

Full classification of anti-van der Waerden numbers of graph products of forests

ZHANAR BERIKKYZY

Mathematics Department
Fairfield University, Fairfield, CT, U.S.A.
zberikkyzy@fairfield.edu

JOE MILLER

Department of Mathematics
Iowa State University, Ames, IA, U.S.A.
jmillero@iastate.edu

NATHAN WARNBERG

Department of Mathematics and Statistics
University of Wisconsin-La Crosse, La Crosse, WI, U.S.A.
nwarnberg@uwlax.edu

Abstract

The anti-van der Waerden number of a graph G is the fewest number of colors needed to guarantee a rainbow 3-term arithmetic progression in any vertex coloring of G , denoted $\text{aw}(G, 3)$. It is known that the anti-van der Waerden number of graph products is $3 \leq \text{aw}(G \square H, 3) \leq 4$. Previous work has been done on classifying families of graph products into $\text{aw}(G \square H, 3) = 3$ and $\text{aw}(G \square H, 3) = 4$. Some of these families include the product of two paths, the product of paths and cycles, the product of two cycles, and the product of odd cycles with any graph. Recently, a partial characterization of the product of two trees was established. This paper completes the characterization for $\text{aw}(T \square T', 3)$ where T and T' are trees. Moreover, this result extends to a full classification of products of forests.

1 Introduction

Ramsey Theory is a branch of mathematics that assigns colors to elements of a set and then determines whether monochromatic substructures exist within the set, whereas

Anti-Ramsey Theory determines whether rainbow (polychromatic) substructures exist within the set. Ramsey Theory has a long history dating back to around 1920 with Schur, Ramsey and van der Waerden making the earliest contributions (see [12, 14, 16]) where the sets considered were integers or graph edges. It was not until 1973 when Erdős, Simonovits, and Sós, in [8], introduced the idea of Anti-Ramsey Theory. Thirty years later, Jungić et al started investigating anti-van der Waerden problems where the sets being colored were $\{1, 2, \dots, n\} = [n]$ and $\mathbb{Z}_n$ and the rainbow substructures were 3-term arithmetic progressions (see [9, 10]). These papers focused on the conditions on the sizes of the color classes that guarantee rainbow arithmetic progressions. The anti-van der Waerden number was first defined in [15] by Uherka in 2013. The question asked was: given a fixed value of n , what is the fewest number of colors needed to guarantee a rainbow k -term arithmetic progression in $[n]$ or $\mathbb{Z}_n$, denoted $\text{aw}([n], k)$ and $\text{aw}(\mathbb{Z}_n, k)$, respectively? In [7], Butler et al. bounded $\text{aw}(\mathbb{Z}_n, 3)$ based on the prime factorization of n and found a logarithmic bound for $\text{aw}([n], 3)$. The exact values of $\text{aw}([n], 3)$ were then determined by Berikkyzy, Schulte and Young in [5]. Young, in [18], determined the anti-van der Waerden numbers for finite abelian groups based on the order of the group. Concurrently, authors noted that a 3-term arithmetic progression $(a_1, a_1 + d, a_1 + 2d)$ satisfies the equation $x_1 + x_3 = 2x_2$ and anti-van der Waerden numbers for Sidon sets and other linear equations were investigated (see [1], [2], [6]).

Investigations then turned to arithmetic progressions in graphs as it was observed that the path P_n behaves like $[n]$ and the cycle C_n behaves like $\mathbb{Z}_n$.

Given a graph G , an *exact r -coloring of G* is a surjective function $c : V(G) \rightarrow [1, \dots, r]$. An *arithmetic progression* in G of length k (k -AP) with common difference d is a list of vertices $(v_1, \dots, v_k)$ such that $d(v_i, v_{i+1}) = d$ for $1 \leq i < k$. An arithmetic progression is *rainbow* if all of the vertices are colored distinctly. The fewest number of colors that guarantees a rainbow k -AP is called the *anti-van der Waerden number of G* and is denoted $\text{aw}(G, k)$. To show $r \leq \text{aw}(G, k)$ we construct an exact $(r - 1)$ -coloring that avoids rainbow k -APs. To show $\text{aw}(G, k) \leq r$, we show that every exact r -coloring gives a rainbow k -AP.

The definition of $\text{aw}(G, k)$ was introduced in [13] and provided the first results about Cartesian graph products. In particular, Theorems 1.1 and 1.2 have proven to be essential in most results about $\text{aw}(G \square H, k)$.

Theorem 1.1 ([13]). *For $m, n \geq 2$,*

$$\text{aw}(P_m \square P_n, 3) = \begin{cases} 3 & \text{if } m = 2 \text{ and } n \text{ is even, or } m = 3 \text{ and } n \text{ is odd,} \\ 4 & \text{otherwise.} \end{cases}$$

A subgraph H of G is *isometric* if for every $u, v \in V(H)$ we have $d_H(u, v) = d_G(u, v)$. Theorem 1.1 is used when we find an isometric subgraph $P_m \square P_n$ within $G \square H$ to either show that our coloring must have a rainbow within the isometric subgraph or that we know we can color the isometric subgraph in some way and avoid rainbows. This result was used extensively when determining $\text{aw}(P_m \square C_n, 3)$ in [11].

