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Pattern-avoiding Cayley permutations via combinatorial species

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Abstract

A Cayley permutation is a word of positive integers such that if a letter appears in this word, then all positive integers smaller than that letter also appear. We initiate a systematic study of pattern avoidance on Cayley permutations adopting a combinatorial species approach. Our methods lead to species equations, generating series, and counting formulas for Cayley permutations avoiding any pattern of length at most three. We also introduce the species of primitive structures as a generalization of Cayley permutations with no "flat steps". Finally, we explore various notions of Wilf equivalence arising in this context.

1 Introduction

A permutation w (written in one-line notation) is said to contain a permutation p as a pattern if some subsequence of the entries of w has the same relative order as all of the entries of p. If w does not contain p, then w is said to avoid p. One of the first notable results in the field of permutation patterns was obtained by MacMahon [52] in 1915 when he proved that the Catalan numbers enumerate the 123-avoiding permutations. The study of permutation patterns began receiving focused attention following Knuth's introduction of stack-sorting in 1969 [48]. Knuth proved that a permutation can be sorted by a stack if and only if it avoids the pattern 231. The Catalan numbers also enumerate the stack-sortable permutations. The

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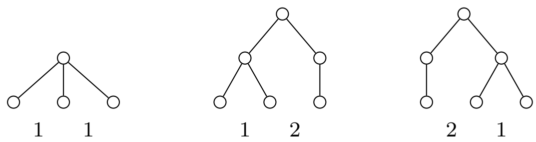
first explicit systematic treatment of pattern avoidance was conducted by Simion and Schmidt [66]. In subsequent years, the notion of pattern avoidance has been extended to numerous combinatorial objects, including set partitions [28, 44, 63], multiset permutations [39, 64], compositions [39, 45, 64], ascent sequences [26, 28, 29], and modified ascent sequences [14, 15]. We refer the reader to the books by Bóna [10] and Kitaev [47] for a comprehensive summary of pattern avoidance in permutations and words.

In this paper, we study pattern avoidance in the setting of Cayley permutations, which were so named by Mor and Fraenkel in 1983 [56]. A Cayley permutation w is a word of positive integers such that if a letter b appears in w then all positive integers $a < b$ also appear in w. Let us provide some background on why these words may be interesting and how they got their name. In a short article from 1859, Cayley [12] counted a loosely-defined class of trees, and the following is an interpretation of their definition. The class in question consists of unlabeled rooted plane trees with a fixed number of leaves. In addition, all leaves are equidistant from the root, and the number of nodes at distance $i + 1$ from the root is either zero or larger than the number of nodes at distance i from the root. For instance, these are all such trees with three leaves:



Written between each pair of adjacent leaves is the shortest distance to a common ancestor of the two leaves. The resulting words are exactly the Cayley permutations of length two: 11, 12, and 21. This connection between the trees and the words seems to have been first noted by MacMahon [51], though he formulated it in terms of compositions of a “multipartite number”. Cayley derived the exponential generating series

$$\frac{1}{2 - e^x} = 1 + x + 3 \cdot \frac{x^2}{2!} + 13 \cdot \frac{x^3}{3!} + 75 \cdot \frac{x^4}{4!} + 541 \cdot \frac{x^5}{5!} + \cdots$$

for the number of trees with $n + 1$ leaves, which is the same as the number of Cayley permutations of length n. The coefficients 1, 1, 3, 13, 75, 541, ... of this series are known as the Fubini numbers, and they appear as entry A000670 in the Online Encyclopedia of Integer Sequences (OEIS) [59].

There are quite a few authors building on the work of Mor and Fraenkel [56]. They include Fraenkel and Mor [32], Fraenkel [31], Bernstein and Sloane [5], Göbel [34], Hoffman [40, 41], Baril [3], Jacques, Wong, and Woo [43], Mütze [57], Cerbai [13, 14, 15], Cerbai and Claesson [16, 17, 18], and Cerbai, Claesson and Sagan [21]. This list is not complete; it is not even complete when restricted to authors using the Cayley permutation terminology. It is even less complete when the multitude of different guises that Cayley permutations have appeared under are taken into account. Synonyms for Cayley permutation include *packed word* [30, 49, 53, 54], *surjective word* [38], *Fubini word* [8, 62, 67], and *initial word* [60].

Cayley permutations may be seen as representatives for equivalence classes of words modulo order isomorphism. For instance, if we take the ambient space of words to be $\{1,2,3\}^3$, then the equivalence class represented by 111 is $\{111, 222, 333\}$, while 112 represents $\{112, 113, 223\}$. This hints at the view that Cayley permutations simply are *patterns* [64] or (in statistics) *generalized ordinal patterns* [65].

A *weak order* [2, 9, 42, 43], also called a *preferential arrangement* [1, 7, 37, 58, 61] or a *race with ties* [55], is a way to order objects where ties are allowed: it is a reflexive, transitive, and total binary relation. To each Cayley permutation $w = w_1 w_2 \dots w_n$ there is an associated binary relation $\preceq$ defined by $i \preceq j \Leftrightarrow w_i \leq w_j$. This sets up a one-to-one correspondence between Cayley permutations and weak orders.

Cayley permutations encode *ballots*, also known as *ordered set partitions*: if ϖ is a ballot on $\{1, \dots, n\}$, then the corresponding Cayley permutation w has n letters and its ith letter equals the unique index j such that i belongs to the jth block of ϖ. While pattern avoidance has been studied in the context of ballots [9, 22, 35, 46], the notion of pattern avoidance we explore in this paper is distinct. The interplay between (equivalence classes of) pattern-avoiding Cayley permutations and modified ascent sequences was explored by Cerbai and Claesson [17]. The same authors introduced Caylerian polynomials [18], a generalization of the Eulerian polynomials that tracks the number of descents over Cayley permutations.

Most of the structures listed so far can be encoded as Cayley permutations satisfying additional properties. For instance, modified ascent sequences are Cayley permutations where an entry is an ascent top if and only if it is the leftmost copy of the corresponding integer in the sequence [17]. Another classical example is given by Stirling permutations [33], defined as 212-avoiding Cayley permutations in which each letter appears twice.

Our approach to pattern-avoiding Cayley permutations involves the use of combinatorial species, defined in Section 2, as formal objects that capture both their structural and enumerative properties. The framework of combinatorial species has proven useful in the context of Cayley permutations [19, 20]. In Section 3, we define the species of Cayley permutations and in Section 4, we introduce the notion of pattern avoidance for Cayley permutations. These sections lay the groundwork for the remainder of the paper. In Sections 5 and 6, we provide species descriptions that lead to generating series and counting formulas for Cayley permutations avoiding any pattern of length two and three. Table 1 summarizes our main results. In Section 7, we introduce the species of primitive structures as a generalization of Cayley permutations with no “flat steps” and study pattern avoidance in this context. The paper concludes in Section 8 with an exploration of various notions of Wilf equivalence in connection with Cayley permutations.

The work contained in this paper was initiated in the fourth author’s 2024 master’s thesis [36]. In an upcoming companion paper, we will tackle sets of patterns.

2 Species

In this section, we mimic the development of species in the book by Bergeron, Labelle and Leroux [4]. See also Claesson’s short introduction to the topic [23]. We utilize two different types of species, namely $\mathbb{B}$-species and $\mathbb{L}$-species. Loosely speaking, a species defines both a class of (labeled) combinatorial objects and how those objects are impacted by relabeling. This mechanism of relabeling is called the transport of structure.

A $\mathbb{B}$-*species* (or simply *species*) F is a rule that produces

- for each finite set U, a finite set $F[U]$;
- for each bijection $\sigma : U \to V$, a bijection $F[\sigma] : F[U] \to F[V]$ such that $F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]$ for all bijections $\sigma : U \to V$, $\tau : V \to W$, and $F[\mathrm{id}_U] = \mathrm{id}_{F[U]}$ for the identity map $\mathrm{id}_U : U \to U$.

An element $s \in F[U]$ is called an F*-structure* on U and the function $F[\sigma]$ is called the *transport of* F*-structures along* σ, or simply *transport of structure* if the context is clear.

In the language of category theory, a $\mathbb{B}$-species is a functor $F : \mathbb{B} \to \mathbb{B}$, where $\mathbb{B}$ is the category of finite sets with bijective functions as morphisms.

Below we list several species that will be used throughout this paper.

(a) E: sets;

(b) E_{even}: sets of even cardinality;

(c) E_{odd}: sets of odd cardinality;

(d) X: singletons;

(e) 1: characteristic of empty set;

(f) L: linear orders;

(g) $\mathcal{S}$: permutations;

(h) $\mathcal{C}$: cyclic permutations;

(i) Par: set partitions;

(j) Bal: ballots.

Let F and G be species. Then G is said to be a *subspecies* of F, and we write $G \subseteq F$, if it satisfies the following two conditions:

- for each finite set U, we have $G[U] \subseteq F[U]$;
- for any bijection $\sigma : U \to V$, we have $G[\sigma] = F[\sigma]|_{G[U]}$.

For a species F, we let F_+, F_{even}, F_{odd}, and F_n denote the subspecies of F consisting of F-structures on nonempty sets, sets with even cardinality, sets with odd cardinality, and sets with cardinality n, respectively.

Throughout this paper, we define $[n] = \{1, 2, \ldots, n\}$ for $n \geq 0$, where $[0] = \emptyset$.

Example 2.1 (Linear orders)**.** Let U be a finite set with $n = |U|$ elements. We identify a linear order of U with a bijection $f : [n] \to U$, which we may represent using one-line notation: $f(1) \ldots f(n)$. Letting B^A denote the set of functions from A to B, we have $L[U] = \{f \in U^{[n]} : f \text{ bijection}\}$. Note that $|L[U]| = n!$. The transport of structure along a bijection $\sigma : U \to V$ is given by $L[\sigma](f) = \sigma \circ f$ or, in one-line notation, $L[\sigma](f) = \sigma(f(1)) \ldots \sigma(f(n))$.

Example 2.2 (Permutations)**.** Next we describe the species $\mathcal{S}$ of permutations. The $\mathcal{S}$-structures are defined by $\mathcal{S}[U] = \{f \in U^U : f \text{ bijection}\}$ while the corresponding transport of structure along a bijection $\sigma : U \to V$ is given by $\mathcal{S}[\sigma](f) = \sigma \circ f \circ \sigma^{-1}$. This reflects the fact that conjugation preserves the cycle type of a permutation. Note that $|\mathcal{S}[U]| = n!$ when $|U| = n$.

To aid in the enumeration of F-structures, we associate an exponential generating series, denoted by $F(x)$. It is easy to see that for any finite set U, the number of F-structures on U depends only on the number of elements of U. For ease of notation, we use $F[n] = F[[n]]$. We define the *(exponential) generating series* of the species F to be the formal power series

$$F(x) = \sum_{n \geq 0} |F[n]| \frac{x^n}{n!}.$$

The generating series associated with some of the species introduced above are provided below.

(a) $E(x) = e^x$;

(b) $E_{\text{even}}(x) = \cosh(x)$;

(c) $E_{\text{odd}}(x) = \sinh(x)$;

(d) $X(x) = x$;

(e) $1(x) = 1$;

(f) $L(x) = 1/(1 - x)$;

(g) $\mathcal{S}(x) = 1/(1 - x)$;

(h) $\mathcal{C}(x) = -\log(1 - x)$;

(i) $\text{Par}(x) = \exp(e^x - 1)$;

(j) $\text{Bal}(x) = 1/(2 - e^x)$.

Two species F and G are *isomorphic* if there is a family of bijections

$$\alpha_U : F[U] \to G[U]$$

such that for any bijection $\sigma : U \to V$ between two finite sets, the diagram in Figure 1 commutes. That is, $\alpha_V \circ F[\sigma] = G[\sigma] \circ \alpha_U$. In the language of category theory, F and G are isomorphic if and only if there exists a natural isomorphism between the functors F and G. As is tradition, we will consider two species as equal if they are isomorphic and use the notation $F = G$ for both concepts. Clearly, if $F = G$, then $F(x) = G(x)$. It is important to note that, for $\mathbb{B}$-species, the converse of this implication is false. Recall the species L (linear orders) and $\mathcal{S}$ (permutations) from Examples 2.1 and 2.2. Despite L and $\mathcal{S}$ not being isomorphic, it should come as no surprise that they have the same generating series.

