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Diagonal graph Ramsey numbers of even cycles with pendant edges

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Abstract

Let G be a simple graph. The diagonal graph Ramsey number $R(G,G)$ is defined to be the minimum n, where every 2-coloring of the edges of K_n contains a red G or a blue G. In this paper, new diagonal graph Ramsey numbers are calculated for some classes of even cycles with pendant edges.

1 Introduction

In 1929, Frank Ramsey [15] established an innocuous-looking result in his groundbreaking paper on formal logic. Although it was not apparent at the time, his theorem would eventually form the cornerstone of Ramsey theory, a vibrant and rich area of extremal combinatorics.

The following general question [4] is investigated in Ramsey theory.

> If a particular mathematical structure (e.g., algebraic, combinatorial, or geometric) is arbitrarily partitioned into finitely many classes, what kinds of substructures must always remain intact in at least one of the classes?

Over many decades, Ramsey-type questions involving the set of integers, graphs, Euclidean space and topological spaces have been investigated. As of this writing, a keyword search for "Ramsey" yields 8121 entries in the MathSciNet database. The interested reader is directed to [4, 5] for a comprehensive overview of Ramsey theory. For gentle introductions to Ramsey theory, [8, 17] are recommended.

The reader should note that the seeds of Ramsey theory were planted even before Ramsey introduced his theorem. Soifer's [20] beautifully written book is filled with deep mathematics and also provides a rich historical context of Ramsey theory.

Interesting applications of Ramsey theory can be found in number theory, algebra, geometry, topology, set theory, logic, ergodic theory, information theory and computer science. The reader is directed to Rosta's [18] survey for a detailed exposition of some of these applications.

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2 Preliminaries

First, we recall some standard definitions and notation from graph theory. In this paper, all graphs are finite and simple.

For a graph G with vertex set $V(G)$ and edge set $E(G)$, the *order* and *size* of G are defined to be $|V(G)|$ and $|E(G)|$, respectively. A *pendant edge* of G is an edge of G where one of its vertices is of degree one. The *complete graph* K_n is the graph on n (≥ 2) vertices, where every pair of vertices is adjacent. The *path* on n vertices is denoted by P_n. For $n \geq 3$, the cycle on n vertices is denoted by C_n. Any graph-theoretic terms which are not explicitly defined in this paper can be found in [4, 22]

Definition. Let $k \geq 2$. A *k-coloring* of graph G is a coloring of $E(G)$, using a maximum of k colors.

Using graph-theoretic language, a simple version of Ramsey's theorem can be stated in the following way.

Theorem 2.1. (*Ramsey [15]). Let $s, t \geq 2$. Then, there exists a smallest positive integer n such that every 2-coloring of K_n contains a red K_s or a blue K_t.*

Definition. Let G and H be simple connected graphs. The *graph Ramsey number* $R(G, H)$ is the minimum n, where every 2-coloring of K_n contains a red G or a blue H. When $G = H$, we say that $R(G, G)$ is the *diagonal* graph Ramsey number of G.

Notation. For brevity, we use the notation $R(G)$ to denote $R(G, G)$.

Considerable work has been done in graph Ramsey theory. In addition to the calculation of Ramsey numbers in the classical theory, many different concepts have been introduced over time. They include Ramsey functions on graphs, many kinds of mixed Ramsey numbers, size Ramsey numbers, connected Ramsey numbers, anti-Ramsey numbers and Gallai-Ramsey numbers. For an overview of classical graph Ramsey theory, the general surveys of Burr [1, 2], Radziszowski [14], Read and Wilson [16], and Sudakov [21] are invaluable. New directions and additional open questions in graph Ramsey theory are addressed in [3, 23, 24].

Calculating graph Ramsey numbers is a difficult problem. In this paper, we compute $R(G)$, where G is in a certain class of connected unicyclic graphs of even girth. This is motivated by the following remarkable conjecture.

Conjecture 2.2. (*Grossman [6]). Let G be a connected unicyclic graph of odd girth and $|V(G)| \geq 4$. Then, $R(G) = 2 \cdot |V(G)| - 1$.*

This conjecture has been proved for various classes of connected unicylic graphs of odd girth. These include:

- C_3 with pendant edges from its three vertices [6]
- C_3 with pendant edges from a vertex and pendant edges and a path from another vertex [6]

- Any odd cycle with pendant edges at a single vertex [9]
- Any odd cycle with a pendant edge at two adjacent vertices [10]
- C_3 with pendant edges and/or a star at a single vertex [12]

Resolving Conjecture 2.2 appears to be a formidable task. We believe that Conjecture 2.2 is true and that its proof is as difficult (if not more) as determining the diagonal graph Ramsey numbers of trees, which is currently an unsolved problem. At this time, there is no "obvious" conjecture which describes the diagonal graph Ramsey numbers of connected unicyclic graphs of even girth.

3 $R(G)$, where G is an even cycle with pendant edges

Notation. Let C_n^k denote a cycle C_n with k pendant edges at a common vertex of the cycle. When there is no danger of confusion, it is convenient to use K_X to denote the complete graph on vertex set X. Similarly, $K_{X,Y}$ can be used to denote the complete bipartite graph with vertex partite sets X and Y.

The diagonal graph Ramsey number of $G = C_n^1$ (for $n \geq 3$) was determined in [12].

Theorem 3.1. (*Low and Kapbasov [12]*). *Let $n \geq 3$. Then,*

$$R(C_n^1) = \begin{cases} 2n+1 & \text{if } n \text{ is odd,} \\ \frac{3n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Theorem 3.2. *Let $n \geq 3$. Then,*

$$R(C_n^2) = \begin{cases} 2n+3 & \text{if } n \text{ is odd,} \\ \frac{3n}{2}+1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $G = C_n^2$. In [12], we see that $R(G) = 7$ when $n = 4$. So, the claim holds when $n = 4$.

Case 1: $R(G)$ where $n \geq 6$ and is even.

First, we show that $R(G) \leq \frac{3n}{2} + 1$. Let $j = \frac{3n}{2} + 1$ and $\mathcal{C}$ be a 2-coloring of K_j. Since $R(C_n^1) = j - 1$ (see [12]), there is a (say) blue subgraph $C_n^1 = v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$ with pendant edge v_nv_{n+1}. Consider the vertex sets $X = \{v_1, v_2, \ldots, v_n\}$ and $Y = V(K_j) - X = \{v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_j\}$ and form the complete bipartite graph $B = K_{X,Y}$. Note that $|X| = n$ and $|Y| = \frac{n+2}{2}$.

