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Examples of diameter-2 graphs with no triangle or $K_{2,t}$

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Abstract

For each $t \geq 1$ let $\mathcal{W}_t$ denote the class of graphs other than stars that have diameter 2 and contain neither a triangle nor a $K_{2,t}$. The famous Hoffman–Singleton Theorem implies that $\mathcal{W}_2$ is finite. Recently Wood suggested the study of $\mathcal{W}_t$ for $t > 2$ and conjectured that $\mathcal{W}_t$ is finite for all $t \geq 2$. In this note we show that (1) $\mathcal{W}_3$ is infinite, (2) $\mathcal{W}_5$ contains infinitely many regular graphs, and (3) $\mathcal{W}_7$ contains infinitely many Cayley graphs. Our $\mathcal{W}_3$ and $\mathcal{W}_5$ examples are based on so-called crooked graphs, first constructed by de Caen, Mathon, and Moorhouse. Our $\mathcal{W}_7$ examples are Cayley graphs with vertex set $\mathbb{F}_p^2$ for prime $p \equiv 11 \pmod{12}$. We also highlight the surprising fact that crooked graphs themselves provide an infinite family of graphs which imply that

$$\mathrm{ex}(n, \{C_3, K_{2,3}\}) = \left(\frac{1}{\sqrt{2}} + o(1)\right) n^{3/2}$$

for an infinite, albeit sparse, set of n's.

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1 Introduction

Let G be a graph. The *diameter* of G is the maximum distance between two vertices in G. The *girth* of G is the length of a shortest cycle in G, or else ∞ if G is acyclic. The girth of a graph can also be defined in terms of forbidden subgraphs. Given a family of graphs $\mathcal{F}$, we say G is *$\mathcal{F}$-free* if G has no subgraph isomorphic to a member of $\mathcal{F}$. Hence G has girth at least g if and only if G is $\{C_3, C_4, \dots, C_{g-1}\}$-free.

A particularly important family of graphs is the class of diameter-2 graphs of girth 5. Such a graph G is known as a *Moore graph* (of diameter 2). The famous Hoffman–Singleton theorem implies that only finitely many such graphs exist.

Theorem 1.1 (Hoffman–Singleton [9, 16])**.** *Let G be a finite graph with girth 5 and diameter 2. Then G is d-regular with $d \in \{2, 3, 7, 57\}$.*

In fact the only such graphs are

(1) the 5-cycle ($d = 2$),
(2) the Petersen graph ($d = 3$),
(3) the Hoffman–Singleton graph ($d = 7$),
(4) possibly, one or more graphs of degree $d = 57$ and order 3250.

The existence of a graph of the fourth type has been a mystery for 65 years.

Attempting to identify a relaxation of the class of Moore graphs that is more suggestive of extremal rather than algebraic combinatorics, Wood proposed the study of $\{C_3, K_{2,t}\}$-free graphs of diameter 2 (see [4]).

Definition 1.2. A graph G is called a *Wood graph (with parameter t)* if it has diameter 2, it is $\{C_3, K_{2,t}\}$-free, and it is not a star. We denote by $\mathcal{W}_t$ the class of Wood graphs with parameter t.

Note that $\mathcal{W}_2$ is precisely the class of Moore graphs, so in particular $\mathcal{W}_2$ is finite. Wood conjectured the following generalization.

Conjecture 1.3. *The class $\mathcal{W}_t$ is finite for all $t \geq 2$. In other words, there is an n_t such that if G is a $\{C_3, K_{2,t}\}$-free graph of diameter 2 with $n > n_t$ vertices then G is isomorphic to the star graph $K_{n-1,1}$.*

This conjecture and particularly the class $\mathcal{W}_3$ were studied by Devillers, Kamčev, McKay, Ó Catháin, Royle, Van de Voorde, Wanless, and Wood [4], who found more than five million graphs in $\mathcal{W}_3$ but no infinite family.

Our main contribution is to present an infinite family of graphs in $\mathcal{W}_3$, which shows that this conjecture is false for all $t \geq 3$, albeit for a sparse set of n's.