One can build new species with operations on previously known species. For species F and G, an $(F + G)$*-structure* is either an F-structure or a G-structure. Denoting disjoint union by $\sqcup$, we have

$$(F + G)[U] = F[U] \sqcup G[U]$$

and for all bijections $\sigma : U \to V$,

$$(F + G)[\sigma](s) = \begin{cases} F[\sigma](s) & \text{if } s \in F[U], \\ G[\sigma](s) & \text{if } s \in G[U]. \end{cases}$$

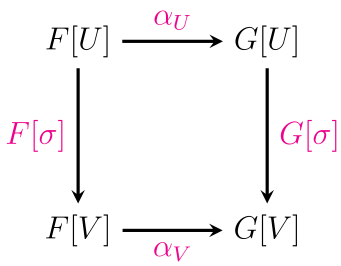


Figure 1: Commutative diagram for isomorphic species.

It is easy to see that $(F+G)(x) = F(x) + G(x)$.

For species F and G, we define an $(F \cdot G)$-*structure* on a finite set U to be a pair (s,t) such that s is an F-structure on a subset $U_1 \subseteq U$ and t is a G-structure on $U_2 = U \setminus U_1$. Formally,

$$(F \cdot G)[U] = \bigsqcup_{(U_1,U_2)} F[U_1] \times G[U_2]$$

with $U = U_1 \sqcup U_2$. Transport of structure is defined by

$$(F \cdot G)[\sigma](s,t) = (F[\sigma_1](s), G[\sigma_2](t)),$$

where $\sigma_1 = \sigma|_{U_1}$ and $\sigma_2 = \sigma|_{U_2}$. It turns out that $(F \cdot G)(x) = F(x) \cdot G(x)$.

Now we define the composition of species. For species F and G, an $(F \circ G)$-*structure* is a generalized partition in which each block of a partition carries a G-structure and blocks are structured by F. Formally, if F and G are two species such that $G[\emptyset] = \emptyset$, we define

$$(F \circ G)[U] = \bigsqcup_{\beta=\{B_1,\ldots,B_k\}} F[\beta] \times G[B_1] \times \cdots \times G[B_k],$$

where $\beta = \{B_1, \ldots, B_k\}$ is a partition of U. The details on the transport of structure can be found in Chapter 1 of the book by Bergeron, Labelle and Leroux [4] or on page 521 of Claesson [23]. As expected, it is also true that $(F \circ G)(x) = F(G(x))$.

Example 2.3 (Cyclic permutations)**.** Any permutation can be written as a product of disjoint cycles. Since the cycles are disjoint they commute and can be written in any order. In other words, a permutation can be viewed as a set of cycles. In terms of composition of species we have

$$\mathcal{S} = E(\mathcal{C}),$$

where $\mathcal{C}$ is the species of *cyclic permutations*; that is, permutations that when written as a product of disjoint cycles consist of a single cycle.

Example 2.4 (Ballots)**.** A *ballot* on a finite set U is a sequence of sets $B_1 B_2 \ldots B_k$, where each B_i is a nonempty subset of U, $B_i \cap B_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^{k} B_i = U$.

Each B_i is referred to as a *block*. The species of ballots is given by $\mathrm{Bal} = L(E_+)$. For a bijection $\sigma : U \to V$, the transport of structure $\mathrm{Bal}[\sigma] : \mathrm{Bal}[U] \to \mathrm{Bal}[V]$ is given by

$$\mathrm{Bal}[\sigma](B_1B_2 \dots B_k) = \sigma(B_1)\sigma(B_2)\dots\sigma(B_k),$$

where $\sigma(B_i) = \{\sigma(x) : x \in B_i\}$. It follows that

$$\mathrm{Bal}(x) = L(E_+(x)) = \frac{1}{1-(e^x-1)} = \frac{1}{2-e^x}.$$

The examples of species we have met thus far are well known [4]. In contrast, the species specification of the alternating group given in the next example is, as far as we know, new.

Example 2.5 (Alternating group)**.** Let a permutation $f \in \mathcal{S}[n]$ be given and assume that when written as a product of disjoint cycles it has c cycles. It is easy to see that the parity of f is the same as the parity of $n - c$. Hence f is even if and only if n and c are both even or n and c are both odd. Thus,

$$\mathrm{Alt} = (E_{\mathrm{even}} \circ \mathcal{C})_{\mathrm{even}} + (E_{\mathrm{odd}} \circ \mathcal{C})_{\mathrm{odd}}$$

is a species specification of the *alternating group*. The integers are commonly defined as equivalence classes of pairs of natural numbers, where $(a, b) \sim (c, d) \Leftrightarrow a+d = b+c$. Virtual species [4, §2.5] are a similar extension allowing for the subtraction of species. Using virtual species, for any species F, we have

$$F_{\mathrm{even}} = F_0 + F_2 + F_4 + \cdots = \frac{1}{2}\big(F + F(-X)\big);$$
$$F_{\mathrm{odd}} = F_1 + F_3 + F_5 + \cdots = \frac{1}{2}\big(F - F(-X)\big).$$

For ease of notation, let $\mathcal{S}_e = E_{\mathrm{even}}(\mathcal{C})$ and $\mathcal{S}_o = E_{\mathrm{odd}}(\mathcal{C})$ be the species of permutations with an even and odd number of cycles, respectively. Then,

$$\begin{aligned}\mathrm{Alt} &= (\mathcal{S}_e)_{\mathrm{even}} + (\mathcal{S}_o)_{\mathrm{odd}}\\ &= \frac{1}{2}\big(\mathcal{S}_e + \mathcal{S}_e(-X)\big) + \frac{1}{2}\big(\mathcal{S}_o - \mathcal{S}_o(-X)\big)\\ &= \frac{1}{2}\big(\mathcal{S} + \mathcal{S}_e(-X) - \mathcal{S}_o(-X)\big). \qquad (1)\end{aligned}$$

We will return to this example once $\mathbb{L}$-species have been introduced.

The *derivative* of a species F, denoted F', is defined via

$$F'[U] = F[U \sqcup \{\star\}]$$

and for a bijective map $\sigma : U \to V$, we define $F'[\sigma] = F[\tau]$, where

$$\tau(x) = \begin{cases}\sigma(x) & \text{if } x \in U,\\ \star & \text{if } x = \star.\end{cases}$$

In terms of generating series, we have $F'(x) = \frac{d}{dx}[F(x)]$.

Example 2.6. To illustrate the derivative of a species, we look at the derivative of the species of linear orders. We claim that $L' = L^2$. Combinatorially, each L'-structure is simply an L-structure preceding $\star$ followed by another L-structure. That is, the derivative of a linear order just separates the linear order into two linearly ordered components. For instance, $41 \star 5263 \mapsto (41, 5263)$. Consequently, we have

$$L'(x) = \frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{1}{(1-x)^2} = L^2(x).$$

The last operation we will define for $\mathbb{B}$-species is *pointing*. For a species F, we define the species $F^\bullet$, called F*-pointed*, via

$$F^\bullet[U] = F[U] \times U.$$

That is, an $F^\bullet$*-structure* on U is a pair (s, u), where s is an F-structure on U and $u \in U$ is a distinguished element that we can think of as being "pointed at". The operations of pointing and differentiation are related by

$$F^\bullet = X \cdot F'.$$

Further, we have $|F^\bullet[n]| = n|F[n]|$.

Next we look at $\mathbb{L}$-species. The key differences are that for $\mathbb{L}$-species, the underlying sets are totally ordered and morphisms have to respect that order.

Let $\ell_1 = (U_1, \preceq_1)$ and $\ell_2 = (U_2, \preceq_2)$ be two totally ordered sets. Their *ordinal sum* is the totally ordered set $\ell = (U, \preceq)$, denoted by $\ell = \ell_1 \oplus \ell_2$, where

$$u \preceq v \iff \begin{cases} u \prec_1 v & \text{if } u, v \in U_1, \\ u \prec_2 v & \text{if } u, v \in U_2, \\ u \in U_1 \text{ and } v \in U_2 & \text{otherwise.} \end{cases}$$

In other words, ℓ respects ℓ_1 and ℓ_2, and all elements of ℓ_1 are smaller than the elements of ℓ_2. The totally ordered set obtained by adding a new minimum element to ℓ is denoted by $1 \oplus \ell$. Similarly, we can append a new maximum element to obtain $\ell \oplus 1$.

An $\mathbb{L}$*-species* is a functor $F : \mathbb{L} \to \mathbb{B}$, where $\mathbb{L}$ is the category of finite totally ordered sets with order preserving bijections. In other words, an $\mathbb{L}$-species is a rule F that associates

- for each finite totally ordered set ℓ, a finite set $F[\ell]$;
- for each order preserving bijection $\sigma : \ell_1 \to \ell_2$, a bijection $F[\sigma] : F[\ell_1] \to F[\ell_2]$ such that $F[\sigma \circ \tau] = F[\sigma] \circ F[\tau]$ for all order preserving bijections $\sigma : \ell_1 \to \ell_2$, $\tau : \ell_2 \to \ell_3$, and $F[\mathrm{id}_\ell] = \mathrm{id}_{F[\ell]}$.

Any $\mathbb{B}$-species F produces an $\mathbb{L}$-species, also denoted by F, defined by setting

$$F[(U, \preceq)] = F[U],$$

for any totally ordered set $\ell = (U, \preceq)$, where the transport of structure is obtained by restriction to order preserving bijections. All of the $\mathbb{B}$-species defined earlier have the same name when interpreted as $\mathbb{L}$-species.

Two $\mathbb{L}$-species F and G are *isomorphic* if there is a family of bijections

$$\alpha_\ell : F[\ell] \to G[\ell],$$

for each totally ordered set ℓ that commutes with the transports of structures.

For an $\mathbb{L}$-species F, the associated generating series is defined in the same way as for $\mathbb{B}$-species: $F(x) = \sum_{n\geq 0} |F[n]| x^n/n!$. Since there is a unique order preserving bijection between any two totally ordered sets with the same cardinality we have that, for $\mathbb{L}$-species, $F = G$ if and only if $F(x) = G(x)$. In particular, it is possible for two nonisomorphic $\mathbb{B}$-species to become isomorphic when looked at as $\mathbb{L}$-species. For example, as $\mathbb{L}$-species, $L = \mathcal{S}$. In this case, $f \in \mathcal{S}[n]$ naturally becomes the word $f(1)f(2)\dots f(n) \in L[n]$.

Operations on $\mathbb{B}$-species can be extended to $\mathbb{L}$-species while new operations such as integration, ordinal product, and convolution also become possible. For an $\mathbb{L}$-species F, we define the *derivative* F' via

$$F'[\ell] = F[1 \oplus \ell].$$

Certainly, one can equivalently append a new maximal element to a totally ordered set ℓ as opposed to a new minimal element, hence we have the alternative definition $F'[\ell] = F[\ell \oplus 1]$. We also define the *integral* of F, denoted $\int_0^X F(T)dT$, or more simply $\int F$, by

$$\left(\int F\right)[\ell] = \begin{cases} \emptyset & \text{if } \ell = \emptyset, \\ F[\ell \setminus \{\min(\ell)\}] & \text{if } \ell \neq \emptyset. \end{cases}$$

Given $\mathbb{L}$-species F and G, we define the *ordinal product* $F \odot G$ via

$$(F \odot G)[\ell] = \sum_{\ell = \ell_1 \oplus \ell_2} F[\ell_1] \times G[\ell_2].$$

In contrast to the product structure, an ordinal product structure $(F \odot G)[\ell]$ is obtained by splitting ℓ into an initial segment ℓ_1 and a terminal segment ℓ_2, where ℓ_1 has an F-structure and ℓ_2 has a G-structure. The *convolution product* $F * G$ is given by

$$F * G = F \odot X \odot G.$$

Details about the corresponding transport of structures for each of these $\mathbb{L}$-species operations can be found in the usual book [4].

According to the standard reference [4], if F and G are $\mathbb{L}$-species, then we have the following:

(a) $(F + G)(x) = F(x) + G(x)$;

(b) $(F \cdot G)(x) = F(x)G(x)$;

(c) $(F \circ G)(x) = F(G(x))$, where $G(0) = 0$;

(d) $F'(x) = \frac{d}{dx}F(x)$;

(e) $\left(\int_0^X F(T)dT\right)(x) = \int_0^x F(t)dt$;

(f) $(F * G)(x) = F(x) * G(x) = \int_0^x F(x-t)G(t)dt$.