The edges v_nv_{n+2}, v_nv_{n+3}, $\ldots v_nv_j$ are all red. If not, then there is a blue G and we are done. Observe that there is at most one blue edge from each vertex in X to the vertices in Y. Otherwise, there is a blue G and we are done. Furthermore, the number of blue edges from X to Y is between 1 and n, inclusive. The remaining edges of B are red. By a straightforward counting argument, one sees the existence of many red copies of G. Here, $\frac{n}{2}$ vertices of the cycle are from X, $\frac{n}{2}$ vertices of the cycle are from Y, and the two pendant vertices are from X. Thus, $R(G) \leq \frac{3n}{2} + 1$.

Finally, we show that $R(G) > \frac{3n}{2}$. In particular, we construct a 2-coloring of K_l where $l = \frac{3n}{2}$, which does not contain a monochromatic G. Let $\mathcal{Q}_1 = \{w_1, w_2, \ldots, w_n, w_{n+1}\}$ be the vertices of a blue C_n^1 in K_l (see [12]) and $\mathcal{Q}_2 = V(K_l) - \mathcal{Q}_1$. Let $B' = K_{\mathcal{Q}_1,\mathcal{Q}_2}$ and color the edges of B' with red. Since $|\mathcal{Q}_2| < \frac{n}{2}$ (as $|\mathcal{Q}_2| = \frac{n-2}{2}$), a red C_n does not exist in B'. In particular, there is no red G in B'. Finally, color all of the edges in the complete subgraphs $K_{\mathcal{Q}_1}$ and $K_{\mathcal{Q}_2}$, induced by $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively, blue. Since $K_{\mathcal{Q}_1}$ and $K_{\mathcal{Q}_2}$ each contain less vertices than G, those complete subgraphs do not contain a blue G. Thus, $R(G) > \frac{3n}{2}$.

Hence, $\frac{3n}{2} < R(G) \leq \frac{3n}{2} + 1$ and we conclude that $R(G) = \frac{3n}{2} + 1$.

Case 2: $R(G)$ where $n \geq 3$ and is odd.

This case follows from a theorem of Köhler ([9], see Theorem A in the Appendix). □

The diagonal graph Ramsey number of $G = C_4^k$ was determined in [12].

Theorem 3.3. (*Low and Kapbasov [12]*). *Let* $k \geq 1$. *Then,*

$$R(C_4^k) = \begin{cases} 2k+4 & \text{if } k \text{ is odd,} \\ 2k+3 & \text{if } k \text{ is even.} \end{cases}$$

Notation. Let $N_R(v)$ denote the set of vertices which are red adjacent to v. Let $N_{\gamma,R}(v)$ denote the set of vertices in γ which are red adjacent to v.

Lemma 3.4. *Let* $k \geq 3$ *be odd. Then,* $R(C_6^k) \leq 2k + 5$.

Proof. Using scientific computing (via a SAT solver) as found in [13], we determined that $R(C_6^3) = 11$. Thus, the lemma holds for $k = 3$.

Let $G = C_6^k$ where $k \geq 5$ is odd, and consider a 2-coloring $\mathcal{C}$ of K_{2k+5} (say, vertices $v_1, v_2, \ldots, v_{2k+4}, v_{2k+5}$). By a theorem of Harary ([7], see Theorem B in the Appendix), there exists a monochromatic $K_{1,k+3}$ (say red, with central vertex v_1 and leaves $v_2, v_3, \ldots, v_{k+4}$) in coloring $\mathcal{C}$ of K_{2k+5}. Let $A = \{v_2, v_3, \ldots, v_{k+4}\}$ and $B = \{v_{k+5}, v_{k+6}, \ldots, v_{2k+5}\}$; $|A| = k + 3$ and $|B| = k + 1$. Furthermore by a theorem of Zhang, Sun, and Wu ([25], see Theorem C in the Appendix), there exists a monochromatic C_6 in $K_{A,B}$ in coloring $\mathcal{C}$. If this C_6 is red, then there exists a red G in $\mathcal{C}$ and the lemma is proved. Therefore, this C_6 is blue (say, with vertices $v_2, v_3, v_4, v_{k+5}, v_{k+6}$ and v_{k+7}). Let $\alpha = A \backslash \{v_2, v_3, v_4\}$ and $\beta = B \backslash \{v_{k+5}, v_{k+6}, v_{k+7}\}$, with $|\alpha| = k$ and $|\beta| = k - 2$. Now, consider (in coloring $\mathcal{C}$) the complete bipartite subgraph $K_{X,Y}$ $(= K_{6,2k-1})$ with vertex partitions $X = \{v_2, v_3, v_4, v_{k+5}, v_{k+6}, v_{k+7}\}$ and $Y = \alpha \cup \beta \cup \{v_1\}$.

From each vertex in $\{v_{k+5}, v_{k+6}, v_{k+7}\}$ to $\alpha \cup \beta \cup \{v_1\}$, there are at most $k - 1$ blue edges (and hence, at least k red edges). Otherwise, the lemma is established. From each vertex in $\{v_2, v_3, v_4\}$ to $\alpha \cup \beta$, there are at most $k - 1$ blue edges (and hence, at least $k - 1$ red edges). Otherwise, the lemma is established.

Now, let us focus on the red edges of $K_{X,Y}$ in coloring $\mathcal{C}$.

Claim: Either a red P_5 with endpoints in α and not containing v_1, or a red C_6 containing v_1 (and not containing v_3, v_4) exists within $K_{X,Y}$. Recall that $k \geq 5$ is odd.

(We first establish that v_{k+5} is red adjacent to vertex $v_{\alpha_1} \in \alpha$ and vertex $v_{\beta_1} \in \beta$). By the Pigeonhole Principle, each vertex in $\{v_{k+5}, v_{k+6}, v_{k+7}\}$ is red adjacent to some (possibly different) vertex in α. Now, if there are no red edges from $\{v_{k+5}, v_{k+6}, v_{k+7}\}$ to β, then one of three possibilities can occur for each vertex in $\{v_{k+5}, v_{k+6}, v_{k+7}\}$: (i). k red edges to all of α and one red edge to v_1. (ii). $k-1$ red edges to vertices in α and one red edge to v_1. (iii). k red edges to all of α and one blue edge to v_1. In all instances, a red P_5 with endpoints in α (and not containing v_1) exists or a red C_6 containing v_1 exists, which establishes the Claim. Thus, we have (WLOG) that v_{k+5} is red adjacent to vertex $v_{\alpha_1} \in \alpha$ and vertex $v_{\beta_1} \in \beta$.

