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Multicolor, multipartite Ramsey numbers for quadrilaterals

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Abstract

The p-partite Ramsey number for a quadrilateral, denoted by $r_p(C_4, k)$, is the least positive integer n such that any coloring of the edges of a complete p-partite graph with n vertices in each partition with k colors will result in a monochromatic copy of C_4. In this paper, we present an upper bound for $r_p(C_4, k)$ and the exact values of $r_p(C_4, 2)$ for all $p \geq 2$. In the tripartite case we show that $r_3(C_4, k) \leq \lfloor (k+1)^2/2 \rfloor - 1$, and the exact value of the 4-color tripartite Ramsey number, $r_3(C_4, 4) = 11$.

1 Introduction

In this paper all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph. The vertex set of G is denoted by $V(G)$, the edge set of G by $E(G)$, and the number of edges in G by $e(G)$. Let $\deg(v)$ be the degree of vertex v and $\deg_i(v)$ denote the number of the edges incident to v colored with color i. The open neighborhood in color i of vertex v is $N_i(v) = \{u \in V(G) | \{u, v\} \in E(G)$ and $\{u, v\}$ is colored with color $i\}$. For vertex $v \in V(G)$ and $U \subset V(G)$, let $\deg_i(v, U)$ be the number of edges in color i between v and vertices from U. Define $G[S]$ to be the subgraph of G induced by a set of vertices $S \subset V(G)$

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and G^i to be the graph containing the edges of G in color i. Let C_n be the cycle on n vertices, K_n be the complete graph on n vertices, and K_n^p be the complete p-partite graph with every partition of size n.

For a given graph G and integer $k \geq 2$, the *multicolor Ramsey number* $R_k(G)$ is the smallest integer n such that if we arbitrarily color with k colors the edges of the complete graph K_n, then it contains a monochromatic copy of G in one of the colors. In this paper we consider the Ramsey number for the quadrilateral ($G = C_4$). In 1972 Chvátal and Harary [4] determined $R_2(C_4) = 6$. Bialostocki and Schönheim [2] proved that $R_3(C_4) = 11$. A lower bound for $R_4(C_4)$ of 18 was determined by Exoo [7] and an upper bound by Sun Yongqi et al. [14]. Lazebnik and Woldar [11] showed that the best known bounds for the 5-color Ramsey number for a quadrilateral are $27 \leq R_5(C_4) \leq 29$. More bounds and general results for Ramsey numbers can be found in the regularly updated survey by Radziszowski [12].

The *p-partite Ramsey number* for the quadrilateral, denoted by $r_p(C_4, k)$, is the least positive integer n such that any coloring of the edges of K_n^p with k colors will result in a monochromatic copy of C_4 in one of the colors.

The study of bipartite Ramsey numbers ($p = 2$) was initiated by Beineke and Schwenk in 1976, and was continued by others, in particular, Exoo [6], Hattingh and Henning [9], Goddard, Henning, and Oellermann [8], and Lazebnik and Mubayi [10]. The exact values of $r_2(C_4, k)$ are known for $k \leq 4$. Beineke and Schwenk [1] proved that $r_2(C_4, 2) = 5$, Exoo [6] found the value $r_2(C_4, 3) = 11$ and, independently, Steinbach and Posthoff [13] and Dybizbański, Dzido and Radziszowski [5] showed that $r_2(C_4, 4) = 19$. We also know that for every integer $k \geq 5$, we have $k^2 + 1 \leq r_2(C_4, k) \leq k^2 + k - 2$ (see [5]).

In the tripartite case ($p = 3$), exact values of Ramsey numbers $r_3(C_4, k)$ are known for $k \leq 3$. In 2014, Buada, Samana and Longani [3] proved that $r_3(C_4, 3) = 7$. In the same paper one can find a proof that $r_3(C_4, 2) = 3$.

In this paper we show an upper bound for $r_p(C_4, k)$ (Theorem 2.2), and exact values of $r_p(C_4, 2)$ for all $p \geq 2$ (Theorem 3.2). In the tripartite case we show that $r_3(C_4, k) \leq \lfloor (k+1)^2/2 \rfloor - 1$ (Theorem 4.1) and we determine the exact value of the 4-color tripartite Ramsey number: $r_3(C_4, 4) = 11$ (Theorem 4.2).