In addition,

(g) $\frac{d}{dx}[F(x) * G(x)] = F(0) \cdot G(x) + F'(x) * G(x)$,

which is sometimes referred to as the *Leibniz rule*.

Example 2.7 (Alternating group as an $\mathbb{L}$-species)**.** Clearly, we have the $\mathbb{B}$-species identity $E = E_{\text{even}} + E_{\text{odd}}$, which is reflected in the familiar hyperbolic functions identity $e^x = \cosh(x) + \sinh(x)$. In a similar vein, consider the combinatorial counterpart of the identity $\cosh(x)^2 - \sinh(x)^2 = 1$, namely $E_{\text{even}}^2 = 1 + E_{\text{odd}}^2$. This holds as an identity among $\mathbb{L}$-species, but it is false as a $\mathbb{B}$-species identity. While there are two "unlabeled" E_{even}^2-structures of size two, namely $(\{\circ,\circ\},\emptyset)$ and $(\emptyset,\{\circ,\circ\})$, there is a single unlabeled $(1+E_{\text{odd}}^2)$-structure of size two, namely $(\{\circ\},\{\circ\})$. Continuing with the $\mathbb{L}$-species setting, we have

$$\begin{aligned} 1 &= E_{\text{even}}^2 - E_{\text{odd}}^2 \\ &= (E_{\text{even}} - E_{\text{odd}})(E_{\text{even}} + E_{\text{odd}}) \\ &= (E_{\text{even}} - E_{\text{odd}}) \cdot E, \end{aligned}$$

which when composing with $\mathcal{C}$ yields $1 = (\mathcal{S}_e - \mathcal{S}_o) \cdot \mathcal{S}$. The multiplicative inverse of $\mathcal{S}$ is the virtual species $1 - X$, and hence $\mathcal{S}_e - \mathcal{S}_o = 1 - X$. Applying this to expression (1) for the species Alt in Example 2.5, we arrive at

$$\mathrm{Alt} = \frac{1}{2}\big(\mathcal{S} + 1 + X\big). \tag{2}$$

Continuing with the topic of permutations, we present the following proposition.

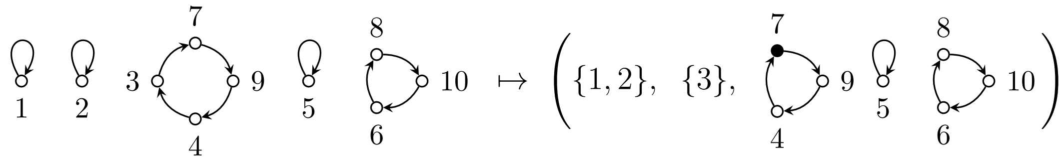
Proposition 2.8. $\mathcal{S} = E + E * \mathcal{S}^\bullet$.

Proof. Let $n \geq 0$. We shall construct a bijection from $\mathcal{S}[n]$ to $(E + E * \mathcal{S}^\bullet)[n]$. Let a permutation $\pi \in \mathcal{S}[n]$ be given. If $\pi = 12\dots n$ is the identity permutation, then we map π to $\{1, 2, \dots, n\}$, the single element of $E[n]$. Assume that π has at least one point that is not fixed. In this case we will have to map π to an element of $(E * \mathcal{S}^\bullet)[n]$. Recall that, by definition of convolution,

$$E * \mathcal{S}^\bullet = E \odot X \odot \mathcal{S}^\bullet.$$

Now, let k be the largest integer such that $1, 2, \ldots, k$ are fixed points in π. Note that $0 \leq k < n-1$. We interpret this maximal prefix of fixed points as the set $\{1, 2, \ldots, k\}$ (an E-structure). Consider the remaining suffix of π. Remove $k+1$ from the cycle it belongs to (this does not alter the number of cycles since $k+1$ is not a fixed point) and interpret $k+1$ as a singleton (an X-structure). From what remains of the suffix we form an $\mathcal{S}^\bullet$-structure by pointing at the element $\pi(k+1)$. Note that $\pi(k+1) > k+1$ by the choice of k. □

Example 2.9. Below is an example of the bijection in the proof of Proposition 2.8.



In this picture, a permutation $\pi \in \mathcal{S}[n]$ is represented by its *functional digraph*: the directed graph with vertex set $[n]$ and arcs $i \to \pi(i)$. The vertex pointed at is black.

Let $\overline{\mathrm{Alt}}$ denote the subspecies of $\mathcal{S}$ consisting of the odd permutations, which is the complement of the alternating group in the sense that $\mathcal{S} = \mathrm{Alt} + \overline{\mathrm{Alt}}$. We have the following corollary of Proposition 2.8, which will turn out to be useful when proving Theorem 6.4.

Corollary 2.10. $\overline{\mathrm{Alt}} = E * \mathrm{Alt}^\bullet$.

Proof. Let π be an odd permutation. Consider the image of π under the bijection from the proof of Proposition 2.8. Removing a maximal prefix 1, 2, …, k of fixed points from π does not alter the parity of π. On the other hand, deleting $k+1$ from its cycle shortens that cycle by one and reverses its parity. It follows that the restriction of this bijection to odd permutations proves our corollary. See Example 2.9, above, for an example of this bijection. □

Example 2.11 (231-avoiding permutations)**.** A permutation or linear order $w = w_1 w_2 \ldots w_n$ on $[n]$ contains 231 (as a pattern) if it contains a subsequence $w_i w_j w_k$ that is order isomorphic to 231; that is, if $w_k < w_i < w_j$. Otherwise, w is 231-avoiding. It is well known that the number of 231-avoiding permutations of $[n]$ is given by the nth Catalan number, $\binom{2n}{n}/(n+1)$. As an illustration of the species approach adopted in this article we shall now give a somewhat unorthodox proof of this fact. Let $F = \mathcal{S}(231)$ be the $\mathbb{L}$-species of 231-avoiding permutations, which is a subspecies of $\mathcal{S}$. Let ℓ be a nonempty totally ordered set and let $w \in \mathcal{S}[\ell]$. Then w belongs to $F[\ell]$ if and only if there are totally ordered sets ℓ_1 and ℓ_2 such that $\ell = \ell_1 \oplus \ell_2 \oplus 1$, and w can be factored as $w = umv$, where $u \in F[\ell_1]$, $v \in F[\ell_2]$, and $m = \max(\ell)$. In other words, F is characterized by the following equation:

$$F = 1 + F \odot X \odot F = 1 + F * F.$$

Our goal is to show that $a_n = |F[n]|$ is the nth Catalan number. Note that $a_n = F^{(n)}(0)$, where $F^{(n)}$ is the nth derivative of F. Now, by the Leibniz rule,

$$F'(x) = (F' * F)(x) + a_0 F(x).$$

Differentiating this expression and, again, using the Leibniz rule, we obtain

$$F''(x) = (F'' * F)(x) + a_0 F'(x) + a_1 F(x).$$

Continuing this way we find that

$$F^{(n+1)}(x) = (F^{(n+1)} * F)(x) + \sum_{i=0}^{n} a_i F^{(n-i)}(x).$$

Note that $(F^{(n+1)} * F)(0) = 0$ and hence, by identifying coefficients in the above identity when $x = 0$, we obtain $a_0 = 1$ and

$$a_{n+1} = \sum_{i=0}^{n} a_i a_{n-i},$$

which is the usual recurrence for the Catalan numbers. We will present a kindred recurrence for the number of 231-avoiding Cayley permutations in Proposition 6.10.

3 Cayley permutations

In the introduction we defined a Cayley permutation as a word w of positive integers such that if a letter b appears in w then all positive integers $a < b$ also appear in w. Expressed differently, a Cayley permutation is a function $w : [n] \to [n]$ such that $\mathrm{Im}(w) = [k]$ for some $k \leq n$. To define the $\mathbb{B}$-species of Cayley permutations we have to generalize this a bit by allowing the domain to be any n element set. Thus, a *Cayley permutation* on a finite set U is a function $w : U \to [n]$ such that $|U| = n$ and $\mathrm{Im}(w) = [k]$ for some $k \leq n$. And we let Cay be the $\mathbb{B}$-species with structures

$$\mathrm{Cay}[U] \;=\; \big\{ w \in [n]^U :\; \mathrm{Im}(w) = [k] \text{ for some } k \leq n \big\},$$

together with the transport of structure along a bijection $\sigma : U \to V$ defined via

$$\mathrm{Cay}[\sigma](w) = w \circ \sigma^{-1}.$$

For $w \in \mathrm{Cay}[n]$, we utilize one-line notation and write $w = w_1 w_2 \dots w_n$, where $w_i = w(i)$. In this case, we say that w is of *length* n.

For example,

- $\mathrm{Cay}[1] = \{1\}$;
- $\mathrm{Cay}[2] = \{11, 12, 21\}$;
- $\mathrm{Cay}[3] = \{111, 112, 121, 122, 123, 132, 211, 212, 213, 221, 231, 312, 321\}$.

For a finite set U, we define $\alpha_U : \mathrm{Cay}[U] \to \mathrm{Bal}[U]$ via

$$\alpha_U(w) = B_1 B_2 \dots B_k, \text{ where } k = |\,\mathrm{Im}(w)| \text{ and } B_i = w^{-1}(\{i\}).$$

This map is clearly bijective and hence $|\,\mathrm{Bal}[n]| = |\,\mathrm{Cay}[n]|$. It also follows (see Example 2.4) that $\mathrm{Cay}(x) = \mathrm{Bal}(x) = 1/(2 - e^x)$. We shall see in Proposition 3.2 below that, via the bijections α_U, the species Cay and Bal are in fact isomorphic.

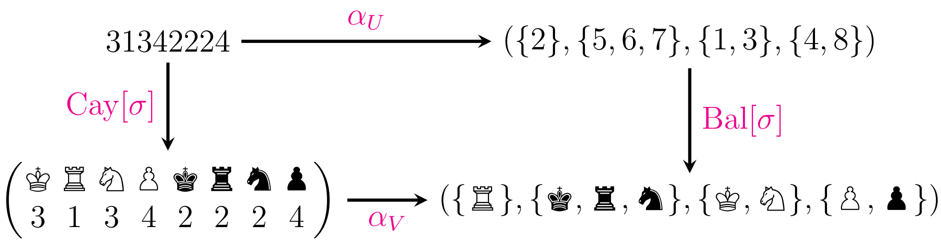


Figure 2: An instance of the compatibility between the transport of structures for Cay and Bal described in Example 3.1.

Example 3.1. We wish to illustrate how the bijections α_U respect the transport of structure induced by Cay and Bal, a fact we will prove formally in Proposition 3.2. Let $U = [8]$ and $V = \{$♔, ♚, ♖, ♜, ♘, ♞, ♙, ♟$\}$. Define a bijection $\sigma : U \longrightarrow V$ by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \text{♔} & \text{♖} & \text{♘} & \text{♙} & \text{♚} & \text{♜} & \text{♞} & \text{♟} \end{pmatrix}.$$

That is, $\sigma(1) =$ ♔, $\sigma(2) =$ ♖, $\sigma(3) =$ ♘, etc. The action of the maps $\mathrm{Bal}[\sigma] \circ \alpha_U$ and $\alpha_V \circ \mathrm{Cay}[\sigma]$ on the $\mathrm{Cay}[U]$-structure $w = 31342224$ is illustrated by the commutative diagram in Figure 2.

Proposition 3.2. *As* $\mathbb{B}$*-species,* Cay = Bal.

Proof. Let $\sigma : U \to V$ be a bijection between finite sets U and V. We aim to show that Figure 1 commutes for species Cay and Bal. Let $w \in \mathrm{Cay}[U]$ with $\mathrm{Im}(w) = [k]$. We see that

$$\begin{aligned}
\big(\alpha_V \circ \mathrm{Cay}\big)[\sigma](w) &= \alpha_V(\mathrm{Cay}[\sigma](w)) \\
&= \alpha_V(w \circ \sigma^{-1}) \\
&= (w \circ \sigma^{-1})^{-1}(\{1\}) \;\ldots\; (w \circ \sigma^{-1})^{-1}(\{k\}) \\
&= \mathrm{Bal}[\sigma](w^{-1}(\{1\})) \;\ldots\; \mathrm{Bal}[\sigma](w^{-1}(\{k\})) \\
&= \mathrm{Bal}[\sigma](\alpha_U(w)) \\
&= \big(\mathrm{Bal}[\sigma] \circ \alpha_U\big)(w).
\end{aligned}$$

□

4 Pattern avoidance in Cayley permutations

In this section, we will introduce pattern avoidance in the context of Cayley permutations and ballots. In order to sensibly define pattern avoidance for Cayley permutations we must restrict the domain to a totally ordered set, which allows us to write Cayley permutations in one-line notation. One consequence of this is that all species of pattern-avoiding Cayley permutations are $\mathbb{L}$-species. Furthermore, since there is a unique order preserving bijection between any pair of finite totally ordered sets of the same cardinality we may, when convenient, assume that the underlying total order is $[n]$ with the standard order.

