2 is neither a cocktail party graph nor a regular exceptional graph by (1). If $G_2 = L(H)$ for a regular graph H, then by Lemma 2.1, $N_2 - 4 + 1 = 7$ divides $4N_2 = 40$, a contradiction. If G_2 is the line graph of a semiregular bipartite graph H, then by (3), we have $r_1 + r_2 = 6$ and both r_1 and r_2 divide 10, which is also not possible.

If $N_1 = 6$, then $G_1 = CP(3)$ (equal to $L(K_4)$). In this case, G_2 is a 4-regular graph with 9 vertices, and obviously G_2 is not a cocktail party graph. When G_2 is a regular exceptional graph, then G_2 is a 4-regular exceptional graph with 9 vertices belongs to the second layer by (1), we have $\overline{G} = CP(3) \cup H$, where H is a 4-regular exceptional graph on 9 vertices from the second layer. If $G_2 = L(H)$, where H is a regular graph, then by Lemma 2.1, H is a 3-regular graph with 6 vertices. $\overline{H}$ must be 2-regular. So $\overline{H} = C_6$ or $2C_3$. Now we can conclude that $\overline{G} = CP(3) \cup L(\overline{C_6})$ or $\overline{G} = CP(3) \cup L(\overline{2C_3})$. If G_2 is the line graph of a semiregular bipartite graph H, then by (3), we have $r_1 + r_2 = 6$ and both r_1 and r_2 divide 9, the only solution is $H = K_{3,3}$.

The case where $N_1 = 7$ and $N_2 = 8$ can be proved similarly, and there exist no such graphs satisfying the conditions.

Case 6. $n = 14$. Then $r_{\overline{G}} = 3$. Firstly, we consider the case when $\overline{G}$ is connected. By (1), $\overline{G}$ is not one of these 187 regular exceptional graphs. If $\overline{G} = L(H)$, where H is a regular graph, then by Lemma 2.1, $n - 10 + 1 = 5$ divides $4n = 56$, a contradiction. If $\overline{G}$ is the line graph of a semiregular bipartite graph, then by (3), we have $r_1 + r_2 = 5$ and both r_1 and r_2 divide 14, again, not possible.

Next, we consider the disconnected cases. Since $r_{\overline{G}} = 3$, each component of $\overline{G}$ has at least 4 vertices, which implies that $\overline{G}$ contains exactly two or three components. We first consider the case that $\overline{G}$ contains two components. Let $\overline{G} = G_1 \cup G_2$, where G_i is a 3-regular graph with N_i vertices. Then, we suppose that $4 \leq N_1 \leq N_2 \leq 10$. Since $r_{\overline{G}} = 3$ is odd, each component of $\overline{G}$ contains an even number of vertices. Thus, we have either $N_1 = 4 < 10 = N_2$ or $N_1 = 6 < 8 = N_2$.

If $N_1 = 4$, then $G_1 = K_4$. In this case, G_2 is a 3-regular graph with 10 vertices, and obviously G_2 is not a cocktail party graph. When G_2 is a regular exceptional graph, then G_2 is a 3-regular exceptional graph with 10 vertices belonging to the first layer by (1), and we have $\overline{G} = K_4 \cup H$, where H is a 3-regular exceptional graph on 10 vertices from the first layer. If $G_2 = L(H)$ is a regular line graph, then by Lemma 2.1, H must be a semiregular bipartite graph. By (3), we have $r_1 + r_2 = 5$, and both r_1 and r_2 divide 10, which is not possible.

The case where $N_1 = 6$ and $N_2 = 8$ can be proved similarly, and there exist no such graphs satisfying the conditions.

Finally, we consider the case where $\overline{G}$ contains three components. Since each component of $\overline{G}$ contains at least four vertices and each component of $\overline{G}$ contains an even number of vertices, the number of vertices of the components can only be $(4, 4, 6)$. Let $\overline{G} = G_1 \cup G_2 \cup G_3$, where G_i is a 3-regular graph with N_i vertices. Let G_1 be the component containing 6 vertices. Obviously G_1 is not a cocktail party

graph nor a regular exceptional graph by (1). If $G_1 = L(H)$, where H is a regular graph, then by Lemma 2.1, $r_H = \frac{3}{2} + 1$, a contradiction. If G_1 is the line graph of a semiregular bipartite graph H, then by (3) we have $r_1 + r_2 = 5$ and both r_1 and r_2 divide 6; by inspection, the only solution is: $r_1 = 2$, $r_2 = 3$, $n_1 = 3$, $n_2 = 2$, and so we have $G_1 = L(K_{2,3})$. Since K_4 is the unique 3-regular graph of order 4, we can conclude that $\overline{G} = L(K_{2,3}) \cup 2K_4$.

The proof is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

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