2 Upper bound

The well-known version of the Cauchy-Schwarz theorem says that if $a_1, \ldots, a_n$ is a sequence of non-negative real numbers and $M = \sum_{i=1}^n a_i$, then $\sum_{i=1}^n a_i^2 \geq M^2/n$. Moreover, $\sum_{i=1}^n a_i^2 = M^2/n$ if and only if $a_i = M/n$ for every i. We will use another (integer) version of this theorem.

Theorem 2.1. *Let $a_1, \ldots, a_n$ be a sequence of non-negative integers and $M = \sum_{i=1}^n a_i$. Then*

$$\sum_{i=1}^{n} \binom{a_i}{2} \geq r\binom{a+1}{2} + (M-r)\binom{a}{2},$$

where a and r are integers such that $M = an + r$ and $0 \le r < n$. Moreover, the minimum is reachable if and only if $|a_i - a_j| \le 1$ for every $1 \le i < j \le n$.

Note that:
(1) $\sum_{i=1}^{n} \binom{a_i}{2}$ has the minimum value if and only if $\sum_{i=1}^{n} a_i^2$ has the minimum value.
(2) If $a_j - a_i > 1$ for some $i \neq j$ then setting $a'_j = a_j - 1$ and $a'_i = a_i + 1$ we obtain a sequence with smaller $\sum_{i=1}^{n} a_i^2$.

Theorem 2.2. *Let $p, k \ge 2$ and $n \ge 1$ be integers, $w = \left\lceil \frac{2\left\lceil n^2p(p-1)/2k \right\rceil}{p} \right\rceil$, and a and r be integers such that $w = a(p-1)n + r$ and $0 \le r < (p-1)n$. If*

$$r\binom{a+1}{2} + ((p-1)n - r)\binom{a}{2} > \binom{n}{2} \tag{1}$$

then $r_p(C_4, k) \le n$.

Proof. Let $G = (V_1 \cup \cdots \cup V_p, E)$ be a p-partite graph of order pn. For every $1 \le i \le p$, V_i is an independent set and $|V_i| = n$. By the pigeonhole principle, there exists i $(1 \le i \le p)$, such that

$$\sum_{v \in V_i} \deg(v) \ge \left\lceil \frac{2|E|}{p} \right\rceil. \tag{2}$$

The complete p-partite graph K_n^p has $n^2p(p-1)/2$ edges, and in every k-edge-coloring there exists a color, say b (blue), containing at least

$$\left\lceil \frac{n^2p(p-1)}{2k} \right\rceil$$

edges.
Moreover, by (2), there exists a partition j, say $j = 1$, such that

$$\sum_{v \in V_1} \deg_b(v) \ge \left\lceil \frac{2\left\lceil n^2p(p-1)/2k \right\rceil}{p} \right\rceil = w.$$

Let $U = V \setminus V_1 = \cup_{l=2}^{p} V_l = \{u_1, u_2, \ldots, u_{(p-1)n}\}$. For every i, $1 \le i \le (p-1)n$, $N_b(u_i) \cap V_1$ is the vertex set of blue neighbors of u_i in V_1. Denote by $a_i = |N_b(u_i) \cap V_1| = \deg_b(u_i, V_1)$. Vertex u_i has a_i blue neighbors in V_1 and $\binom{a_i}{2}$ pairs of blue neighbors in V_1. If $\sum_{i=1}^{(p-1)n} \binom{a_i}{2} > \binom{n}{2}$ then, by the pigeonhole principle, there exists two vertices u_i and u_j with a common pair of blue neighbors, say y_1, y_2 in V_1, and there is blue cycle (u_i, y_1, u_j, y_2) in the graph. Since $\sum_{i=1}^{(p-1)n} a_i = \sum_{u \in U} \deg_b(u, V_1) = \sum_{v \in V_1} \deg_b(v) \ge w$, by Theorem 2.1,

$$\sum_{i=1}^{(p-1)n} \binom{a_i}{2} \ge r\binom{a+1}{2} + (a(p-1)n)\binom{a}{2}$$

$$\geq r\binom{a+1}{2} + ((p-1)n - r)\binom{a}{2}.$$

Thus, if $r\binom{a+1}{2} + ((p-1)n - r)\binom{a}{2} > \binom{n}{2}$ then every k-edge-coloring of K_n^p contains a monochromatic cycle C_4. □