Consider the Cayley permutations $w = w_1w_2 \dots w_n \in \mathrm{Cay}[n]$ and $p = p_1p_2 \dots p_k \in \mathrm{Cay}[k]$. We say that w *contains* p if there exists a subsequence $w_{i_1}w_{i_2} \dots w_{i_k}$ with $i_1 < i_2 < \dots < i_k$ that is order isomorphic to p (i.e., for every $a < b$ we have $w_{i_a} < w_{i_b} \Leftrightarrow p_a < p_b$ and $w_{i_a} = w_{i_b} \Leftrightarrow p_a = p_b$). If w does not contain a pattern p, then we say that w *avoids* p.

Let $\mathrm{Cay}(p)$ be the $\mathbb{L}$-species of p-avoiding Cayley permutations. The corresponding structures are

$$\mathrm{Cay}(p)[n] = \{w \in \mathrm{Cay}[n] : w \text{ avoids } p\}.$$

The transport of structure is inherited from Cay. We also define $\mathrm{Bal}(p)[n]$ to be the image of $\mathrm{Cay}(p)[n]$ under the natural bijection between Cayley permutations and ballots described earlier. It follows that $\mathrm{Cay}(p) = \mathrm{Bal}(p)$ as $\mathbb{L}$-species.

Example 4.1. Consider $\mathrm{Cay}(1^k)$ for $k \geq 2$, where $1^k = 11 \dots 1$ consists of k copies of 1. A 1^k-avoiding ballot has blocks of sizes at most $k-1$. That is, blocks may be any size between 1 and $k-1$, and hence

$$\mathrm{Cay}(1^k) = \mathrm{Bal}(1^k) = L(E_1 + E_2 + \dots + E_{k-1}).$$

It follows that

$$\begin{aligned}\mathrm{Cay}(1^k)(x) &= L(E_1 + \dots + E_{k-1})(x)\\ &= L(E_1(x) + \dots + E_{k-1}(x)) = \left(1 - \sum_{i=1}^{k-1} \frac{x^i}{i!}\right)^{-1}.\end{aligned}$$

We now extend the idea of Wilf equivalence [6] to Cayley permutations. We say that two patterns p and q are *(Wilf) Cayley-equivalent* if $|\mathrm{Cay}(p)[n]| = |\mathrm{Cay}(q)[n]|$ for all n; equivalently, if $\mathrm{Cay}(p) = \mathrm{Cay}(q)$ as $\mathbb{L}$-species.

The standard reverse and complement maps on permutations are easily generalized to Cayley permutations. Given a Cayley permutation $w = w_1 \dots w_n$, the *reverse map* is defined via

$$r(w_1 \dots w_n) = w_n \dots w_1$$

and the *complement map* is given by

$$c(w_1 \dots w_n) = (\max(w) + 1 - w_1) \dots (\max(w) + 1 - w_n),$$

where $\max(w) = \max\{w_i : i \in [n]\}$. Certainly, the reverse and complement maps are bijections on $\mathrm{Cay}[n]$. This implies that p, $r(p)$, $c(p)$, and $(r \circ c)(p)$ are all Cayley-equivalent for a pattern p. The class generated by reverse and complement of a single pattern p is the *symmetry class* of p. Note that two patterns may be Cayley-equivalent without being in the same symmetry class. Note also that there is no counterpart to the inverse map on $\mathrm{Cay}[n]$, a reason for which Cayley-equivalence does not correspond to the usual Wilf equivalence on permutations. For instance, let $p = 1342$ and $q = 1423$. Then p and q are Wilf equivalent on permutations since $p = q^{-1}$. On the other hand, p and q are not Cayley-equivalent:

$$|\mathrm{Cay}(1342)[7]| = 33712 \neq 33710 = |\mathrm{Cay}(1423)[7]|.$$

We will further explore this and other notions of equivalence in Section 8.

5 Patterns of length two

In this section, we will investigate Cayley permutations avoiding patterns of length two and we start with the pattern 11. A Cayley permutation avoiding 11 is simply a permutation. Or, in terms of ballots, a Bal(11)-structure is a ballot with singleton blocks only. We also note that the pattern 11 is a special case of the pattern 1^k studied in Example 4.1, and that $\mathrm{Cay}(11) = L(E_1) = L$ and $\mathrm{Cay}(11)(x) = 1/(1-x)$.

Next we consider the species $\mathrm{Cay}(21) = \mathrm{Bal}(21)$. Note that one could easily obtain the counting formula and corresponding generating series for Cay(12), but we have opted for a species-first approach, in part to illustrate the utility of combinatorial species. A nonempty ballot on $[n]$ avoids 21 precisely when the elements of its first block form an initial segment $[k]$ of $[n]$ and the remaining blocks form a ballot that also avoids 21. Letting $F = \mathrm{Bal}(21)$ this characterization is captured by the equation

$$F = 1 + E_+ \odot F = 1 + E * F.$$

Using induction it immediately follows that

$$F = 1 + 1 * \sum_{k\geq 1} E^{*k}, \tag{3}$$

in which $E^{*k} = E * \cdots * E$ is the convolution of k copies of E.

Lemma 5.1. *For any $k \geq 1$, we have $E^{*k} = E_{k-1}E$.*

Proof. For ease of notation let $k = 3$. By definition of convolution,

$$E^{*3} = E \odot X \odot E \odot X \odot E.$$

Consequently, an E^{*3}-structure on a totally ordered set ℓ is a 5-tuple

$$\gamma = (\ell_1, x_1, \ell_2, x_2, \ell_3) \text{ such that } \ell = \ell_1 \oplus \{x_1\} \oplus \ell_2 \oplus \{x_2\} \oplus \ell_3.$$

Clearly, γ is uniquely determined by the pair $\big(\{x_1, x_2\}, \ell_1 \cup \ell_2 \cup \ell_3\big)$, and this sets up a bijection proving $E^{*3} = E_2E$. The general case is analogous. $\square$

Continuing with derivation (3) prior to this lemma, we find that

$$\begin{aligned} F &= 1 + 1 * \sum_{k\geq 1} E^{*k} \\ &= 1 + 1 * E \sum_{k\geq 1} E_{k-1} = 1 + 1 * E^2 = 1 + \int E^2. \end{aligned}$$

Proposition 5.2. $\mathrm{Cay}(21) = 1 + \int E^2 = E_{\mathrm{even}} \cdot E.$

Proof. Let $F = 1 + \int E^2$ and $G = E_{\text{even}} \cdot E$. We have already established that $\text{Cay}(21) = F$. To show that $F = G$ we note that $F(0) = 1 = G(0)$, $F' = E^2$, and

$$G' = E'_{\text{even}}E + E_{\text{even}}E' = E_{\text{odd}}E + E_{\text{even}}E = E^2.$$

Alternatively, we may prove $F = G$ using an explicit bijection: Let ℓ be a finite totally ordered set. Define $\alpha_\ell : F[\ell] \to G[\ell]$ as follows. For $\ell = \emptyset$ let $\alpha_\ell(\emptyset) = (\emptyset, \emptyset)$. Assume ℓ is nonempty and let $x = \min(\ell)$. An F-structure on ℓ is a pair of disjoint sets (S, T) such that $S \cup T = \ell \setminus \{x\}$ and we define

$$\alpha_\ell(S,T) = \begin{cases} (S, T \cup \{x\}) & \text{if } |S| \text{ is even,} \\ (S \cup \{x\}, T) & \text{if } |S| \text{ is odd.} \end{cases}$$

It is easy to verify that α_ℓ indeed is a bijection. □

Corollary 5.3. $\text{Cay}(21)(x) = \cosh(x)e^x = \frac{1}{2}(e^{2x} + 1)$.

On extracting the coefficient of $x^n/n!$ in $\frac{1}{2}(e^{2x} + 1)$ we find that

$$\big|\text{Cay}(21)[\emptyset]\big| = 1 \quad \text{and} \quad \big|\text{Cay}(21)[n]\big| = 2^{n-1} \quad \text{for } n \geq 1.$$

We could of course have derived this more directly by noting that any 21-avoiding Cayley permutation is weakly increasing and hence consists of a segment of 1s, followed by a segment of 2s, followed by a segment of 3s, etc. Thus there are as many 21-avoiding Cayley permutations as there are integer compositions of n.

6 Patterns of length three

Next we will study Cayley permutations that avoid patterns of length three. It follows from Example 4.1 that $\text{Cay}(111) = L(E_1 + E_2)$ and $\text{Cay}(111)(x) = 2/(2 - 2x - x^2)$. Using a partial fraction decomposition of $2/(2 - 2x - x^2)$ we may also derive the following explicit counting formula for all $n \geq 0$:

$$|\,\text{Cay}(111)[n]| = n! \cdot \frac{(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}}{2^{n+1}\sqrt{3}}.$$

Although 112 and 212 are Cayley-equivalent as a consequence of the results proved by Jelínek and Mansour on k-ary words [45] (see also Section 8), we will prove it here independently by showing that $\text{Cay}(112) = \text{Cay}(212)$ as $\mathbb{L}$-species. We will first provide a species equation for $\text{Cay}(112)$.

Proposition 6.1. $\text{Cay}(112)' = L \cdot \text{Cay}(112) + L \cdot \text{Cay}_+(112)$.

Proof. Let $n \geq 0$. We shall verify the result in terms of ballots by exhibiting a bijection from $\text{Bal}(112)'[n] = \text{Bal}(112)[n+1]$ to $(L \cdot \text{Bal}(112) + L \cdot \text{Bal}_+(112))[n]$. Consider a ballot $\varpi = B_1 B_2 \ldots B_k$ in $\text{Bal}(112)[n+1]$. For any $j \in [n+1]$, each block preceding the block containing j contains at most one value less than j. In particular, if the maximum value $n+1$ belongs to the ith block, then all preceding blocks must be singletons, say $B_1 = \{a_1\}$, $B_2 = \{a_2\}$, …, $B_{i-1} = \{a_{i-1}\}$. Note that we may view $a_1a_2\dots a_{i-1}$ as an L-structure. Let $\tilde{B}_i = B_i \setminus \{n+1\}$. It is easy to check that

$$\varpi \longmapsto \begin{cases} \big(a_1a_2\dots a_{i-1}, B_{i+1}\dots B_k\big) & \text{if } \tilde{B}_i = \emptyset, \\ \big(a_1a_2\dots a_{i-1}, \tilde{B}_iB_{i+1}\dots B_k\big) & \text{otherwise} \end{cases}$$

is the sought bijection. □

Theorem 6.2. $\mathrm{Cay}(112) = \mathrm{Alt}'$.

Proof. Clearly, $\mathrm{Cay}(112)[\emptyset] = \mathrm{Alt}'[\emptyset]$ and hence it suffices to show that $F = \mathrm{Alt}'$ satisfies the differential equation $F' = L\cdot(F+F_+) = L\cdot(2F-1)$ from Proposition 6.1. Recall from equation (2) that $\mathrm{Alt} = (\mathcal{S}+1+X)/2 = (L+1+X)/2$ and hence $\mathrm{Alt}' = (1+L')/2$. Finally, using $L' = L^2$, we get $F' = \mathrm{Alt}'' = L^3 = L\cdot(2F-1)$. □

By the preceding theorem we have $\mathrm{Cay}(112) = \mathrm{Alt}' = (1+L^2)/2$ and since $L(x) = 1/(1-x)$, we arrive at the following result.

Corollary 6.3. $\mathrm{Cay}(112)(x) = \dfrac{1}{2}\left(1+\dfrac{1}{(1-x)^2}\right) = \dfrac{x^2-2x+2}{2(x-1)^2}$.