(Now, we establish that v_{k+6} is red adjacent to a vertex v_{β_2} ($\neq v_{\beta_1}$) in β). We have that v_{k+5} is red adjacent to $v_{\alpha_1} \in \alpha$ and $v_{\beta_1} \in \beta$. If there are no red edges from $\{v_{k+6}, v_{k+7}\}$ to β, then one of three possibilities can occur for each vertex in $\{v_{k+6}, v_{k+7}\}$: (i). k red edges joining v_{k+6} to all of α and k red edges joining v_{k+7} to all of α. (ii). k red edges joining v_{k+6} to all of α and $k-1$ red edges joining v_{k+7} to vertices of α (and one red edge $v_1 v_{k+7}$). (iii). $k-1$ red edges joining v_{k+6} to vertices of α (and one red edge $v_1 v_{k+6}$) and $k-1$ red edges joining v_{k+7} to vertices of α (and one red edge $v_1 v_{k+7}$). In all instances, a red P_5 with endpoints in α (and not containing v_1) exists or a red C_6 containing v_1 exists, which establishes the Claim. Thus, we have (WLOG) that v_{k+6} is red adjacent to a vertex $v_{\beta_2} \in \beta$. If $v_{\beta_2} \neq v_{\beta_1}$, then this bullet point is established. If $v_{\beta_2} = v_{\beta_1}$, then v_{k+6} must only be red adjacent to v_{α_1} in α. Otherwise, a red P_5 exists and the Claim is proved. Consequently, v_{k+5} has exactly one red neighbor in α, namely v_{α_1}. Otherwise, a red P_5 would exist and the Claim is proved. Hence, the red neighborhoods of v_{k+5} and v_{k+6} are identical, namely $\{v_{\alpha_1}, v_1\} \cup \beta$. So, now choose a "new" $v_{\beta_2} \in \beta$ ($\neq v_{\beta_1}$) which is red adjacent to v_{k+6}, which exists since $k \geq 5$ is odd.

Case 1. Both v_{k+6} and v_{k+7} are red adjacent to v_{β_1}. Then by the argument in the second bullet point, the red neighborhoods of v_{k+5}, v_{k+6} and v_{k+7} are identical, namely $\{v_{\alpha_1}, v_1\} \cup \beta$. Here, a red C_6 exists and the Claim is established.

Case 2. Vertex v_{k+6} is red adjacent to v_{β_1} and v_{k+7} is blue adjacent to v_{β_1}. Then the maximum number of red edges from v_{k+7} to β is $k-3$. By the Pigeonhole Principle, there exist two vertices (let $v_{k+4} \neq v_{\alpha_1}$ be one of them) in α which are red adjacent to v_{k+7}. By the Pigeonhole Principle, the red neighborhood of v_{k+6} is $\{v_{\alpha_1}, v_1\} \cup \beta$. Otherwise, a red P_5 exists and the Claim is proved. Thus, v_{k+7} is blue adjacent to all of β. Otherwise, a red P_5 would exist and the Claim is proved. Also, v_{k+5} is blue adjacent to all of $\alpha - \{v_{\alpha_1}\}$ and hence, the red neighborhood of v_{k+5} is $\{v_{\alpha_1}, v_1\} \cup \beta$. Otherwise, a red P_5 exists and the Claim is proved. If $v_1 v_{k+7}$ is red, then the red neighborhood of v_{k+7} is $(v_1 \cup \alpha) - \{v_{\alpha_1}\}$ (since a red $v_{\alpha_1} v_{k+7}$ would give a red C_6 and establish the Claim). On the other hand, if v_1v_{k+7} is blue, then the red neighborhood of v_{k+7} is α. Thus, $\alpha - \{v_{\alpha_1}\}$ is a subset of v_{k+7}'s red neighborhood. Note that v_2 is red adjacent to at least one vertex in α. If v_2 is red adjacent to at least two vertices in $\alpha - \{v_{\alpha_1}\}$, then a red P_5 exists and the Claim is proved. If v_2 is red adjacent to v_{α_1} and v_j in α, then a red P_5 exists and the Claim is proved. Thus, all vertices of β are red-adjacent to v_2, v_3 and v_4, since each vertex in $\{v_2, v_3, v_4\}$ to $\alpha \cup \beta$ has at least $k-1$ red edges. Here, a red $C_6'^k$ exists and the lemma is proved.

Case 3. Vertex v_{k+6} is blue adjacent to v_{β_1}. Then the maximum number of red edges from v_{k+6} to β is $k-3$. By the Pigeonhole Principle, there exist two vertices (let v_{α_2} $[\neq v_{\alpha_1}]$ be one of them) in α which are red adjacent to v_{k+6}. Furthermore, $v_{k+5}v_{\beta_2}$ is blue. Otherwise, a red P_5 exists and the Claim is proved.

(Here, we establish that $v_{k+5}v_{\alpha_2}$ is blue). Assume that $v_{k+5}v_{\alpha_2}$ is red. Then, v_{k+6} is not red adjacent to any vertex in $\alpha - \{v_{\alpha_1}, v_{\alpha_2}\}$. Otherwise, there would be a red P_5 and the Claim is established. Now, examine the red adjacent vertices to v_{k+6}. By the Pigeonhole Principle, we must have that $N_R(v_{k+6}) = \{v_{\alpha_1}, v_{\alpha_2}, v_1\} \cup (\beta - \{v_{\beta_1}\})$. Furthermore, v_{k+5} is not red adjacent to any vertices in $\alpha - \{v_{\alpha_1}, v_{\alpha_2}\}$. Otherwise, a red P_5 would exist and the Claim is proved. By the Pigeonhole Principle, v_{k+5} and v_{k+6} both share a red vertex neighbor in β. Here, a red P_5 exists and the Claim is established. We therefore conclude that $v_{k+5}v_{\alpha_2}$ is blue.