For every $p, k \geq 2$ inequality (1) is satisfied if n is big enough (for example, $n = k^2$ for $p \geq 3$). For $p = 3$ and $k = 4$, $n = 11$ is the smallest integer satisfying (1), so $r_3(C_4, 4) \leq 11$. In a similar way we can bound other numbers. Table 1 presents examples of upper bounds for k-colored, p-partite Ramsey numbers $r_p(C_4, k)$, which can be obtained from Theorem 2.2 (for $2 \leq p \leq 6$ and $2 \leq k \leq 10$). In the bipartite case, we obtain results that are equal to the exact value for $k \leq 4$ and one worse than the best known bound ($k^2 + k + 1$, [5]) for $k \geq 5$.

$k \setminus p$	bipartite	tripartite	4-partite	5-partite	6-partite
2 colors	**5**	**3**	3	3	3
3	**11**	**7**	5	**4**	4
4	**19**	**11**	9	7	6
5	29	17	12	11	9
6	41	23	17	14	13
7	55	31	23	18	16
8	71	39	28	23	20
9	89	49	35	29	24
10	109	59	43	34	29

Table 1: Upper bounds for $r_p(C_4, k)$

3 Two-color multipartite Ramsey numbers

Lemma 3.1. *For positive integer p,*

$$p \geq R_k(C_4) \text{ if and only if } r_p(C_4, k) = 1.$$

Proof. If $p \geq R_k(C_4)$ then every k-edge-coloring of $K_p = K_1^p$ contains a monochromatic C_4 so $r_p(C_4, k) = 1$.

If $1 \leq p < R_k(C_4)$, then there exists a k-edge-coloring of $K_1^p = K_p$ without a monochromatic C_4, so $r_p(C_4, k) > 1$. □

Since K_n^{p-1} is a subgraph of K_n^p and the latter is a subgraph of K_{n+1}^p, the following two monotonic properties hold:

- If $k_1 > k_2$ then $r_p(C_4, k_1) \geq r_p(C_4, k_2)$.
- If $p_1 > p_2$ then $r_{p_1}(C_4, k) \leq r_{p_2}(C_4, k)$.

Theorem 3.2. *Two-color multipartite Ramsey number for a quadrilateral are:*

$$r_p(C_4, 2) = \begin{cases} 5 & \text{for } p = 2, \\ 3 & \text{for } p = 3, \\ 2 & \text{for } p = 4 \text{ or } 5, \\ 1 & \text{for } p \geq 6. \end{cases}$$

Proof. Beineke and Schwenk in [1] showed that $r_2(C_4, 2) = 5$. Buada et al. [3] showed that $r_3(C_4, 2) = 3$. Since $R_2(C_4) = 6$ [4], by Lemma 3.1 we have $r_p(C_4, 2) = 1$, for $p \geq 6$, and (by Lemma 3.1 and monotonicity) $r_4(C_4, 2) \geq r_5(C_4, 2) > 1$.

To finish the proof, we shall show that every 2-edge-coloring of K_2^4 with partition sets $X_1 = \{x_1, x_2\}$, $X_2 = \{w_1, w_2\}$, $X_3 = \{y_1, y_2\}$, and $X_4 = \{z_1, z_2\}$, contains a monochromatic C_4. Consider any 2-edge-coloring of $G = K_2^4 = (\{x_1, x_2, w_1, w_2, y_1, y_2, z_1, z_2\}, E(G))$, say $(G^r = (V(G), E_r),\ G^b = (V(G), E_b))$, where

$$E_r \cup E_b = E(G) = \binom{V(G)}{2} \setminus \{\{x_1, x_2\}, \{w_1, w_2\}, \{y_1, y_2\}, \{z_1, z_2\}\}.$$

Now, we have the following claims:

Claim 3.3. *If there exists a vertex of $V(G)$, say $x_1 \in X_1$, and a color, say red, such that $\deg_r(x) \geq 4$, then the coloring contains a monochromatic C_4.*