We now transition to 212-avoiding Cayley permutations.

Proposition 6.4. $\mathrm{Cay}(212) = 1 + E * \mathrm{Cay}(212) + E * \mathrm{Cay}(212)^\bullet$.

Proof. Let $n \geq 0$. For any species F we have $E * F = E \odot X \odot F = E_+ \odot F$. It thus suffices to provide a bijection from $\mathrm{Cay}(212)[n]$ to

$$\big(1 \,+\, E_+ \odot \mathrm{Cay}(212) \,+\, E_+ \odot \mathrm{Cay}(212)^\bullet\big)[n]$$

For $n=0$ we map the empty Cayley permutation to the empty set. Assume $n>0$ and $w \in \mathrm{Cay}(212)[n]$. Observe that the occurrences of the largest value $m = \max(w)$ in w must occur as a single contiguous string. That is, we can write

$$w \;=\; w_1\dots w_i m\dots m w_{i+d+1}\dots w_n,$$

where d is the number of occurrences of m in w and both $w_1\dots w_i$ and $w_{i+d+1}\dots w_n$ are 212-avoiding Cayley permutations in their own right. The size of the E_+-structure will be d and its role is simply to keep track of the number of occurrences of m. We consider two cases according to whether or not the contiguous string $m\dots m$ occurs on the far right. In the first case,

$$w \;\longmapsto\; \big(\{1,\dots,d\},\, w_1\dots w_i\big).$$

In the second case, we can identify a distinguished position immediately to the right of the contiguous string $m\dots m$:

$$w \;\longmapsto\; \big(\{1,\dots,d\},\, w_1\dots w_i w_j^\bullet \dots w_n\big).$$ □

Having established an equation characterizing Cay(212) we shall show in Theorem 6.6 that Cay(212) = Alt′ satisfies that equation, but first a simple lemma.

Recall that $\overline{\mathrm{Alt}} = \mathcal{S} - \mathrm{Alt}$ denotes the species of odd permutations.

Lemma 6.5. $\mathrm{Alt} = 1 + X + \overline{\mathrm{Alt}}$.

Proof. This can be proved bijectively by fixing a transposition, say τ, and observing that multiplication by τ reverses the parity. Alternatively, from equation (2) we see that $2\,\mathrm{Alt} = 1 + X + \mathcal{S} = 1 + X + \mathrm{Alt} + \overline{\mathrm{Alt}}$ and hence $\mathrm{Alt} = 1 + X + \overline{\mathrm{Alt}}$. □

Theorem 6.6. $\mathrm{Cay}(212) = \mathrm{Alt}'$.

Proof. If we let $F = \mathrm{Cay}(212)$, then Proposition 6.4 reads

$$F = 1 + E * F + E * F^{\bullet}. \tag{4}$$

Any $\mathbb{L}$-species admits a canonical antiderivative and differential equations admit unique solutions [4, Chapter 5]. Thus, it suffices to prove that $F = \mathrm{Alt}'$ satisfies this equation.

Starting from the right-hand side we have

$$\begin{aligned}
1 + E * \mathrm{Alt}' + E * (\mathrm{Alt}')^{\bullet} &= 1 + E * (\mathrm{Alt}' + X\,\mathrm{Alt}'') \\
&= 1 + E * (X\,\mathrm{Alt}')' \\
&= 1 + E * (\mathrm{Alt}^{\bullet})' \\
&= (1 + X + E * \mathrm{Alt}^{\bullet})' && \text{(by the Leibniz rule)} \\
&= (1 + X + \overline{\mathrm{Alt}})' && \text{(by Corollary 2.10)} \\
&= \mathrm{Alt}' && \text{(by Lemma 6.5),}
\end{aligned}$$

which concludes the proof. □

Corollary 6.7. $\mathrm{Cay}(212)(x) = \dfrac{1}{2}\left(1 + \dfrac{1}{(1-x)^2}\right) = \dfrac{x^2 - 2x + 2}{2(x-1)^2}.$

For $n \geq 1$, the number of alternating permutations of $[n]$ is $n!/2$ and since $\mathrm{Cay}(112) = \mathrm{Cay}(212) = \mathrm{Alt}'$, by Theorems 6.2 and 6.6, we immediately get the following counting formula for Cay(112) and Cay(212).

Corollary 6.8. *We have* $|\,\mathrm{Cay}(112)[\emptyset]| = |\,\mathrm{Cay}(212)[\emptyset]| = 1$ *and, for* $n \geq 1$,

$$\big|\mathrm{Cay}(112)[n]\big| = \big|\mathrm{Cay}(212)[n]\big| = \frac{1}{2}(n+1)!$$

Using the reverse and complement maps, we have the $\mathbb{L}$-species identities

$$\mathrm{Cay}(112) = \mathrm{Cay}(221) = \mathrm{Cay}(211) = \mathrm{Cay}(122) \quad \text{and} \quad \mathrm{Cay}(212) = \mathrm{Cay}(121).$$

However, the species Cay(112) and Cay(212) are the same according to Theorems 6.2 and 6.6, which implies that all six of the patterns are Cayley-equivalent.

Birmajer, Gil, Kenepp, and Weiner [9] counted weak orderings subject to various "stopping conditions". In particular, they showed that the number of weak orderings of size $n \geq 1$ subject to $x_{i_1} < x_{i_2} = x_{i_3}$ is $(n+1)!/2$. Translating this to our terminology, they showed that $|\operatorname{Cay}(122)[n]| = (n+1)!/2$. Now, the reverse-complement of 122 is 112, and hence the formula $|\operatorname{Cay}(112)[n]| = (n+1)!/2$ of Corollary 6.8 was already known.

Let us now consider the species Cay(231). We wish to provide a geometric decomposition of 231-avoiding Cayley permutations that has the same flavor as the well-known recursive description of $\mathcal{S}(231)$. Our technique is similar to the one used in Proposition 6.4 and it leads to an integral equation for Cay(231).

Proposition 6.9. $\operatorname{Cay}(231) = 1 + X + (4\operatorname{Cay}(231) - E) * \operatorname{Cay}(231)_+$.

Proof. Any nonempty $w = w_1 w_2 \dots w_n \in \operatorname{Cay}(231)[n]$ factors as

$$w \;=\; u \max(w) v,$$

where $w_i = \max(w)$ is the leftmost copy of $\max(w)$ in w, $u = w_1 \dots w_{i-1}$ is the prefix of w preceding w_i, and $v = w_{i+1} \dots w_n$ is the suffix of w succeeding w_i. Note that $\max(u) < \max(w)$ by our choice of $i \in [n]$. Note also that $\max(u) \leq \min(v)$, or else the triple $\max(u), \max(w), \min(v)$ would form an occurrence of 231 in w. Further, both u and v are (order isomorphic to) 231-avoiding Cayley permutations. Now, any $w \in \operatorname{Cay}(231)[n]$ falls into exactly one of the following cases:

1. $w = \epsilon$, the empty Cayley permutation.
2. $v = \epsilon$. That is, there is only one copy of $\max(w)$ and it is the last entry of w. This includes the single letter Cayley permutation $w = 1$, where $u = v = \epsilon$.
3. $u = \epsilon$ and $v \neq \epsilon$. In this case, either $\max(w) = \max(v)$ or $\max(w) = \max(v)+1$.
4. $u \neq \epsilon$ and $v \neq \epsilon$. Here, there are four sub-cases to be considered, each giving rise to distinct Cayley permutations. In the first three, u and v are allowed to be (order isomorphic to) any 231-avoiding Cayley permutations, while the fourth case needs some additional care.
   (a) $\min(v) = \max(u) + 1$ and $\max(w) = \max(v) + 1$;
   (b) $\min(v) = \max(u) + 1$ and $\max(w) = \max(v)$;
   (c) $\min(v) = \max(u)$ and $\max(w) = \max(v) + 1$;
   (d) $\min(v) = \max(u)$ and $\max(w) = \max(v)$; here, the case where $\max(v) = \min(v)$ is forbidden since otherwise we would have $\max(u) = \max(w)$, which is impossible.

Let $F = \operatorname{Cay}(231)$. The above case analysis leads to the following equation

$$F \;=\; \underbrace{1}_{1} \;+\; \overbrace{\int F}^{2} \;+\; \overbrace{2\int F_+}^{3} \;+\; \underbrace{3F_+ * F_+}_{4(a)+4(b)+4(c)} \;+\; \underbrace{F_+ * (F - E)}_{4(d)}, \tag{5}$$

where we have annotated the terms with the corresponding cases. Each term can be straightforwardly derived using the techniques developed in this article. We omit most of the details and only consider the last term, corresponding to case 4(d):

$$F_+ * (F - E) = F_+ \odot X \odot (F - E).$$

Consider $w = u\max(w)v \in \mathrm{Cay}(231)[n]$ as above. In case 4(d), u is nonempty, but otherwise unconstrained, which gives the F_+ factor. The singleton $\max(w)$ corresponds to the X factor. Further, v is nonempty and $\max(v) > \min(v)$, resulting in the $F - E$ factor. The reason for subtracting E from F is that there is exactly one Cayley permutation of fixed length and fixed underlying alphabet, where $\max(v) = \min(v)$, namely the constant sequence. Finally, the total contribution of 4(d) is obtained as the ordinal product $F_+ \odot X \odot (F - E)$ of the three factors, and on simplifying (5) we obtain the claimed equation. □

Recall that in Example 2.11 we explored the species $F = \mathcal{S}(231)$ of 231-avoiding permutations. It satisfies $F = 1 + F * F$ and by repeated differentiation we derived a familiar recursion for the Catalan numbers. We can apply the same technique to the species $F = \mathrm{Cay}(231)$ of 231-avoiding Cayley permutations. Our starting point is the equation $F = 1 + X + (4F - E) * F_+$ from the preceding proposition. By repeated differentiation, and use of the Leibniz rule, we find that

$$F^{(n+1)}(x) = F^{(n+1)}(x) * \big(4F(x) - E(x)\big) + \sum_{i=0}^{n-1}\big(4F^{(i)}(x) - E(x)\big)a_{n-i},$$

where $a_n = F^n(0) = |\,\mathrm{Cay}(231)[n]|$ is the number of 231-avoiding Cayley permutations on $[n]$. Through identifying coefficients in the above identity when $x = 0$, we obtain the recurrence relation presented below.

Proposition 6.10. *Let $a_n = |\,\mathrm{Cay}(231)[n]|$ be the number of 231-avoiding Cayley permutations. Then $a_0 = a_1 = 1$ and, for $n \geq 1$, we have*

$$a_{n+1} = \sum_{i=0}^{n-1}(4a_i - 1)a_{n-i}.$$

In 1985 Simion and Schmidt [66] gave a bijection between $\mathcal{S}(123)[n]$ and $\mathcal{S}(132)[n]$. It is considered a classic result in the permutation patterns literature. Simion and Schmidt presented their bijection as an algorithm, but it can also be stated in terms of equivalence classes induced by positions and values of left-to-right minima as follows [25]. Given a permutation $w = w_1 \dots w_n$, define

$$\mathrm{LMIN}(w) = \{(i, w_i) : w_j > w_i \text{ for every } j < i\}.$$

Further, let the equivalence class of w be the set of permutations w' where $\mathrm{LMIN}(w') = \mathrm{LMIN}(w)$. Each equivalence class defined this way contains exactly one permutation u avoiding 123 and one permutation v avoiding 132; Simion-Schmidt's bijection is the mapping $u \mapsto v$. More explicitly, the entries of u and v are uniquely determined as follows. Let $w \in \mathcal{S}[n]$. For $i = 1, 2, \dots, n$, if $(i, a) \in \mathrm{LMIN}(w)$, then let $u_i = v_i = a$. Otherwise:

- Let u_i be equal to the largest letter in $[n]$ that has not been used before.
- Let v_i be equal to the smallest letter in $[n]$ that has not been used before and is greater than all the letters used thus far.

Picking the largest letter not used at every step guarantees that the resulting permutation u avoids 123; indeed, a permutation avoids 123 if and only if the entries that are not left-to-right minima form a decreasing sequence. Similarly, picking the smallest letter not used ensures that v avoids 132. Finally, the additional constraint that the letter picked at every step is greater than all the letters seen before ensures that this construction gives the desired set of left-to-right minima positions and values $\textsc{lmin}(u) = \textsc{lmin}(v)$.