(Now, we establish that $v_{k+6}v_{\alpha_1}$ is blue). Assume that $v_{k+6}v_{\alpha_1}$ is red. Then, v_{k+5} is not red adjacent to any vertex in $\alpha - \{v_{\alpha_1}, v_{\alpha_2}\}$. Otherwise, there would be a red P_5 and the Claim is established. Thus, $|N_R(v_{k+5})|$ is at most $(k-3)+1+1$, which gives a desired contradiction. We therefore conclude that $v_{k+6}v_{\alpha_1}$ is blue.

Thus far, we have the following: In $K_{X,Y}$, red edges – $v_{k+5}v_{\alpha_1}$, $v_{k+5}v_{\beta_1}$, $v_{k+6}v_{\beta_2}$, $v_{k+6}v_{\alpha_2}$; blue edges – $v_{k+5}v_{\beta_2}$, $v_{k+5}v_{\alpha_2}$, $v_{k+6}v_{\beta_1}$, $v_{k+6}v_{\alpha_1}$. If vertices v_{k+5} and v_{k+6} have a common red neighbor in $\alpha \cup \beta$, then a red P_5 exists and the Claim is proved. So, $N_{\alpha\cup\beta,R}(v_{k+5}) \cap N_{\alpha\cup\beta,R}(v_{k+6}) = \emptyset$. Since $|\alpha \cup \beta \cup \{v_1\}| = 2k-1$ and $|N_R(v_{k+5})| + |N_R(v_{k+6})| \geq 2k$, this implies that v_1 is the only shared red neighbor of v_{k+5} and v_{k+6}. Also, note that $|N_{\alpha\cup\beta,R}(v_{k+5})| = k-1 = |N_{\alpha\cup\beta,R}(v_{k+6})|$ and that each vertex in $\alpha\cup\beta$ is red adjacent to either v_{k+5} or v_{k+6}, but not both. We also have that $|N_{\alpha,R}(v_{k+5})| \geq 2$ and $|N_{\alpha,R}(v_{k+6})| \geq 2$. Let $v_{\alpha_1}, v_{\alpha_3} \in N_{\alpha,R}(v_{k+5})$ and $v_{\alpha_2}, v_{\alpha_4} \in N_{\alpha,R}(v_{k+6})$.

Case 1. Let $v' \in N_{\alpha\cup\beta,R}(v_{k+7}) \cap N_{\alpha\cup\beta,R}(v_{k+6})$ and $v'' \in N_{\alpha\cup\beta,R}(v_{k+7}) \cap N_{\alpha\cup\beta,R}(v_{k+5})$. Then, the red $C_6 = v_1v_{k+5}v''v_{k+7}v'v_{k+6}v_1$ exists and the Claim is proved.

Case 2. Let $N_R(v_{k+7}) = N_R(v_{k+5})$. Then, the red $P_5 = v_{\alpha_1}v_{k+7}v_{\beta_1}v_{k+5}\,v_{\alpha_3}$ exists and the Claim is proved.

Case 3. Let $N_R(v_{k+7}) = N_R(v_{k+6})$. Then, the red $P_5 = v_{\alpha_2}v_{k+7}v_{\beta_2}v_{k+6}\,v_{\alpha_4}$ exists and the Claim is proved.

All possible scenarios lead to a red C_6 or a red P_5. Thus, the Claim is established. $\Diamond$

It follows immediately from the Claim that a red C_6^k exists in $\mathcal{C}$. Thus, the lemma is established. □

Lemma 3.5. *Let $k \geq 2$ be even. Then, $R(C_6^k) \leq 2k+6$.*

Proof. By Theorem 3.2, we see that $R(C_6^2) = 10$. So the claim holds when $k = 2$.

Let $G = C_6^k$, where $k \geq 4$ is even, and $\mathcal{C}$ be a 2-coloring of K_{2k+6} (say, vertices $v_1, v_2, \ldots, v_{2k+5}, v_{2k+6}$). By [7] (see Theorem B in the Appendix), there exists a monochromatic $K_{1,k+3}$ (say red, with central vertex v_1 and leaves $v_2, v_3, \ldots, v_{k+4}$) in coloring $\mathcal{C}$ of K_{2k+6}. Consider the edge-coloring of the complete bipartite subgraph with vertex sets $\{v_2, v_3, \ldots, v_7\}$ and $\{v_{2k+1}, v_{2k+2}, \ldots, v_{2k+6}\}$. By [25] (see Theorem C in the Appendix), there exists a monochromatic C_6 containing three vertices from each vertex set. If this C_6 is red, then there exists a red G in $\mathcal{C}$ and the lemma is proved. Therefore, this C_6 is blue (say, with vertices say $v_2, v_3, v_4, v_{2k+4}, v_{2k+5}, v_{2k+6}$). Let $X = \{v_{2k+4}, v_{2k+5}, v_{2k+6}\}$, $\alpha = \{v_5, v_6, \ldots, v_{k+4}\}$, and $\beta = \{v_{k+5}, v_{k+6}, \ldots, v_{2k+3}\}$. Consider (in coloring $\mathcal{C}$) the complete bipartite subgraph $K_{X,Y}$ $(= K_{3,2k})$ with vertex partitions X and $Y = \alpha \cup \beta \cup \{v_1\}$.

From each vertex in X to Y, there are at most $k-1$ blue edges (and hence, at least $k+1$ red edges). Otherwise, a blue C_6^k exists and the lemma is established. Now, let us focus on the red edges of $K_{X,Y}$ in coloring $\mathcal{C}$.

<u>Claim</u>: Either a red P_5 with endpoints in α and not containing v_1, or a red C_6 containing v_1 (and not containing v_3, v_4) exists. Recall that $k \geq 4$ and even.

(We establish that v_{2k+6} is red adjacent to vertex $v_{\alpha_1} \in \alpha$ and vertex $v_{\beta_1} \in \beta$). By the Pigeonhole Principle, each vertex in $\{v_{2k+4}, v_{2k+5}, v_{2k+6}\}$ is red adjacent to a (possibly different) vertex in α. Now, if there are no red edges from $\{v_{2k+4}, v_{2k+5}, v_{2k+6}\}$ to β, then only one possibility can occur for each vertex in $\{v_{2k+4}, v_{2k+5}, v_{2k+6}\}$, namely k red edges to all of α and one red edge to v_1. In this instance, a red P_5 with endpoints in α (and not containing v_1) exists and the Claim is established. Thus, we have (WLOG) that v_{2k+6} is red adjacent to vertex $v_{\alpha_1} \in \alpha$ and vertex $v_{\beta_1} \in \beta$.