Proof. Consider vertex x_2. Since we aim to avoid C_4, at least three edges from x_2 to $N_r(x_1)$ must be colored blue (denoted as color b). Consider three vertices $\{v_1, v_2, v_3\} \subset N_r(x_1) \cap N_b(x_2)$.

a) If those vertices are in a different partition then consider the triangle $\{v_1, v_2, v_3\}$. Two of the edges of this triangle must have the same color. These two edges create a monochromatic C_4 with x_1 (if they are red) or x_2 (if blue).

b) Suppose that two of those vertices are in the same partition. Without loss of generality we can assume that $\{v_1, v_2, v_3\} = \{y_1, y_2, z_1\}$. Consider vertex w_1 and colors of the edges $\{w_1, y_1\}$, $\{w_1, y_2\}$ and $\{w_1, z_1\}$. Two of these edges must have the same color, creating a monochromatic C_4 with x_1 or x_2. □

In the sequel we assume that every vertex in each color has degree 3.

Claim 3.4. *If there exists a vertex of $V(G)$, say $x_1 \in X_1$, and a color say b, such that $X_i \subseteq N_b(x_1)$ for one $i \in \{2, 3, 4\}$, then the coloring contains a monochromatic C_4.*

Proof. Without loss of generality let $N_b(x_1) = \{w_1, w_2, y_1\}$. Therefore $N_r(x_1) = \{y_2, z_1, z_2\}$. Now considering x_2, one can check that $\{w_1, w_2\} \subseteq N_r(x_2)$ and $\{z_1, z_2\} \subseteq N_b(x_2)$. Otherwise, on the contrary assume that $x_2z_1 \in E_r$ (for the other case the proof is the same). Hence as $\deg_b(z_1) = \deg_r(z_1) = 3$, and $x_iz_1 \in E_r$, one can say that x_1, z_1 and two vertices from $N_b(x_1) \cap N_b(z_1)$ create a blue C_4. If $x_2y_1 \in E_b$ then as $\deg_r(y_1) = \deg_b(y_1) = 3$, and $x_iy_1 \in E_b$, one can say that either x_1, y_1 and two vertices from $N_r(x_1) \cap N_r(y_1)$ create a red C_4, or x_2, y_1 and two vertices from $N_r(x_2) \cap N_r(y_1)$ create a red C_4. So assume that $N_r(x_1) = N_b(x_2)$ and $N_r(x_2) = N_b(x_1)$. For one $i = 1, 2$, $y_2z_i \in E_b$, or otherwise $C_4 \subseteq G^r$. Without loss of generality let $y_2z_1 \in E_b$. Therefore as $\deg_r(z_1) = 3$, and $x_2z_1, y_2z_1 \in E_b$, one can say that x_2, z_1 and two vertices from $N_r(x_2) \cap N_r(z_1)$ create a red C_4. Hence the claim holds. □

Now by Claim 3.4, without loss of generality we may suppose that $N_b(x_1) = \{w_1, y_1, z_1\}$. Assume that $|N_b(x_1) \cap N_b(x_2)| \neq 0$, and without loss of generality let $w_1 \in N_b(x_2)$. Hence by Claim 3.4, $x_1w_2, x_2w_2 \in E_r$. Now, as $\deg_r(w_1) = 3$ and $w_1x_i \in E_b$, either x_1, w_1 and two vertices from $N_r(x_1) \cap N_r(w_1)$ create a red C_4, or x_2, w_1 and two vertices from $N_r(x_2) \cap N_r(w_1)$ create a red C_4. So, let $N_b(x_2) = N_r(x_1)$ and $N_r(x_2) = N_b(x_1)$. Consider w_1 and without loss of generality assume that $w_1z_1 \in E_b$. Therefore by Claim 3.4 we have $w_1z_2, w_2z_1 \in E_r$ and $w_2z_2 \in E_b$. If either $w_1y_1 \in E_b$ or $y_1z_1 \in E_b$ then one can check that $C_4 \subseteq G^b[\{x_1, w_1, y_1, z_1\}]$. So we may suppose that w_1y_1, $y_1z_1 \in E_r$, and in this case we have $C_4 \subseteq G^r[\{x_2, w_1, y_1, z_1\}]$; hence the proof is complete. □