Theorem 1.4. *For each integer $e \geq 3$ there is a graph $G \in \mathcal{W}_3$ of order $2^{4e-1}+2^{2e}+1$.*

Our proof of Theorem 1.4 uses a family of antipodal, distance-regular graphs of diameter 3 called *crooked graphs*, defined by Bending and Fon-Der-Flaass [2] generalizing previous work of de Caen, Mathon, and Moorhouse [6]. We modify these graphs suitably to reduce the diameter to 2 without creating any triangles or $K_{2,3}$'s.

Notably, the resulting graphs are not regular, and it would be interesting to find an infinite family of regular graphs in $\mathcal{W}_3$. We have not found such a family, but we can construct an infinite sequence of regular graphs in $\mathcal{W}_5$. The construction is again based on crooked graphs.

Theorem 1.5. *For each integer $e \geq 2$ there is a regular graph $G \in \mathcal{W}_5$ of order 2^{2^e-1}.*

In [6], the automorphism group of the first family of crooked graphs was computed and it follows that this family is not vertex-transitive. Embracing the view that a family of vertex-transitive Wood graphs would be even better, we give an unrelated construction of an infinite sequence of Cayley graphs in $\mathcal{W}_7$. These graphs were first described on the first author's blog [7] and arose in discussions between him and Padraig Ó Catháin.

Theorem 1.6. *For every prime $p \equiv 11 \pmod{12}$ there is a symmetric subset $A \subset \mathbb{F}_p^2$ such that the Cayley graph $G = \mathrm{Cay}(\mathbb{F}_p^2, A) \in \mathcal{W}_7$.*

Finally, we observe an application of crooked graphs to the Turán number $\mathrm{ex}(n, \{C_3, K_{2,3}\})$ which equals the maximum number of edges that a graph of order n which is C_3-free, and $K_{2,3}$-free can have. Until now, the best bounds stated in the literature are

$$\left(\frac{1}{\sqrt{3}} + o(1)\right) n^{3/2} \leq \mathrm{ex}(n, \{C_3, K_{2,3}\}) \leq \left(\frac{1}{\sqrt{2}} + o(1)\right) n^{3/2}.$$

The lower bound follows from the work of Allen, Keevash, Sudakov, and Verstraëte [1] who studied Turán problems on forbidding odd cycles together with at least one bipartite graph. The upper bound holds for all n as the Kővári–Sós–Turán Theorem [13] implies $\mathrm{ex}(n, K_{s,t}) \leq \frac{1}{2}(t-1)^{1/s} n^{2-1/s} + \frac{sn}{2}$ for $2 \leq s \leq t$.

The following fact, which follows as an immediate consequence of the existence of crooked graphs, was brought to the attention of the second author by Sam Mattheus.

Theorem 1.7. *Let $k > 1$ be a positive integer and $n = 2^{2k+1}$. Then*

$$\mathrm{ex}(n, \{C_3, K_{2,3}\}) = \left(\frac{1}{\sqrt{2}} + o(1)\right) n^{3/2}.$$

2 Crooked graphs

Our proof of Theorem 1.4 uses a remarkable family of antipodal distance-regular graphs of diameter 3 called *crooked graphs* (see [2, 6]). Let V be an n-dimensional vector space over $\mathbb{F}_2$. A function $Q : V \to V$ is *crooked* if

(1) $Q(0) = 0$,

(2) $\sum_{i=1}^{4} Q(x_i) \neq 0$ for all distinct $x_1, x_2, x_3, x_4 \in V$ with $x_1 + x_2 + x_3 + x_4 = 0$, and

(3) $\sum_{i=1}^{3}(Q(x_i) + Q(x_i + a)) \neq 0$ for all $x_1, x_2, x_3 \in V$ and $a \in V \setminus \{0\}$.

The prototypical example of a crooked function is $Q(x) = x^3$ where $V = \mathbb{F}_{2^e}$ with e odd. Given a crooked function Q, the corresponding *crooked graph* G_Q is the graph G_Q with vertex set $V \times \mathbb{F}_2 \times V$ where distinct vertices (a, i, α) and (b, j, β) are adjacent if and only if

$$\alpha + \beta = Q(a + b) + (i + j + 1)(Q(a) + Q(b)). \tag{1}$$

Many properties of crooked graphs were established in [2, 6] (see particularly [2, Section 2]).[1] We summarize these properties as a lemma.