As the reader may have guessed, we can modify the Simion-Schmidt construction to obtain a bijection between $\mathrm{Cay}(123)[n]$ and $\mathrm{Cay}(132)[n]$ for $n \geq 0$. Given $w \in \mathrm{Cay}[n]$, let

$$\textsc{wlmin}(w) = \{(i, w_i) : w_j \geq w_i \text{ for every } j < i\}$$

be the set of weak left-to-right minima positions and values of w. Further, let the *filling* of w be the multiset of its letters that are not weak left-to-right minima

$$\textsc{fill}(w) = \{w_i : (i, w_i) \notin \textsc{wlmin}(w)\}.$$

Now, define the equivalence class of w as the set of Cayley permutations $w' \in \mathrm{Cay}[n]$, where $\textsc{wlmin}(w') = \textsc{wlmin}(w)$ and $\textsc{fill}(w') = \textsc{fill}(w)$. Once again, we shall see that each equivalence class defined this way contains exactly one Cayley permutation $u \in \mathrm{Cay}(123)[n]$ and one Cayley permutation $v \in \mathrm{Cay}(132)[n]$, which are defined as follows. Let $w \in \mathrm{Cay}[n]$. For $i = 1, 2, \dots, n$, if $(i, a) \in \textsc{wlmin}(w)$, then let $u_i = v_i = a$. Otherwise:

- Let u_i be equal to the largest letter in $\textsc{fill}(w)$ that has not been used before.
- Let v_i be equal to the smallest letter in $\textsc{fill}(w)$ that has not been used before and is (strictly) greater than all the letters used thus far.

The additional requirement that $\textsc{fill}(u) = \textsc{fill}(v)$ is necessary as otherwise, due to the possibility of having repeated entries, there could be more than one 123-avoiding Cayley permutation or more than one 132-avoiding Cayley permutation with the same positions and values of weak left-to-right minima. To illustrate this construction, let

$$w = \underline{7}\,\underline{7}\,9\,8\,\underline{5}\,9\,9\,\underline{5}\,6\,7\,\underline{4}\,\underline{1}\,2\,6\,3\,\underline{1}\,3\,3,$$

where the underscored entries are the weak left-to-right minima of w. We have:

$$\begin{aligned}\textsc{wlmin}(w) &= \{(1,7),(2,7),(5,5),(8,5),(11,4),(12,1),(16,1)\};\\ \textsc{fill}(w) &= \{2,3,3,3,6,6,7,8,9,9,9\}.\end{aligned}$$

To obtain the only 123-avoiding Cayley permutation u in the equivalence class of w, we keep the same positions and values of weak left-to-right minima and arrange the letters of $\textsc{fill}(w)$ in weakly decreasing order:

$$u = \underline{7}\,\underline{7}\,9\,9\,\underline{5}\,9\,8\,\underline{5}\,7\,6\,\underline{4}\,\underline{1}\,6\,3\,3\,\underline{1}\,3\,2.$$

Finally, the corresponding 132-avoiding Cayley permutation is

$$v = \underline{7}\,\underline{7}\,8\,9\,\underline{5}\,6\,6\,\underline{5}\,7\,9\,\underline{4}\,\underline{1}\,2\,3\,3\,\underline{1}\,3\,9.$$

To see that v avoids 132, suppose for a contradiction that there are three entries v_i, v_j, v_k with $i < j < k$ and $v_i < v_k < v_j$. Without loss of generality, we can assume that $(i, v_i) \in \text{WLMIN}(v)$. Note that (j, v_j) and (k, v_k) are not in $\text{WLMIN}(v)$. In particular, both v_j and v_k are in the filling of v, which contradicts $v_k < v_j$ since the procedure defined above should pick the smaller entry v_k before v_j.

Using the reverse and complement maps, together with the extension of the Simion-Schmidt bijection to Cayley permutations shown above, it follows that all permutations in $\mathcal{S}[3] = \{123, 321, 312, 213, 132, 231\}$ are Cayley-equivalent. An alternative way to obtain this result consists of combining results by Chen, Dai and Zhou [22] and Kasraoui [46] to obtain the following ordinary generating series for any $p \in \mathcal{S}[3]$:

$$\sum_{n\geq 0} \big|\text{Cay}(p)[n]\big| x^n = \frac{1}{2} + \frac{1}{1 + \sqrt{1 - 8x + 8x^2}}. \tag{6}$$

We also have the explicit formula [46, Proposition 1.1]:

$$|\operatorname{Cay}(p)[n]| = \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} |[j]^n(p)|. \tag{7}$$

The number of p-avoiding words over an alphabet of k letters was determined by Burstein [11, Theorem 3.2]:

$$|[k]^n(p)| = 2^{n-2(k-2)} \sum_{m=0}^{k-2} \sum_{j=m}^{k-2} C_j \binom{2(k-2-j)}{k-2-j} \binom{n+2m}{n}, \tag{8}$$

where C_j is the jth Catalan number. Combining formulas (7) and (8) leads to a rather unwieldy quadruple sum for $|\operatorname{Cay}(p)[n]|$. Luckily there is a more compact way of expressing the result. Birmajer et al. [9] enumerated weak orderings subject to the "stopping condition" $x_{i_1} < x_{i_2} < x_{i_3}$. In our terminology they showed that

$$\big|\text{Cay}(123)[n]\big| = \sum_{j=0}^{n} (-1)^j 2^{n-j-1} \binom{n-j}{j} C_{n-j} \tag{9}$$

for $n \geq 1$, where C_n is the nth Catalan number.

The results from Sections 5 and 6 are summarized in Table 1.

7 Primitive Cayley permutations

A Cayley permutation $w = w_1 \dots w_n$ over $[n]$ is said to be *primitive* if it is nonempty ($n \geq 1$) and $w_i \neq w_{i+1}$ for $1 \leq i \leq n-1$. That is, a primitive Cayley permutation has no "flat steps". Under the designation *multipermutations*, primitive Cayley permutations have figured in the work of Lam and Pylyavskyy [50], as well as in the work of Marberg [54]. We let Prim denote the $\mathbb{L}$-species of primitive Cayley permutations.

Pattern	Species	Series	Enumeration ($n \geq 1$)	OEIS
11	L	$\dfrac{1}{1-x}$	$n!$	A000142
12	$E_{\text{even}} \cdot E$	$\dfrac{e^{2x}+1}{2}$	2^{n-1}	A011782
21				
111	$L(E_1 + E_2)$	$\dfrac{2}{2-2x-x^2}$	$n! \cdot \dfrac{(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}}{2^{n+1}\sqrt{3}}$	A080599
212	Alt'	$\dfrac{x^2-2x+2}{2(x-1)^2}$	$\dfrac{(n+1)!}{2}$	A001710
121				
112				
211				
221				
122				
123			$\displaystyle\sum_{j=0}^{n} (-1)^j 2^{n-j-1} \binom{n-j}{j} C_{n-j}$	A226316
132				
213				
231				
321				
312				

Table 1: Results for patterns of lengths two and three.

Lemma 7.1. $\mathrm{Cay} = 1 + \int(E \cdot \mathrm{Prim}')$.

Proof. Let $F = 1 + \int(E \cdot \mathrm{Prim}')$ and let n be a nonnegative integer. We shall define a bijection $\alpha : F[n] \to \mathrm{Cay}[n]$. Since there is a unique F-structure on the empty set, which can be mapped to the empty Cayley permutation, we may assume that $n > 0$. Now, an F-structure on $[n]$ is an $(E{\cdot}\mathrm{Prim}')$-structure on $[n]\setminus\{1\} = \{2, 3, \ldots, n\}$. Such a structure is a pair (S, v), where S is a subset of $\{2, 3, \ldots, n\}$ and v is a primitive Cayley permutation on $1 \oplus T = \{1\} \cup T$ with $T = \{2, 3, \ldots, n\} \setminus S$. Let k be the number of letters of v and write $v = v_1v_2 \ldots v_k$. Then $\alpha(v) = w = w_1w_2 \ldots w_n$ is the Cayley permutation obtained from filling n slots as follows. Write down the letters $v_1, v_2, \ldots, v_k$ of v on the slots belonging to T. Note that $w_1 = v_1$. Moving from left to right, fill in the vacant slots by duplicating the letter to its left. For instance, let $S = \{2, 3, 7\}$ and $v = 325154$. We start by filling the slots $\{1, 4, 5, 6, 8, 9\}$ with the letters of v, obtaining 3 _ _ 2 5 1 _ 5 4, then we fill in the vacant slots and arrive at $w = 333251154$. It is easy to see how to reverse this process and thus α is a bijection. □

Theorem 7.2. $\mathrm{Prim}' = \mathrm{Cay}^2$.

Proof. Differentiating the equation given in Lemma 7.1 yields $\mathrm{Cay}' = E{\cdot}\mathrm{Prim}'$. Next we solve for Prim' by multiplying by the inverse of E. It is the virtual species

$$E^{-1} = (1 + E_+)^{-1} = \sum_{k\geq 0}(-1)^k E_+^k.$$

Thus, an E^{-1}-structure is a ballot such that if it has an even number of blocks it is considered positive, while if it has an odd number of blocks it is considered negative. Continuing with our derivation we get

$$\begin{aligned}\mathrm{Prim}' &= E^{-1} \cdot \mathrm{Cay}' \\ &= E^{-1} \cdot \mathrm{Bal}' \\ &= E^{-1} \cdot (L(E_+))' \\ &= E^{-1} \cdot E_+' \cdot L^2(E_+) \\ &= L^2(E_+) \\ &= \mathrm{Bal}^2 \\ &= \mathrm{Cay}^2\,.\end{aligned}$$ □

The OEIS entry for the coefficients of $\mathrm{Cay}^2(x)$ is A005649.

A consequence of Lemma 7.1 is $\mathrm{Cay}' = E \cdot \mathrm{Prim}'$. Inspired by this identity, we more generally say that if two species F and G are related by $F(0) = 1$, $G(0) = 0$, and

$$F' = E \cdot G',$$

then G is the species of *primitive F-structures*. For $\mathbb{L}$-species, this is equivalent to

$$F = 1 + \int(E \cdot G').$$

Examples include:

- Primitive Cay is $\int \mathrm{Cay}^2$ (above).
- Primitive Par is $\int \mathrm{Par}$ (similar to above).
- Primitive (modified) ascent sequences are nonempty (modified) ascent sequences with no flat steps [14, 27].
- Primitive $\mathcal{S}$ is $\int(\mathrm{Der} + \mathrm{Der}') = \int \mathrm{Der} + \mathrm{Der}_+$ as follows from differentiating $\mathcal{S} = E \cdot \mathrm{Der}$, in which Der is the species of derangements. Alternatively, Claesson [24] proved that primitive permutations are the integral of those avoiding the Hertzsprung pattern 12.

The equation $F' = E \cdot G'$ yields the formula

$$|F[n]| = \sum_{j=1}^{n} \binom{n-1}{j-1} |G[j]| \qquad \text{for } n \geq 1, \tag{10}$$

where $|F[0]| = 1$. Similarly, the equation $G' = E^{-1} \cdot F'$ yields the formula

$$|G[n]| = \sum_{j=1}^{n} (-1)^{n-j} \binom{n-1}{j-1} |F[j]| \qquad \text{for } n \geq 0. \tag{11}$$

Let $\mathrm{Prim}(p)$ be the species of primitive Cayley permutations avoiding the pattern p. The proof of the following proposition is almost identical to that of Lemma 7.1 and we omit the details. Note, however, that it is crucial for p to be primitive. Otherwise, an occurrence of p could be created in the process of duplicating the letters of a $\mathrm{Prim}(p)'$-structure.

Proposition 7.3. *If p is a primitive Cayley permutation, then*

$$\mathrm{Cay}(p) = 1 + \int (E \cdot \mathrm{Prim}(p)')$$

and

$$\mathrm{Prim}(p) = \int (E^{-1} \cdot \mathrm{Cay}(p)').$$

For instance, $\mathrm{Cay}(21)' = E^2$ by Proposition 5.2 and hence

$$\mathrm{Prim}(21) = \int (E^{-1} \cdot \mathrm{Cay}(21)') = \int E = E_+.$$

This should come as no surprise since, for each $n \geq 1$, the only primitive 21-avoiding Cayley permutation on $[n]$ is the identity permutation $12 \ldots n$.