4 Tripartite Ramsey numbers

Theorem 4.1. *For every integer $k \geq 2$,*

$$r_3(C_4, k) \leq \begin{cases} \frac{(k+1)^2-1}{2} - 1 & \text{for even } k, \\ \frac{(k+1)^2}{2} - 1 & \text{for odd } k. \end{cases}$$

Proof. For $k = 2$ we know (from [3]) that $r_3(C_4, 2) = 3 = \frac{(2+1)^2-1}{2} - 1$. For even $k > 2$, we use Theorem 2.2. Note that, for $n = \frac{(k+1)^2-1}{2} - 1$:

- $\left\lceil \frac{n^2p(p-1)}{2k} \right\rceil = \frac{3}{4}k^3 + 3k^2 - 5$;
- $w = \frac{1}{2}k^3 + 2k^2 - 3$;
- $a = \frac{k}{2}$;
- $r = k^2 + k - 3$;

and the inequality $r\binom{a+1}{2} + ((p-1)n - r)\binom{a}{2} > \binom{n}{2}$ can be simplified to $k > 2$, so that $r_p(C_4, k) \leq n$ for even $k \geq 2$.

For odd $k > 1$ note that, for $n = \frac{(k+1)^2}{2} - 1$:

- $\left\lceil \frac{n^2p(p-1)}{2k} \right\rceil = \frac{3}{4}k^3 + 3k^2 + \frac{3}{2}k - \frac{9}{4} - \frac{1}{4}((k-1) \mod 4)$;
- $w = \frac{k^3}{2} + 2k^2 + k - \frac{3}{2}$;
- $a = \frac{k+1}{2}$;
- $r = \frac{k^2+k-2}{2}$;

and the inequality $r\binom{a+1}{2} + ((p-1)n - r)\binom{a}{2} > \binom{n}{2}$ can be simplified to $k(k+2) > 3$, so $r_p(C_4, k) \le n$ for odd $k \ge 3$. □

From [3] we know the exact values $r_3(C_4, 2) = 3$ and $r_3(C_4, 3) = 7$. The next theorem shows that the upper bound presented in Theorem 4.1 is sharp for $k = 4$.

Theorem 4.2. $r_3(C_4, 4) = 11$.

Proof. From Theorem 4.1 we have $r_3(C_4, 4) \le 11$. For the lower bound we present the adjacency matrix of a 4-coloring of K^3_{10} (see Table 2). The entry in the i-th row and j-th column of the matrix refers to the color of edge (i, j). Value 0 means that there is no edge, and values greater than 0 are colors.

The matrix in Table 2 is written in partitioned form as

$$\begin{bmatrix} 0 & 0 & M_{1,2} & M_{1,3} & M_{1,4} & M_{1,5} \\ 0 & 0 & M_{2,2} & M_{2,3} & M_{2,4} & M_{2,5} \\ M^T_{1,2} & M^T_{2,2} & 0 & 0 & M_{3,4} & M_{3,5} \\ M^T_{1,3} & M^T_{2,3} & 0 & 0 & M_{4,4} & M_{4,5} \\ M^T_{1,4} & M^T_{2,4} & M^T_{3,4} & M^T_{4,4} & 0 & 0 \\ M^T_{1,5} & M^T_{2,5} & M^T_{3,5} & M^T_{4,5} & 0 & 0 \end{bmatrix},$$

where each block $M_{i,j}$ in this partitioned matrix is the 5×5 matrix of the form:

$$\begin{bmatrix} a & b & c & d & e \\ e & a & b & c & d \\ d & e & a & b & c \\ c & d & e & a & b \\ b & c & d & e & a \end{bmatrix},$$