(We establish that v_{2k+5} is red adjacent to a vertex v_{β_2} $(\neq v_{\beta_1})$ in β). If there are no red edges from $\{v_{2k+4}, v_{2k+5}\}$ to β, then only one possibility can occur for each vertex in $\{v_{2k+4}, v_{2k+5}\}$, namely k red edges to all of α and one red edge to v_1. Here, a red P_5 exists and the Claim is established. Thus, we have (WLOG) that v_{2k+5} is red adjacent to a vertex $v_{\beta_2} \in \beta$. If $v_{\beta_2} \neq v_{\beta_1}$, then this bullet point is established. If $v_{\beta_2} = v_{\beta_1}$, then v_{2k+5} must only be red adjacent to v_{α_1} in α. Otherwise, a red P_5 exists and the Claim is proved. Consequently, v_{2k+6} has exactly one red neighbor in α, namely v_{α_1}. Otherwise, a red P_5 would exist and the Claim is proved. Hence, the red neighborhoods of v_{2k+5} and v_{2k+6} are identical, namely $\{v_{\alpha_1}, v_1\} \cup \beta$. So, now choose a "new" $v_{\beta_2} \in \beta$ $(\neq v_{\beta_1})$ which is red adjacent to v_{2k+5}, which exists since $k \geq 4$ even.

(We establish that $v_{2k+5}v_{\alpha_2}$ is red $(\neq v_{\alpha_1})$). If there is no v_{α_2} which is red adjacent to v_{2k+5}, then $v_{2k+5}v_{\alpha_1}$ is red and $N_R(v_{2k+5}) = \beta \cup \{v_{\alpha_1}\} \cup \{v_1\}$, by the Pigeonhole

Principle. This implies that that $N_R(v_{2k+6}) = \beta \cup \{v_{\alpha_1}\} \cup \{v_1\}$. Otherwise, a red P_5 exists and the Claim is proved. Now, every vertex in $\{v_2, v_3, v_4\}$ has at most $k-1$ blue edges (and hence, at least k red edges) going to vertices in $\alpha \cup \beta$. If v_2 is red adjacent to a vertex v_{β_t}, then a red C_6^k (underlying red $C_6 = v_1 v_2 v_{\beta_s} v_{2k+6} v_{\beta_t} v_{2k+5} v_1$) exists and the lemma is proved. Thus, $v_2 v_{\alpha_1}$ is red. However, now a red C_6^k (underlying red $C_6 = v_1 v_2 v_{\alpha_1} v_{2k+6} v_{\beta_t} v_{2k+5} v_1$) exists and the lemma is proved. Therefore, we conclude that $v_{2k+5} v_{\alpha_2}$ is red ($\neq v_{\alpha_1}$).

(We establish that $v_{2k+5} v_{\beta_1}$ is blue and $v_{2k+6} v_{\beta_2}$ is blue). If either $v_{2k+5} v_{\beta_1}$ is red or $v_{2k+6} v_{\beta_2}$ is red, then a red P_5 exists and the Claim is proved.

(We establish the following: $v_{2k+5} v_{\alpha_1}$ is blue and $v_{2k+5} v_{\alpha_3}$ is red ($\neq v_{\alpha_1}, v_{\alpha_2}$)). Suppose that $v_{2k+5} v_{\alpha_1}$ is red. By the Pigeonhole Principle, v_{2k+6} has at least two red neighbors in α. They must only be v_{α_1} and v_{α_2}. Otherwise, a red P_5 exists and the Claim is proved. Thus, $N_R(v_{2k+6}) = (\beta - \{v_{\beta_2}\}) \cup \{v_1, v_{\alpha_1}, v_{\alpha_2}\}$. By the Pigeonhole Principle, v_{2k+5} has at least two red neighbors in α. They must only be v_{α_1} and v_{α_2}. Otherwise, a red P_5 exists and the Claim is proved. Thus, $N_R(v_{2k+5}) = (\beta - \{v_{\beta_2}\}) \cup \{v_1, v_{\alpha_1}, v_{\alpha_2}\}$. However, now a red P_5 exists and the Claim is proved. Therefore, we conclude that $v_{2k+5} v_{\alpha_1}$ is blue. Since v_{2k+5} has at most $k-2$ red neighbors in β, v_{2k+5} must have at least two red neighbors in α. Let v_{α_3} ($\neq v_{\alpha_1}, v_{\alpha_2}$) be a red neighbor of v_{2k+5}.

Finally, recall that there are at least $2k+2$ red edges to $\alpha \cup \beta \cup \{v_1\}$ from v_{2k+5} and v_{2k+6}. Furthermore, $|\alpha \cup \beta \cup \{v_1\}| = 2k$. Thus, there are only two cases left to consider.

<u>Case 1</u>. $|N_{\alpha,R}(v_{2k+5}) \cap N_{\alpha,R}(v_{2k+6})| \geq 1$. Here, a red P_5 exists and the Claim is proved.

<u>Case 2</u>. $|N_{\beta,R}(v_{2k+5}) \cap N_{\beta,R}(v_{2k+6})| \geq 1$. Here, a red P_5 exists and the Claim is proved.

All possible scenarios lead to a red C_6 or a red P_5. Thus, the Claim is established. $\Diamond$

It follows immediately from the Claim that a red C_6^k exists in $\mathcal{C}$. Thus, the lemma is proved. $\square$

Theorem 3.6. *Let $k \geq 1$ and $n \geq 4$ be even. Then, $R(C_n^k) > 2(k + \lfloor \frac{n-1}{2} \rfloor)$.*

Proof. Let $G = C_n^k$, $l = 2(k + \lfloor \frac{n-1}{2} \rfloor)$ and consider the following 2-coloring of K_l with vertices $\{v_1, v_2, \ldots, v_l\}$: Let $X = \{v_1, v_2, \ldots, v_{\frac{l}{2}}\}$ and $Y = \{v_{\frac{l}{2}+1}, v_{\frac{l}{2}+2}, \ldots, v_l\}$. Color all of the edges of K_X and K_Y red. Since $|X| = |Y| = k + \lfloor \frac{n-1}{2} \rfloor$ and $|V(G)| = k + n$, a red G does not exist within K_X nor K_Y. Now, color all the edges of $K_{X,Y}$ blue. Since $\frac{n}{2}$ vertices (from X) and $\frac{n}{2}$ vertices (from Y) are needed for a blue C_n, at most a blue C_n^{k-1} exists. Thus, $R(C_n^k) > 2(k + \lfloor \frac{n-1}{2} \rfloor)$. $\square$