For another instance, 212 is a primitive pattern and $\mathrm{Cay}(212) = \mathrm{Alt}'$ by Theorem 6.6. Thus $\mathrm{Prim}(212) = \int(E^{-1}L^3)$ and the corresponding generating series is

$$\begin{aligned} \mathrm{Prim}(212)(x) &= \int_0^x \frac{e^{-t}}{(1-t)^3}\, dt \\ &= x + 2 \cdot \frac{x^2}{2!} + 7 \cdot \frac{x^3}{3!} + 32 \cdot \frac{x^4}{4!} + 181 \cdot \frac{x^5}{5!} + 1214 \cdot \frac{x^6}{6!} + \cdots \end{aligned}$$

whose coefficients form sequence A000153 in the OEIS.

Since any permutation is a primitive Cayley permutation, Proposition 7.3 applies whenever the pattern p is a permutation. Consider $p = 231$, or any other permutation in $\mathcal{S}[3]$; they are all Cayley-equivalent by the work in Section 6. To calculate

$$\mathrm{Prim}(231)(x) = \int_0^x e^{-t}\,\mathrm{Cay}(231)(t)\,dt \tag{12}$$

we would need to know $\mathrm{Cay}(231)(x)$, and it may be possible to find an expression for $\mathrm{Cay}(231)(x)$ by solving the equation in Proposition 6.9, but we have failed to do so. It is, however, easy to calculate the numbers $\big|\mathrm{Cay}(231)[n]\big|$; e.g., using formula (9). The result is sequence A226316:

$$1, 1, 3, 12, 56, 284, 1516, 8384, 47600, 275808, 1624352, 9694912, \ldots$$

Using equation (11) we may calculate the corresponding numbers $\big|\mathrm{Prim}(231)[n]\big|$ for primitive Cayley permutations. They turn out to be

$$0, 1, 2, 7, 28, 121, 550, 2591, 12536, 61921, 310954, 1582791, \ldots$$

and match sequence A010683, whose ordinary generating series is the square of that for the Schröder–Hipparchus numbers (A001003), shifted by one. We have arrived at an educated guess:

$$\sum_{n\geq 0}\big|\mathrm{Prim}(231)[n]\big|x^n \;=\; x\left(\frac{2}{1+x+\sqrt{1-6x+x^2}}\right)^2. \tag{13}$$

Can we prove this? Yes, but first we need to translate the relationship (12) for the exponential generating series to a relationship between ordinary generating series. For this discussion, let $a_0, a_1, a_2, \ldots$ be a sequence of numbers and let $A(x)$ and $\hat{A}(x)$ be the corresponding exponential and ordinary generating series. Furthermore, let $b_0, b_1, b_2, \ldots$ be another sequence of numbers and define $B(x)$ and $\hat{B}(x)$ in the same manner. The integral "transform", $A(x) \mapsto \int_0^x A(t)\,dt$, corresponds to a shift of the coefficients, $a_n \mapsto a_{n-1}$, which in turn corresponds to $\hat{A}(x) \mapsto x\hat{A}(x)$. The derivative, $A(x) \mapsto A'(x)$, gives a shift in the other direction, $a_n \mapsto a_{n+1}$, which corresponds to $\hat{A}(x) \mapsto (\hat{A}(x) - 1)/x$. Finally, the so-called inverse binomial transform, $a_n = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}b_j$, or $B(x) = e^{-x}A(x)$, corresponds to $\hat{A}(x) = \hat{B}\big(x/(1+x)\big)/(1+x)$. Putting this all together we can translate equation (12) and the result is

$$\begin{aligned}\sum_{n\geq 0}\big|\mathrm{Prim}(231)[n]\big|x^n \;&=\; \frac{x}{1+x}\left[\frac{1}{t}\left(\sum_{n\geq 0}\big|\mathrm{Cay}(231)[n]\big|t^n - 1\right)\right]_{t=\frac{x}{1+x}} \\ &=\; \frac{1+x}{1+x+\sqrt{1-6x+x^2}} - \frac{1}{2},\end{aligned}$$

where we have used (6) to calculate the explicit expression. It is easy to verify that this answer is consistent with our guess (13).

Despite its simplicity, we could not find a bijective proof of the species equality $\mathrm{Prim}' = \mathrm{Cay}^2$ of Theorem 7.2. The interplay between Prim' and Cay is not limited to this equality. Indeed, we also conjecture the following two equalities.

Conjecture 7.4. *For all $n \geq 1$, we have*

$$\frac{1}{2}|\operatorname{Prim}'[n]| = \frac{1}{n}\sum_{w\in\operatorname{Cay}[n]}\sum_{i=1}^{n} w_i = \sum_{w\in\operatorname{Cay}[n]}\sum_{i\in\text{FIX}(w)} i,$$

where $\text{FIX}(w) = \{i : w_i = i\}$ *is the set of fixed points of* w.

8 Wilf equivalences

Recall from Section 4 that two Cayley permutations p and q are Cayley-equivalent if $|\operatorname{Cay}(p)[n]| = |\operatorname{Cay}(q)[n]|$ for every $n \geq 0$. The following proposition summarizes the Cayley-equivalence results from Section 6.

Proposition 8.1. *Cayley-equivalence partitions patterns of length three into the following three classes:*

- $\{111\}$;
- $\{112, 121, 122, 211, 212, 221\}$;
- $\{123, 132, 213, 231, 312, 321\}$.

Jelínek and Mansour [45] considered two alternative notions of equivalence on k-ary words: strong (word) equivalence and word equivalence. We define them below, together with their natural counterparts on Cayley permutations. We also introduce an analogue to Cayley-equivalence on k-ary words. We expand a bit on the topic of Wilf equivalences on words and Cayley permutations by relating some of the equivalences defined this way. Finally, we state some open problems for future investigation.

Let the *content* of a word be the multiset of its letters. Further, denote by $\operatorname{Cay}^k(p)[n]$ the set of Cayley permutations over $[n]$ that avoid p and whose maximum value is equal to k. Two Cayley permutations p and q are

- *strong-word-equivalent* ($p \sim_{sw} q$) if for every k, n there is a bijection between $[k]^n(p)$ and $[k]^n(q)$ that preserves the content;
- *word-equivalent* ($p \sim_w q$) if $|[k]^n(p)| = |[k]^n(q)|$ for every k, n;
- *endo-equivalent* ($p \sim_e q$) if $|[n]^n(p)| = |[n]^n(q)|$ for every n,

where “endo” is short for endofunction. Similarly, p and q are

- *strong-Cayley-equivalent* ($p \sim_{sc} q$) if for every k, n there is a bijection between $\operatorname{Cay}^k(p)[n]$ and $\operatorname{Cay}^k(q)[n]$ that preserves the content;
- *Cayley-max-equivalent* ($p \sim_{cm} q$) if $|\operatorname{Cay}^k(p)[n]| = |\operatorname{Cay}^k(q)[n]|$ for every k, n;
- *Cayley-equivalent* ($p \sim_c q$) if $|\operatorname{Cay}(p)[n]| = |\operatorname{Cay}(q)[n]|$ for every n.

Clearly, strong-word-equivalence implies word-equivalence, which in turn implies endo-equivalence. Similarly, strong-Cayley-equivalence implies Cayley-max-equivalence, which in turn implies Cayley-equivalence. Furthermore, Kasraoui [46, Proposition 1.1] showed that if p and q are permutations and $p \sim_w q$, then $p \sim_c q$. We shall prove that strong-word-equivalence coincides with strong-Cayley-equivalence, and that word-equivalence coincides with Cayley-max-equivalence. To do that, let us consider the following *standardization* map. Let $w \in [n]^n$ and let A be a subset of $[n]$ with $|A| = |\operatorname{Im}(w)|$. Then $\operatorname{std}_A(w)$ is the word obtained by replacing each copy of the ith smallest letter of w with the ith smallest element of A. As an example, let $w = 337217813 \in [9]^9$ and let $A = \{2, 5, 6, 7, 9\}$. Then $\operatorname{std}_A(w) = 667527926$. By choosing $A = [k]$ with $k = |\operatorname{Im}(w)|$, we obtain a Cayley permutation $\operatorname{std}_A(w)$ whose maximum value is equal to k. Moreover, the standardization map preserves pattern avoidance and containment; more explicitly, for any pattern p we have that w contains p if and only if $\operatorname{std}_A(w)$ contains p.

Lemma 8.2. *For each Cayley permutation p and for each $k, n \geq 0$, we have*

$$|[k]^n(p)| = \sum_{i=0}^{k} \binom{k}{i} |\operatorname{Cay}^i(p)[n]| \tag{14}$$

and

$$|\operatorname{Cay}^k(p)[n]| = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} |[i]^n(p)|. \tag{15}$$

Proof. Let $n, k \geq 0$. For any given i, the map $w \mapsto \big(\operatorname{Im}(w), \operatorname{std}_{[i]}(w)\big)$ is a bijection

$$\big\{w \in [k]^n(p) : |\operatorname{Im}(w)| = i\big\} \;\longrightarrow\; \binom{[k]}{i} \times \big\{w \in \operatorname{Cay}^i(p)[n] : \max(w) = i\big\}.$$

Its inverse is given by $(A, v) \mapsto \operatorname{std}_A(v)$. Thus,

$$\big|[k]^n(p)\big| \;=\; \sum_{i=0}^{k} \big|\big\{w \in [k]^n(p) : |\operatorname{Im}(w)| = i\big\}\big| \;=\; \sum_{i=0}^{k} \binom{k}{i} \big|\operatorname{Cay}^i(p)[n]\big|,$$

proving equation (14). This relation between the numbers $|\operatorname{Cay}^i(p)[n]|$ and $|[k]^n(p)|$ can easily be inverted to obtain (15). Indeed, letting $A(x) = \sum_{k \geq 0} |\operatorname{Cay}^k(p)[n]| x^k/k!$ and $B(x) = \sum_{k \geq 0} |[k]^n(p)| x^k/k!$, equation (14) amounts to $B(x) = e^x A(x)$, or, equivalently, $A(x) = e^{-x} B(x)$, which gives equation (15). □

Proposition 8.3. *Let p and q be Cayley permutations. Then*

$$p \sim_w q \iff p \sim_{cm} q.$$

Proof. If $p \sim_w q$, then $|[i]^n(p)| = |[i]^n(q)|$ for every i and thus $p \sim_{cm} q$ by equation (15). Similarly, if $p \sim_{cm} q$, then $|\operatorname{Cay}^k(p)[n]| = |\operatorname{Cay}^k(q)[n]|$ for every k, and $p \sim_w q$ by equation (14). □

Proposition 8.4. *Let p and q be Cayley permutations. Then*

$$p \sim_{sw} q \iff p \sim_{sc} q.$$

Proof. Suppose first that $p \sim_{sw} q$. That is, for each n, k there is a content-preserving bijection

$$f_{k,n} : [k]^n(p) \longmapsto [k]^n(q).$$

Then, we obtain a content-preserving bijection $g_{k,n}$ between $\operatorname{Cay}^k(p)[n]$ and $\operatorname{Cay}^k(q)[n]$ by simply letting $g_{k,n}$ be the restriction of $f_{k,n}$ to the subset $\operatorname{Cay}^k(p)[n]$ of $[k]^n(p)$. This proves $p \sim_{sc} q$. To prove the converse implication, suppose that $p \sim_{sc} q$. Equivalently, for each n, k we have a content-preserving bijection

$$g_{k,n} : \operatorname{Cay}^k(p)[n] \longmapsto \operatorname{Cay}^k(q)[n].$$

Then we define a content-preserving bijection $f_{k,n}$ from $[k]^n(p)$ to $[k]^n(q)$ by letting

$$f_{k,n}(w) = \left(\operatorname{std}_{\operatorname{Im}(w)} \circ g_{j,n} \circ \operatorname{std}_{[j]}\right)(w),$$

where $j = |\operatorname{Im}(w)|$. In other words, $f_{k,n}(w)$ is obtained by first standardizing w under $\operatorname{std}_{[j]}$, then applying the suitable content-preserving bijection $g_{j,n}$, and finally applying the inverse standardization (i.e., $\operatorname{std}_{\operatorname{Im}(w)}$) to obtain a word that has the same content as w. Or, even less formally, by applying $g_{j,n}$ to w pretending that the numbers that appear in w are $1, 2, \ldots, j$. It is easy to see that $f_{k,n}$ preserves the content, as well as the avoidance of p. The proof that $f_{k,n}$ is a bijection is left to the reader. □

We end this section with a list of open problems. Let p and q be two Cayley permutations of the same length. Can we prove or refute any of the following conjectures?