Lemma 2.1. *Let Q be a crooked function over $\mathbb{F}_{2^e}$ and G_Q the corresponding crooked graph. Then*

(1) e is odd;
(2) G_Q is distance-regular of order 2^{2e+1}, and degree $2^{e+1} - 1$, and diameter 3;
(3) G_Q is a triangle-free;
(4) any pair of vertices at distance two have exactly two common neighbors;
(5) G_Q is antipodal, i.e., if $u, v, w \in V(G_Q)$ are distinct vertices and $d(u, v) = d(u, w) = 3$ then $d(v, w) = 3$;
(6) the map $(a, i, \alpha) \mapsto (a, i)$ defines a q-fold cover $G_Q \to K_{2q}$ with fibers $I_1, I_2, \ldots, I_{2q}$. Between any two fibers exists a perfect matching. Two distinct vertices are in the same fiber if and only if they are at distance 3.

Proof of Theorem 1.4. Let $q = 2^e$ and let $Q : \mathbb{F}_q \to \mathbb{F}_q$ be any crooked function. Let G_Q be the corresponding crooked graph. Define a new graph G'_Q by adding $2q + 1$ vertices $v_1, \ldots, v_{2q}, v$ to G_Q with the following adjacency rules. For $1 \leq j \leq 2q$, the neighborhood of v_j is $I_j \cup \{v\}$. The neighborhood of v is $\{v_1, \ldots, v_{2q}\}$. See Figure 1 below. Then G'_Q has $2q^2 + 2q + 1$ vertices and it contains G_Q as an induced subgraph.

We claim that $G'_Q \in \mathcal{W}_3$. If G'_Q contains a triangle, then this triangle must contain at least one of the new vertices $v_1, \ldots, v_{2q}, v$. There is no triangle containing v because the neighborhood of v is the independent set $\{v_1, v_2, \ldots v_{2q}\}$. Similarly there is no triangle containing v_j because the neighborhood of v_j is $I_j \cup \{v\}$, which is also an independent set. Therefore G'_Q is triangle-free.

Next we claim that G'_Q does not contain a $K_{2,3}$. Equivalently, any two distinct vertices $x, y \in V(G'_Q)$ have at most two common neighbors. We may assume x and y are not adjacent, since G'_Q is triangle-free. There are a number of cases. If $x = v$ and $y \in I_j$ then x and y have a unique common neighbor v_j. If $x = v_j$ and $y = v_{j'}$

[1] Conversely, Godsil and Roy [8] characterized crooked functions in terms of the distance-regularity of the corresponding graph defined by (1).

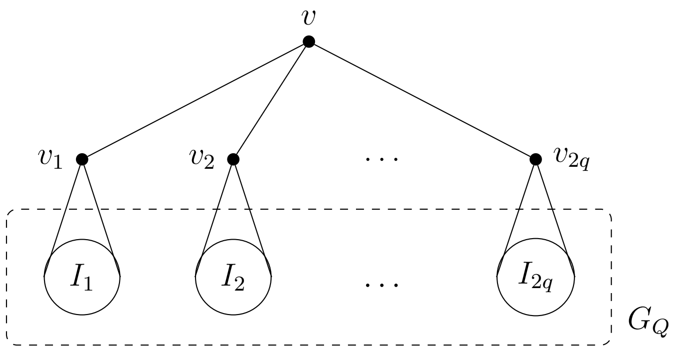

Figure 1: A visual of the graph G_Q.

then x and y have a unique common neighbor v. If $x = v_j$ and $y \in I_{j'}$ then x and y have a unique common neighbor $z \in I_j$, because the edges between I_j and $I_{j'}$ form a perfect matching. If x and y lie in a common fiber I_j then they have a unique common neighbor v_j. Finally, if $x \in I_j$ and $y \in I_{j'}$ with $j \neq j'$ then their common neighbors are exactly their common neighbors in G_Q, of which there are exactly two by parts (4) and (6) of Lemma 2.1

The case analysis of the previous paragraph also shows that every two nonadjacent vertices have at least one common neighbor, so G'_Q has diameter 2. Thus $G'_Q \in \mathcal{W}_3$. □

We now turn our attention to proving Theorem 1.5, which yields an infinite family of regular graphs $G \in \mathcal{W}_5$. As in the proof of Theorem 1.4, we start with G_Q, but instead of adding vertices we add edges within each fiber.