Corollary 3.7. *Let $k \geq 2$ be even. Then, $2k + 5 \leq R(C_6^k) \leq 2k + 6$.*

Proof. The lower bound is established by Theorem 3.6, with $n = 6$. The upper bound is established by Lemma 3.5. □

Theorem 3.8. *Let $k \geq 3$ be odd. Then, $R(C_6^k) = 2k + 5$.*

Proof. Using Theorem 3.6 with $n = 6$, one obtains $R(C_6^k) > 2k+4$. This, along with Lemma 3.4, establishes the claim. □

Theorem 3.9. *Let $n \geq 6$ be even and $k \geq 1$. Then, $R(C_n^k) > R(C_n) + k - 1 = \frac{3n}{2} + k - 2$.*

Proof. Let $G = C_n^k$. For $k \geq 1$, $n \geq 6$ even and $k > \frac{n}{2}$, the claim follows from Theorem 3.6 since $\frac{3n}{2} + k - 2 < 2(k + \lfloor \frac{n-1}{2} \rfloor)$. So, let $1 \leq k \leq \frac{n}{2}$ and $l = \frac{3n}{2} + k - 2$. Consider the following 2-coloring of K_l with vertices $\{v_1, v_2, \ldots, v_l\}$: Let $X = \{v_1, v_2, \ldots, v_{n+k-1}\}$ and $Y = \{v_{n+k}, v_{n+k+1}, \ldots, v_l\}$. Color all of the edges of K_X and K_Y red. Since $|X| = n + k - 1$, $|Y| = \frac{n}{2} - 1$ and $|V(G)| = k + n$, a red G does not exist within K_X nor K_Y. Now, color all the edges of $K_{X,Y}$ blue. Since $\frac{n}{2}$ vertices (from X) and $\frac{n}{2}$ vertices (from Y) are needed for a blue C_n, there is no blue C_n. In particular, there is no blue G. Thus, $R(C_n^k) > \frac{3n}{2} + k - 2$. □

<u>Remark</u>. With regards to Theorems 3.6 and 3.9 (for $n \geq 6$ even), note the following:

- $[k = \frac{n}{2}]$: $\frac{3n}{2} + k - 2 = 2n - 2 = 2(\frac{n}{2} + \frac{n}{2} - 1) = 2(k + \lfloor \frac{n-1}{2} \rfloor)$.
- $[k < \frac{n}{2}]$: Since $k+n < \frac{n}{2}+n$, we have $2k+n < \frac{3n}{2}+k$ and $2k+n-2 < \frac{3n}{2}+k-2$. Hence, $2(k + \lfloor \frac{n-1}{2} \rfloor) < \frac{3n}{2} + k - 2$.
- $[k > \frac{n}{2}]$: Since $\frac{n}{2} + n < k + n$, we have $\frac{3n}{2} + k < 2k + n$. Hence, $\frac{3n}{2} + k - 2 < 2(k + \lfloor \frac{n-1}{2} \rfloor)$.

Thus, $\frac{3n}{2}+k-2$ is a better lower bound of $R(C_n^k)$, for "smaller" k, whereas $2(k+\lfloor \frac{n-1}{2} \rfloor)$ is a better lower bound for "larger" k.

Theorem 3.10. *Let $n \geq 6$ be even and $1 \leq k \leq \lceil \frac{n}{4} \rceil$. Then, $R(C_n^k) \leq R(C_n) + k = \frac{3n}{2} - 1 + k$.*

Proof. Fix even $n \geq 6$ and $1 \leq k \leq \lceil \frac{n}{4} \rceil$. Let $c = \frac{3n}{2} - 1 + k$ and $\mathcal{C}$ be a 2-coloring of $K_c = \{v_1, v_2, \ldots, v_c\}$. Since $k \geq 1$, there exists a monochromatic (say, red) C_n with edge set $\{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$ in $\mathcal{C}$. Let $X = \{v_1, v_2, \ldots, v_n\}$ and $Y = V(K_c) \backslash X = \{v_{n+1}, v_{n+2}, \ldots, v_c\}$. Note that $|Y| = \frac{n}{2} - 1 + k$. In particular, if $k = \lceil \frac{n}{4} \rceil$, then

$$|Y| = \begin{cases} \frac{3n-4}{4} & \text{if } \frac{n}{2} \text{ is even,} \\ \frac{3n-2}{4} & \text{if } \frac{n}{2} \text{ is odd.} \end{cases}$$

In $\mathcal{C}$, consider the 2-coloring of $K_{X,Y}$. From each vertex in X to vertex set Y, there are at most $k-1$ red edges. Otherwise, a red C_n^k exists and the theorem is established. Hence, there are at least $\frac{n}{2}$ blue edges from each vertex in X to vertex set Y. Thus, there are at least $\frac{n^2}{2}$ blue edges in $K_{X,Y}$.

Using a result by Li and Ning ([11], see Theorem E in the Appendix), let $n = |X|$, $\frac{n}{2} \leq m = |Y| \leq \frac{3n-4}{4}$ or $\frac{3n-2}{4}$, and $t = \frac{n}{2}$. For the range of m, it is straightforward to verify that $t \leq m \leq 2t - 2$. Furthermore,

$$(t-1)(n-1) + m \leq \begin{cases} \frac{n^2}{2} - \frac{3n}{4} & \text{if } m \text{ is even,} \\ \frac{n^2}{2} - \frac{3n}{4} + \frac{1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

In both cases, $(t-1)(n-1)+m < |E_B(K_{X,Y})|$, where $|E_B(K_{X,Y})|$ denotes the number of blue edges in $K_{X,Y}$. Thus by Theorem E, a blue C_n exists in $K_{X,Y}$ in coloring $\mathcal{C}$.

From X, $\frac{n}{2}$ vertices are needed and from Y, $\frac{n}{2}$ vertices are needed to form the blue C_n. Let X' and Y' denote the sets containing the required vertices, respectively. Then, let $X'' = X \backslash X'$ and $Y'' = Y \backslash Y'$. Note that $|X'| = |Y'| = |X''| = \frac{n}{2}$ and $|Y''| = k - 1$.