Conjecture 8.5. *If $p \sim_{cm} q$, then $p \sim_{sc} q$.*

Jelínek and Mansour [45] showed that strong-word-equivalence and word-equivalence coincide on patterns of length at most six. By Propositions 8.3 and 8.4, Conjecture 8.5 holds up to length six.

Conjecture 8.6. *If $p \sim_c q$, then $p \sim_{cm} q$.*

The smallest candidate for a counterexample to Conjecture 8.6 is the pair $p = 13442$ and $q = 14233$. Here, $p \not\sim_{cm} q$ since e.g., $|\operatorname{Cay}^5(p)[9]| = 742943 \neq 742944 = |\operatorname{Cay}^5(q)[9]|$. If the conjecture is true, then $|\operatorname{Cay}(p)[n]|$ and $|\operatorname{Cay}(q)[n]|$ will differ for some n. We have, however, checked that $|\operatorname{Cay}(p)[n]| = |\operatorname{Cay}(q)[n]|$ for $n \leq 9$.

It is easy to see that if $\max(p) \neq \max(q)$, then $p \not\sim_{cm} q$. Indeed, suppose that $\max(p) = m < \max(q)$, for $p, q \in \operatorname{Cay}[n]$. Then there is only one Cayley permutation in $\operatorname{Cay}^m[n]$ that contains p, namely p itself; on the other hand, no Cayley permutation in $\operatorname{Cay}^m[n]$ contains q since we assumed $\max(q) > m$. Thus,

$$|\operatorname{Cay}^m(p)[n]| = |\operatorname{Cay}^m[n]| - 1 < |\operatorname{Cay}^m[n]| = |\operatorname{Cay}^m(q)[n]|,$$

showing that $p \not\sim_{cm} q$. Since $p \sim_{sc} q$ implies $p \sim_{cm} q$, the same property holds for strong-Cayley-equivalence. What about Cayley-equivalence?

Conjecture 8.7. *If* $p \sim_c q$*, then* $\max(p) = \max(q)$.

Finally, we conjecture that $\max(p) \leq \max(q)$ turns into the opposite inequality for the number of Cayley permutations avoiding the two patterns.

Conjecture 8.8. *If* $\max(p) \leq \max(q)$*, then* $|\operatorname{Cay}_n(p)| \geq |\operatorname{Cay}_n(q)|$.

References

[1] C. Ahlbach, J. Usatine and N. Pippenger, Barred preferential arrangements, *Electron. J. Combin.* **20**(2) (2013), #P55.

[2] R. W. Bailey, The number of weak orderings of a finite set, *Soc. Choice Welf.* **15**(4) (1998), 559–562.

[3] J.-L. Baril, Gray code for Cayley permutations, *Computer Science* **11**(2) (2003), 32.

[4] F. Bergeron, G. Labelle and P. Leroux, *Combinatorial species and tree-like structures*, Cambridge University Press, 1998.

[5] M. Bernstein and N. J. A. Sloane, Some canonical sequences of integers, *Linear Algebra Appl.* **226/228** (1995), 57–72.

[6] D. Bevan, Permutation patterns: basic definitions and notation, `arXiv:1506.06673` (2015).

[7] Y. Biers-Ariel, Y. Zhang and A. Godbole, Some results on superpatterns for preferential arrangements, *Adv. Appl. Math.* **81** (2016), 202–211.

[8] S. C. Billey and S. Ryan, Brewing Fubini-Bruhat orders, *Sém. Lothar. Combin.*, in press.

[9] D. Birmajer, J. B. Gil, D. S. Kenepp and M. D. Weiner, Restricted generating trees for weak orderings, *Discrete Math. Theor. Comput. Sci.* **24**(1) (2022).

[10] M. Bóna, *Combinatorics of Permutations*, 3rd ed., Chapman and Hall/CRC (2022).

[11] A. Burstein, *Enumeration of words with forbidden patterns*, PhD Thesis, University of Pennsylvania (1998).

[12] A. Cayley, On the analytic forms called trees, second part, *Philos. Mag.* **18** (1859), 374–378.

[13] G. Cerbai, Sorting Cayley permutations with pattern-avoiding machines, *Australas. J. Combin.* **80** (2021), 322–341.

[14] G. Cerbai, Modified ascent sequences and Bell numbers, *Electron. J. Combin.* **31**(4) (2024), #P31.

[15] G. Cerbai, Pattern-avoiding modified ascent sequences, *Electron. J. Combin.* **32**(3) (2025), #P3.6.

[16] G. Cerbai and A. Claesson, Fishburn trees, *Adv. Appl. Math.* **151** (2023).

[17] G. Cerbai and A. Claesson, Transport of patterns by Burge transpose, *European J. Combin.* **108** (2023).

[18] G. Cerbai and A. Claesson, Caylerian polynomials, *Discrete Math.* **347**(12) (2024).

[19] G. Cerbai and A. Claesson, Counting fixed-point-free Cayley permutations, 2025, `arXiv:2507.09304`.

[20] G. Cerbai and A. Claesson, Enumerative aspects of Caylerian polynomials, 2025, `arXiv:2411.08426`.

[21] G. Cerbai, A. Claesson and B. E. Sagan, Modified difference ascent sequences and Fishburn structures, *Discrete Math.* **349**(1) (2026).

[22] W. Y. C. Chen, A. Y. L. Dai and R. D. P. Zhou, Ordered partitions avoiding a permutation pattern of length 3, *European J. Combin.* **36** (2014), 416–424.

[23] A. Claesson, A species approach to Rota's twelvefold way, *Expo. Math.* (2019).

[24] A. Claesson, From Hertzsprung's problem to pattern-rewriting systems, *Algebr. Comb.* **5**(6) (2022), 1257–1277.

[25] A. Claesson and S. Kitaev, Classification of bijections between 321- and 132-avoiding permutations, *Sém. Lothar. Combin.* **60** (2008).

[26] A. R. Conway, M. Conway, A. Elvey Price and A. J. Guttmann, Pattern-avoiding ascent sequences of length 3, *Electron. J. Combin.* **29**(4) (2022), #P25.

[27] M. Dukes, S. Kitaev, J.B. Remmel and E. Steingrímsson, Enumerating (2+2)-free posets by indistinguishable elements, *J. Comb.* **2**(1) (2011), 139–163.

[28] P. Duncan and E. Steingrímsson, Pattern avoidance in ascent sequences, *Electron. J. Combin.* **18**(1) (2011), #P226.

[29] E. S. Egge, Pattern-avoiding Fishburn permutations and ascent sequences, 2022, `arXiv:2208.01484`.

[30] L. Foissy, J.-C. Novelli and J.-Y. Thibon, Polynomial realizations of some combinatorial Hopf algebras, *J. Noncommut. Geom.* **8**(1) (2014), 141–162.

[31] A. S. Fraenkel, Systems of numeration, *Amer. Math. Monthly* **92**(2) (1985), 105–114.

[32] A. S. Fraenkel and M. Mor, Combinatorial compression and partitioning of large dictionaries, *Comput. J.* **26**(4) (1983), 336–343.

[33] I. Gessel and R. P. Stanley, Stirling polynomials, *J. Combin. Theory Ser. A* **24**(1) (1978), 25–33.

[34] M. Göbel, On the number of special permutation-invariant orbits and terms, *Appl. Algebra Engrg. Comm. Comput.* **8**(6) (1997), 505–509.

[35] A. Godbole, A. Goyt, J. Herdan and L. Pudwell, Pattern avoidance in ordered set partitions, *Ann. Comb.* **18** (2014).

[36] H. Golab, *Pattern Avoidance in Cayley Permutations*, MS Thesis, Northern Arizona University (2024).

[37] O. A. Gross, Preferential arrangements, *Amer. Math. Monthly* **69** (1962), 4–8.

[38] M. Hazewinkel, Hopf algebras of endomorphisms of Hopf algebras, *Arab. J. Sci. Eng. Sect. C Theme Issues* **33**(2) (2008), 239–272.

[39] S. Heubach and T. Mansour, Avoiding patterns of length three in compositions and multiset permutations, *Adv. Appl. Math.* **36**(2) (2006).

[40] M. E. Hoffman, An analogue of covering space theory for ranked posets, *Electron. J. Combin.* **8**(1) (2001), #R32.

[41] M. E. Hoffman, Updown categories: generating functions and universal covers, *Discrete Math.* **339**(2) (2016), 906–922.

[42] V. Horan and G. Hurlbert, Universal cycles for weak orders, *SIAM J. Discrete Math.* **27**(3) (2013), 1360–1371.

[43] M. Jacques, D. Wong and K. Woo, Generating Gray codes for weak orders in constant amortized time, *Discrete Math.* **343**(10) (2020), 111992.

[44] V. Jelínek, T. Mansour and M. Shattuck, On multiple pattern avoiding set partitions, *Adv. Appl. Math.* **50**(2) (2013).

[45] V. Jelínek and T. Mansour, Wilf-equivalence on k-ary words, compositions, and parking functions, *Electron. J. Combin.* **16**(1) (2009), #R58.

[46] A. Kasraoui, Pattern avoidance in ordered set partitions and words, *Adv. Appl. Math.* **61** (2014), 85–101.

[47] S. Kitaev, *Patterns in permutations and words*, Springer-Verlag, 2011.

[48] D. E. Knuth, *The Art of Computer Programming*, vol. 1, Addison-Wesley (1969).

[49] D. Kroes and B. Rhoades, Packed words and quotient rings, *Discrete Math.* **345**(9) (2022).

[50] T. Lam and P. Pylyavskyy, Combinatorial Hopf algebras and K-homology of Grassmannians, *Int. Math. Res. Not. IMRN* (24) (2007).

[51] P. A. MacMahon, Yoke-chains and multipartite compositions in connexion with the analytical forms called 'trees', *Proc. Lond. Math. Soc.* **22** (1890/91), 330–346.

[52] P. A. MacMahon, *Combinatory Analysis*, vol. 1, Cambridge University Press, 1915.

[53] E. Marberg, Bialgebras for Stanley symmetric functions, *Discrete Math.* **343**(4) (2020).

[54] E. Marberg, Linear compactness and combinatorial bialgebras, *Electron. J. Combin.* **28**(3) (2021), #P9.

[55] E. Mendelson, Races with ties, *Math. Mag.* **55**(3) (1982), 170–175.

[56] M. Mor and A. S. Fraenkel, Cayley permutations, *Discrete Math.* **48**(1) (1984), 101–112.

[57] T. Mütze, Combinatorial Gray codes—an updated survey, *Electron. J. Combin.* (2023), #DS26.

[58] S. Nkonkobe, B. Bényi, R. B. Corcino and C. B. Corcino, A combinatorial analysis of higher order generalised geometric polynomials: a generalisation of barred preferential arrangements, *Discrete Math.* **343**(3) (2020), 111729.

[59] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, published electronically at `https://oeis.org/` (2024).

[60] R. Patrias and P. Pylyavskyy, Combinatorics of K-theory via a K-theoretic Poirier-Reutenauer bialgebra, *Discrete Math.* **339**(3) (2016), 1095–1115.

[61] N. Pippenger, The hypercube of resistors, asymptotic expansions, and preferential arrangements, *Math. Mag.* **83**(5) (2010), 331–346.

[62] B. Rhoades, Generalizations of the flag variety tied to the Macdonald-theoretic delta operators, `arXiv:2204.03386` (2022).

[63] B. E. Sagan, Pattern avoidance in set partitions, *Ars Combin.* **94** (2006).

[64] C. D. Savage and H. S. Wilf, Pattern avoidance in compositions and multiset permutations, *Adv. Appl. Math.* **36**(2) (2006), 194–201.

[65] A. Schnurr and S. Fischer, Generalized ordinal patterns allowing for ties and their applications in hydrology, *Comput. Statist. Data Anal.* **171** (2022).

[66] R. Simion and F. W. Schmidt, Restricted permutations, *European J. Combin.* **6** (1985), 383–406.

[67] A. T. Wilson, Torus link homology and the nabla operator, *J. Combin. Theory Ser. A* **154** (2018), 129–144.

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