Lemma 2.2. *Let G_Q be a crooked graph on $2q^2$ vertices where $q = 2^{2e-1}$ and e is a positive integer. Let $H \in \mathcal{W}_5$ have order q. If one arbitrarily embeds a copy of H into each fiber of G_Q, then the resultant graph is again in $\mathcal{W}_5$.*

Proof. Let G''_Q be a graph resulting from embedding a copy of H into each fiber of G_Q. For each index j let $H_j \cong H$ denote the subgraph of G''_Q induced by I_j. Note that since no edges were added between any two distinct fibers, the edges between them still form a perfect matching in G''_Q. Also, G''_Q has G_Q as a subgraph.

Since H_j has diameter 2, any two vertices in the same fiber H_j are at distance at most 2. Any two vertices in distinct fibers are at distance at most 2 in G_Q, and so also have distance at most 2 in G''_Q. This shows G''_Q has diameter 2.

There is no triangle in G''_Q with all three vertices contained in the same fiber H_j, because H_j is triangle-free. Similarly, there is no triangle in G''_Q with vertices in distinct fibers, because G_Q is triangle-free and no edges were added between fibers. Finally, there is no triangle in G''_Q with one vertex in a fiber H_i and the other two vertices in another fiber H_j, because the edges between H_i and H_j form a perfect matching. Thus G''_Q is triangle-free.

Now suppose that G''_Q contains a $K_{2,5}$ with vertex classes $\{r_1, r_2\}$ and $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$. We argue similarly as above. It cannot be that all 7 vertices are contained in a single fiber H_j, because H_j is $K_{2,5}$-free. If $r_1, r_2 \in H_i$ for some i, then some ℓ_k must be contained in a different fiber H_j, but this is impossible because the edges between H_i and H_j form a perfect matching. Thus r_1 and r_2 must be in distinct fibers, say H_1 and H_2. Similarly, $\ell_1, \ldots, \ell_5$ must be in distinct fibers. Therefore at least 3 of the vertices ℓ_i are in fibers distinct from H_1 and H_2, say $\ell_3 \in H_3, \ell_4 \in H_4, \ell_5 \in H_5$ without loss of generality. But this is impossible because G_Q is $K_{2,3}$-free. Thus G''_Q is $K_{2,5}$-free, which finishes the proof that $G''_Q \in \mathcal{W}_5$. □

Proof of Theorem 1.5. We use induction on e. The base case $e = 2$ is established by observing that $K_{4,4} \in \mathcal{W}_5$. Now suppose inductively that $H \in \mathcal{W}_5$ is a regular graph of order $q = 2^{2^e-1}$. Let $Q : \mathbb{F}_q \to \mathbb{F}_q$ be a crooked function and let G_Q be the corresponding crooked graph of order $2q^2 = 2^{2^{e+1}-1}$. Embed copies of H into the fibers of G_Q, obtaining G''_Q, which is again a regular graph. By Lemma 2.2, G''_Q is again in $\mathcal{W}_5$. This completes the induction. □

Finally, we comment that parts (2) and (3) of Lemma 2.1 imply that a crooked graph of order n is C_3-free, $K_{2,3}$-free and has $\frac{1}{\sqrt{2}}n^{3/2} - \frac{n}{2}$ edges. Thus, crooked graphs yield an infinite family of graphs whose edge count meets the known upper bound for $\mathrm{ex}(n, \{C_3, K_{2,3}\})$.

3 Cayley graphs

Finally we prove Theorem 1.6.