<u>Claim</u>: There exists a blue C_n^k, where the cycle is formed by the $\frac{n}{2}$ vertices of X', the $\frac{n}{2}$ vertices of Y' and the k pendants are from a vertex $v \in Y'$ to vertices in X''.

If $\frac{n}{2}$ is even, then $|Y| \leq \frac{3n-4}{4}$, $|Y''| \leq \frac{3n-4}{4} - \frac{n}{2} = \frac{n-4}{4}$ and $1 \leq k \leq \frac{n}{4}$. If $\frac{n}{2}$ is odd, then $|Y| \leq \frac{3n-2}{4}$, $|Y''| \leq \frac{3n-2}{4} - \frac{n}{2} = \frac{n-2}{4}$ and $1 \leq k \leq \frac{n+2}{4}$.

<u>Case 1</u>. $\frac{n}{2}$ is even. Then, $|E_B(K_{X'',Y''})| \leq \frac{n}{2}(\frac{n-4}{4}) = \frac{n^2}{8} - \frac{n}{2}$. So, $|E_B(K_{X'',Y'})| \geq \frac{n^2}{4} - (\frac{n^2}{8} - \frac{n}{2}) = \frac{n}{2}(\frac{n}{4} + 1) \geq \frac{n}{2}(k+1)$. Thus, there is a vertex $v \in Y'$ with at least k blue edges joined to vertices of X''.

<u>Case 2</u>. $\frac{n}{2}$ is odd. Then, $|E_B(K_{X'',Y''})| \leq \frac{n}{2}(\frac{n-2}{4}) = \frac{n^2}{8} - \frac{n}{4}$. So, $|E_B(K_{X'',Y'})| \geq \frac{n^2}{4} - (\frac{n^2}{8} - \frac{n}{4}) = \frac{n}{2}(\frac{n}{4} + \frac{1}{2}) \geq \frac{n}{2}(k)$. Thus, there is a vertex $v \in Y'$ with at least k blue edges joined to vertices of X''.

This finishes the proof of the Claim. ◊

Hence, coloring $\mathcal{C}$ must contain a red C_n^k or a blue C_n^k. Therefore, $R(C_n^k) \leq R(C_n) + k = \frac{3n}{2} - 1 + k$. □

Theorem 3.11. *Let $n \geq 8$, $n \equiv 0 \pmod 4$ and $1 \leq k \leq \lceil \frac{n}{4} \rceil$. Then, $R(C_n^k, C_n^{k+1}) = \frac{3n}{2} - 1 + k$.*

Proof. From Case 1 (of Claim) in the proof of Theorem 3.10, there is a vertex $v \in Y'$ with at least $k+1$ blue edges joined to vertices of X''. This yields a blue C_n^{k+1}. Thus, $R(C_n^k, C_n^{k+1}) \leq \frac{3n}{2} - 1 + k$. The 2-coloring of K_l (found in the proof of Theorem 3.9), where $l = \frac{3n}{2} + k - 2$, does not contain a red C_n^k nor a blue C_n^{k+1}. □

Theorem 3.12. *Let $n \geq 8$, $n \equiv 0 \pmod 4$ and $k = \frac{n}{4} + 1$. Then, $R(C_n^k) \leq R(C_n) + k = \frac{3n}{2} - 1 + k$.*

Proof. Fix even $n \geq 8$, where $n \equiv 0 \pmod 4$. Let $k = \frac{n}{4} + 1$, $c = \frac{3n}{2} - 1 + k = \frac{7n}{4}$ and $\mathcal{C}$ be a two-coloring of $K_c = \{v_1, v_2, \ldots, v_c\}$. There exists a monochromatic (say, red) C_n in $\mathcal{C}$. Let $X = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices of this particular C_n

and $Y = V(K_c) - X$. Note that $|Y| = \frac{n}{2} - 1 + k = \frac{3n}{4}$. In $\mathcal{C}$, consider the 2-coloring of $K_{X,Y}$. From each vertex in X to vertex set Y, there are at most $k-1$ red edges. Otherwise, a red C_n^k exists and the theorem is established. Hence, there are at least $\frac{n}{2}$ blue edges from each vertex in X to vertex set Y. Thus, there are at least $\frac{n^2}{2}$ blue edges in $K_{X,Y}$.

Using a result by Li and Ning ([11], see Theorem E in the Appendix), since $\frac{n^2}{2} > (\frac{n}{2} - 1)(n-1) + \frac{3n}{4}$, there exists a blue C_n with vertex set X' in X and vertex set Y' in Y. Let $X'' = X \backslash X'$ and $Y'' = Y \backslash Y'$. Note that $|X''| = |X'| = |Y'| = \frac{n}{2}$ and $|Y''| = k - 1 = \frac{n}{4}$.

Each vertex in Y' is blue adjacent to at most $k-1 = \frac{n}{4}$ vertices in X''. Otherwise, a blue C_n^k exists and the theorem is proved. Thus, each vertex in Y' is red adjacent to at least $\frac{n}{4}$ vertices in X''. So, a minimum of $\frac{n}{2} \cdot \frac{n}{4} = \frac{n^2}{8}$ red edges exist in $K_{X'',Y'}$. Also, there are at least $\frac{n}{2} \cdot \frac{n}{4}$ blue edges. If this was not the case, then there are at most $\frac{n^2}{8} - 1$ blue edges (implying at least $\frac{n^2}{8} + 1$ red edges). This results in a red C_n^k and the theorem is proved.

Since $|K_{X'',Y'}| = \frac{n^2}{4}$, this implies that each vertex in Y' has exactly $\frac{n}{4}$ red edges and exactly $\frac{n}{4}$ blue edges to vertex set X''. Furthermore, because each vertex in X is red adjacent to at most $\frac{n}{4}$ vertices in Y, this implies that $K_{X'',Y''}$ is blue.