□

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X 0 0 0 0	0 0 0 0 0	1 4 2 3 2	2 1 3 4 4	1 3 2 2 3	4 1 1 4 3
0 X 0 0 0	0 0 0 0 0	2 1 4 2 3	4 2 1 3 4	3 1 3 2 2	3 4 1 1 4
0 0 X 0 0	0 0 0 0 0	3 2 1 4 2	4 4 2 1 3	2 3 1 3 2	4 3 4 1 1
0 0 0 X 0	0 0 0 0 0	2 3 2 1 4	3 4 4 2 1	2 2 3 1 3	1 4 3 4 1
0 0 0 0 X	0 0 0 0 0	4 2 3 2 1	1 3 4 4 2	3 2 2 3 1	1 1 4 3 4
0 0 0 0 0	X 0 0 0 0	3 2 3 1 4	3 1 1 2 4	4 4 1 2 1	3 2 4 2 3
0 0 0 0 0	0 X 0 0 0	4 3 2 3 1	4 3 1 1 2	1 4 4 1 2	3 3 2 4 2
0 0 0 0 0	0 0 X 0 0	1 4 3 2 3	2 4 3 1 1	2 1 4 4 1	2 3 3 2 4
0 0 0 0 0	0 0 0 X 0	3 1 4 3 2	1 2 4 3 1	1 2 1 4 4	4 2 3 3 2
0 0 0 0 0	0 0 0 0 X	2 3 1 4 3	1 1 2 4 3	4 1 2 1 4	2 4 2 3 3
1 2 3 2 4	3 4 1 3 2	X 0 0 0 0	0 0 0 0 0	3 3 4 1 1	4 4 1 2 2
4 1 2 3 2	2 3 4 1 3	0 X 0 0 0	0 0 0 0 0	1 3 3 4 1	2 4 4 1 2
2 4 1 2 3	3 2 3 4 1	0 0 X 0 0	0 0 0 0 0	1 1 3 3 4	2 2 4 4 1
3 2 4 1 2	1 3 2 3 4	0 0 0 X 0	0 0 0 0 0	4 1 1 3 3	1 2 2 4 4
2 3 2 4 1	4 1 3 2 3	0 0 0 0 X	0 0 0 0 0	3 4 1 1 3	4 1 2 2 4
2 4 4 3 1	3 4 2 1 1	0 0 0 0 0	X 0 0 0 0	2 4 2 4 3	1 3 1 3 2
1 2 4 4 3	1 3 4 2 1	0 0 0 0 0	0 X 0 0 0	3 2 4 2 4	2 1 3 1 3
3 1 2 4 4	1 1 3 4 2	0 0 0 0 0	0 0 X 0 0	4 3 2 4 2	3 2 1 3 1
4 3 1 2 4	2 1 1 3 4	0 0 0 0 0	0 0 0 X 0	2 4 3 2 4	1 3 2 1 3
4 4 3 1 2	4 2 1 1 3	0 0 0 0 0	0 0 0 0 X	4 2 4 3 2	3 1 3 2 1
1 3 2 2 3	4 1 2 1 4	3 1 1 4 3	2 3 4 2 4	X 0 0 0 0	0 0 0 0 0
3 1 3 2 2	4 4 1 2 1	3 3 1 1 4	4 2 3 4 2	0 X 0 0 0	0 0 0 0 0
2 3 1 3 2	1 4 4 1 2	4 3 3 1 1	2 4 2 3 4	0 0 X 0 0	0 0 0 0 0
2 2 3 1 3	2 1 4 4 1	1 4 3 3 1	4 2 4 2 3	0 0 0 X 0	0 0 0 0 0
3 2 2 3 1	1 2 1 4 4	1 1 4 3 3	3 4 2 4 2	0 0 0 0 X	0 0 0 0 0
4 3 4 1 1	3 3 2 4 2	4 2 2 1 4	1 2 3 1 3	0 0 0 0 0	X 0 0 0 0
1 4 3 4 1	2 3 3 2 4	4 4 2 2 1	3 1 2 3 1	0 0 0 0 0	0 X 0 0 0
1 1 4 3 4	4 2 3 3 2	1 4 4 2 2	1 3 1 2 3	0 0 0 0 0	0 0 X 0 0
4 1 1 4 3	2 4 2 3 3	2 1 4 4 2	3 1 3 1 2	0 0 0 0 0	0 0 0 X 0
3 4 1 1 4	3 2 4 2 3	2 2 1 4 4	2 3 1 3 1	0 0 0 0 0	0 0 0 0 X

Table 2: Matrix of 4-edge-coloring of K^3_{10} without monochromatic C_4.

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