Proof of Theorem 1.6. Let $p \equiv 11 \pmod{12}$ be a prime. By quadratic reciprocity, this condition ensures that -1 and -3 are quadratic nonresidues in $\mathbb{F}_p$. Let V be the group $\mathbb{F}_p^2$ where the operation is component-wise addition. Define $A \subset V$ by

$$A = \{(x, \pm x^2) : x \in \mathbb{F}_p \backslash \{0\}\}.$$

Let $G = \mathrm{Cay}(V, A)$. By definition, distinct vertices $(x_1, y_1), (x_2, y_2) \in V$ are adjacent if and only if

$$y_1 - y_2 = \pm(x_1 - x_2)^2 \neq 0.$$

We claim that G is $\{C_3, K_{2,7}\}$-free and has diameter 2. Since G is vertex-transitive, it is enough to show that the vertex $(0, 0)$ is not in a triangle, and that for any nonzero $(a, b) \in \mathbb{F}_p^2 \setminus A$, there is at least one but at most six paths of length 2 from $(0, 0)$ to (a, b).

The neighborhood of $(0, 0)$ is the set A. Suppose a pair of distinct vertices in A are adjacent. This implies that there exists $x, y \in \mathbb{F}_p\backslash\{0\}$ such that one of the following equations holds: $x^2 + y^2 = (x - y)^2$, $x^2 + y^2 = -(x - y)^2$, $x^2 - y^2 = (x - y)^2$, or $x^2 - y^2 = -(x - y)^2$. The first, third, and fourth cannot occur since x and y are distinct and not zero. The second implies $x^2 - xy + y^2 = 0$ which implies x/y is a solution to the quadratic equation $X^2 - X + 1 = 0$. This discriminant of this quadratic is -3 which is a quadratic nonresidue in $\mathbb{F}_p$. We conclude that there can be no such

x and y, so the neighborhood of $(0,0)$ is an independent set. By vertex-transitivity, G is C_3-free.

We now complete the proof of Theorem 1.6 by showing that for any nonzero $(a,b) \in \mathbb{F}_p^2 \setminus A$, the number of paths of length 2 from $(0,0)$ to (a,b) is at least one and at most six. Fix such a vertex $(a,b) \notin A \cup \{(0,0)\}$. Suppose that (x,y) is the middle vertex on a path of length 2 from $(0,0)$ to (a,b). Since (x,y) is adjacent to $(0,0)$, we have $x \neq 0$ and $y = \pm x^2$. Since (x,y) is adjacent to (a,b), we have $x \neq a$ and $b - y = \pm(a-x)^2$. The four possibilities reduce to two quadratic equations $2x^2 - 2ax + a^2 \pm b = 0$ and the two linear equations $2ax - a^2 \pm b = 0$. The quadratic equations have at most two solutions each.

Case 1: If $a \neq 0$ then the two linear equations each have a unique solution x. Moreover, $x \notin \{0,a\}$ because $(a,b) \notin A$. This gives at least one and at most six paths of length 2 from $(0,0)$ to (a,b).

Case 2: If $a = 0$ then the two linear equations are impossible, for otherwise we get $a = b = 0$, but $(a,b) \neq (0,0)$. The quadratic equations reduce to $b = 2x^2$ and $-b = 2x^2$. Since -1 is a quadratic nonresidue, one of these equations has two solutions while the other has none. Thus, in Case 2 we have exactly two paths of length 2 from $(0,0)$ to (a,b).

Once again using the fact that G is vertex-transitive, we can say that G is $K_{2,7}$-free and has diameter 2, so $G \in \mathcal{W}_7$. □

4 Concluding remarks

Crooked graphs were first named by Bending and Fon-Der-Flaass [2] after abstracting the key properties of the crooked functions $Q(x) = x^{2^k+1}$ considered by de Caen, Mathon, and Moorhouse [6]. There are now many known crooked functions. It follows from [2, Proposition 11] that any quadratic almost perfect nonlinear (APN) permutation function is crooked. In 2008, Budaghyan, Carlet and Leander [3] discovered the first new infinite family of quadratic APN permutations inequivalent to power APN functions. More recently, Li and Kaleyski [15] constructed another infinite class of quadratic APN permutations that appear to be inequivalent to those in previously known classes.

Crooked graphs have found a myriad of applications both in spectral graph theory [11, 12, 14] and extremal graph theory [5, 10]. In light of Theorem 1.7, it seems reasonable to conjecture that in fact

$$\mathrm{ex}(n, \{C_3, K_{2,3}\}) = \left(\frac{1}{\sqrt{2}} + o(1)\right) n^{3/2}$$

for all n. Do the crooked graphs contain dense induced subgraphs which would finally put this problem to rest?

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