(Lifting Process). We now assert that $\mathcal{C}$ contains another blue C_n with $\frac{n}{2} - 2$ vertices in X', two vertices in X'', $\frac{n}{2} - 1$ vertices in Y' and one vertex in Y''. This can be seen from the following:

- There is a blue P_{n-3} in $K_{X',Y'}$ with $\frac{n}{2} - 2$ vertices in X' and $\frac{n}{2} - 1$ vertices in Y'.
- Each vertex in Y' has exactly $\frac{n}{4}$ blue edges to vertex set X''. Each vertex in X'' has exactly $\frac{n}{4}$ blue edges to vertex set Y'.
- There is a blue P_3 in $K_{X'',Y''}$ with two vertices in X'' and one vertex in Y''.

In the case where $n \geq 12$, there is a blue C_n^k (since $\frac{n}{2} - 2 \geq k$) and the theorem is established.

To complete the proof of the theorem, we must analyze the case where $n = 8$ (and $k = \frac{n}{4} + 1 = 3$). Let $Y'' = \{v_r, v_s\}$. Here, we only have the existence of a blue C_n^{k-1} ($= C_8^2$) so far. Furthermore after applying the Lifting Process once, v_r (WLOG) has two red neighbors (not contained in the lifted blue C_8) in X' and one red neighbor in Y' (not contained in the lifted blue C_8). Otherwise, a blue C_8^3 exists and the theorem is proved. Note that the original blue C_8 (containing the four vertices from X' and four vertices from Y') contains four blue P_5s, where the end-vertices are in Y'. By applying the Lifting Process repeatedly (using each of these four blue P_5s), we can conclude that v_r is red adjacent to all the vertices in X' and Y'. An identical argument shows that v_s is also red adjacent to all the vertices in X' and Y'.

Now, consider the red $C_8 = \{v_1, v_2, \ldots, v_8\}$, where $V(C_8) = X$. Without loss of generality, let $X' = \{v_1, v_2, v_3, v_4\}$ and $X'' = \{v_5, v_6, v_7, v_8\}$.

Case 1. If there is a red $P_3 = v_x \overline{v} v_z$ with endpoints in X', then the red C_8 can be lifted to another red C_8 which contains one vertex in Y'' and the vertices of the original red C_8 except $\overline{v}$. From this, a red C_8^3 exists and the theorem is proved.

Case 2. Without loss of generality, if edge v_1v_2 is red, then edges v_1v_5 and v_2v_6 are red, since there is no red P_3 in X'. If in addition, v_3v_6 is red, then the original red C_8 can be lifted to another red C_8 which contains one vertex in Y'' and vertices $v_1, v_2, v_3, v_4, v_5, v_7$ and v_8. From this, a red C_8^3 exists and the theorem is proved. So thus far, we have that edges v_1v_5, v_2v_6 and v_6v_7 are red. Furthermore, edge v_5v_8 is in the original red C_8. Otherwise there is a similar lifting (as described above) to another red C_8 which leads to a red C_8^3, proving the theorem. Since edge v_5v_8 is in the original red C_8, this implies that edge v_3v_4 is in the original red C_8. To summarize up to this point, we have determined that the original red $C_8 = v_1v_2, v_2v_6, v_6v_7, v_7v_3, v_3v_4, v_4v_8, v_8v_5, v_5v_1$. Now in this red C_8, replace the red $P_5 = v_2v_6, v_6v_7, v_7v_3, v_3v_4$ with the red $P_5 = v_2v_r, v_rv_3, v_3v_s, v_sv_4$. This gives a lifting of the original red C_8 to another red C_8 which contains the two vertices of Y'' and vertices v_1, v_2, v_3, v_4, v_5 and v_8. From this, a red C_8^3 exists and the theorem is proved.

Case 3. If edge v_1v_5 is red (WLOG), then edge v_1v_6 is red (WLOG). Otherwise, we would be in Case 2 and hence, done. Furthermore, edges v_5v_7 and v_6v_8 are edges of the original red C_8. Otherwise, we would be in Case 1 and hence, done. So this implies that v_2v_3 and v_3v_4 are edges in the original red C_8. This is a red P_3 with endpoints in X'. We are now back in Case 1 and the theorem is proved. □

Theorem 3.13. *Let $n \geq 6$ be even and $1 \leq k \leq \lceil \frac{n+1}{4} \rceil$. Then, $R(C_n^k) = \frac{3n}{2} - 1 + k$.*

Proof. This follows immediately from Theorems 3.9, 3.10 and 3.12. □

4 Directions for further research

Conjecture 4.1. *Let $k \geq 2$ be even. Then, $R(C_6^k) = 2k + 6$.*

Conjecture 4.2. *Let $n \geq 6$ be even and $k \in \mathbb{N}$. Then,*

$$R(C_n^k) = \begin{cases} R(K_{1,k+\frac{n}{2}}) & \text{if } k \geq \frac{n}{2}, \\ R(C_n) + k & \text{if } 1 \leq k \leq \frac{n}{2} - 1. \end{cases}$$

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Appendix

Some auxiliary concepts and theorems were used in this paper. For the sake of completeness, we include them in this section.

Theorem A is used in the proof of Theorem 3.2.

Theorem A. (*Köhler [9]*). *Let G be an odd cycle with $k \geq 1$ pendant edges at a single vertex of G. Then, $R(G) = 2 \cdot |V(G)| - 1$.*

Theorems B and C are used in the proofs of Lemmas 3.4 and 3.5.

Theorem B. (*Harary [7]*). *$R(K_{1,n}, K_{1,m}) = n + m - \epsilon$, where $\epsilon = 1$ for even n and m, and $\epsilon = 0$ otherwise.*

Definition. Let G and H be simple connected bipartite graphs. The *bipartite Ramsey number* $BR(G, H)$ is the minimum n, where every 2-coloring of $K_{n,n}$ contains a monochromatic red G or a monochromatic blue H.

Theorem C. (*Zhang, Sun and Wu [25]*). *$BR(C_6, C_6) = 6$.*

Theorem D is alluded to in the statements of Theorem 3.9 and Conjecture 4.2.

Theorem D. (*Rosta [19]*). *Let $n \geq 6$ be even. Then, $R(C_n) = \frac{3n}{2} - 1$.*

Theorem E is used in the proof of Theorem 3.10.

Theorem E. (*Li and Ning [11]*). *Let $t \geq 1$ and G be a bipartite graph with vertex partitions X and Y, where $|X| = m$ and $|Y| = n$. Suppose that $n \geq m$ and $t \leq m \leq 2t - 2$. If $|E(G)| > (t - 1)(n - 1) + m$, then G contains a cycle of length $2t$.*

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