Theorem 1.2 ([13]). *If G and H are connected graphs and $|G|, |H| \geq 2$, then $\text{aw}(G \square H, 3) \leq 4$.*

Authors investigated $\text{aw}(G, k)$ on trees and graphs with small diameter in [3], graph products of paths and cycles in [11], and graph products of trees T and T' where $\text{diam}(T \square T')$ is odd in [4].

This paper completes the classification of $\text{aw}(T \square T', 3)$ by considering when $\text{diam}(T \square T')$ is even. The following is the main result of this paper.

Theorem 1.3. *Let T and T' be nontrivial trees, where $\text{diam}(T \square T')$ is even. Then*

$$\text{aw}(T \square T', 3) = \begin{cases} 3 & \text{if } T \text{ or } T' \text{ is weakly non-3-peripheral or isomorphic to } P_2, \\ 4 & \text{if } T \text{ and } T' \text{ are both strongly non-3-peripheral and not } P_2. \end{cases}$$

We summarize the full classification of anti-van der Waerden numbers of graph products of trees in the following corollary.

Corollary 1.4. *Let T and T' be trees. Then, $\text{aw}(T \square T', 3) =$*

$$\begin{cases} 3 & \text{if } T \text{ or } T' \text{ is 3-peripheral, or} \\ & \text{if } \text{diam}(T \square T') \text{ is even and either } T \text{ or } T' \text{ is } P_2, \text{ or} \\ & \text{if } \text{diam}(T \square T') \text{ is even and either } T \text{ or } T' \text{ is weakly non-3-peripheral;} \\ 4 & \text{otherwise.} \end{cases}$$

The paper is organized as follows. In Section 2, definitions, notation and conventions are established along with several results that support the main theorem of the paper. Section 3 provides a case-analysis for $\text{aw}(T \square T', 3)$ based on properties of T and T' , e.g. whether the trees are strongly or weakly 3-peripheral (which we define in Section 2), and concludes with the main result.

2 Preliminary Results

This section introduces the tools needed to prove the main result. We begin with basic definitions and known results.

If $G = (V, E)$ and $H = (V', E')$ then the *Cartesian product*, written $G \square H$, has vertex set $\{(x, y) : x \in V \text{ and } y \in V'\}$ and (x, y) and (x', y') are adjacent in $G \square H$ if either $x = x'$ and $yy' \in E'$ or $y = y'$ and $xx' \in E$. This paper will use the convention that if

$$V(G) = \{u_1, \dots, u_{n_1}\} \quad \text{and} \quad V(H) = \{w_1, \dots, w_{n_2}\},$$

then $V(G \square H) = \{v_{1,1}, \dots, v_{n_1,n_2}\}$ where $v_{i,j}$ corresponds to the vertices $u_i \in V(G)$ and $w_j \in V(H)$. Also, if $1 \leq i \leq n_2$, then G_i denotes the i th labeled copy of G in $G \square H$. Likewise, if $1 \leq j \leq n_1$, then H_j denotes the j th labeled copy of H in $G \square H$. In other words, G_i is the induced subgraph $G_i = G \square H[\{v_{1,i}, \dots, v_{n_2,i}\}]$, and H_j is

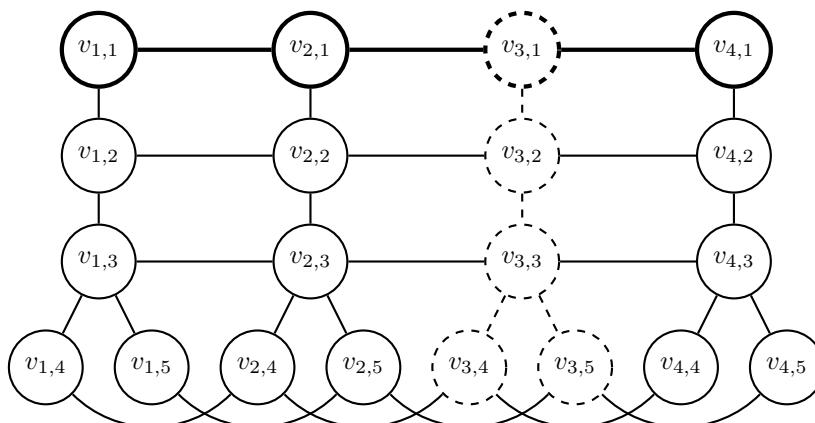


Figure 1: The product of $G = P_4$ and broom graph H . The subgraph G_1 is bolded and the subgraph H_3 is dashed. Graph H is an example of a weakly non-3-peripheral tree of odd diameter (see Definition 2.17) while the path G is an example of a strongly non-3-peripheral graph of odd diameter.

the induced subgraph $H_j = G \square H[\{v_{j,1}, \dots, v_{j,n_1}\}]$. Notice that the i subscript in G_i corresponds to the i th vertex of H and the j in the subscript in H_j corresponds to the j th vertex of G . See Figure 1 below for an example where $G = P_4$ and H is a broom graph.

Proposition 2.1 ([11]). *If $v_{i,j}, v_{h,k} \in V(G \square H)$, then*

$$d_{G \square H}(v_{i,j}, v_{h,k}) = d_G(u_i, u_h) + d_H(w_j, w_k).$$

Note that if G and H are graphs, then a direct consequence of Proposition 2.1 is that

$$\text{diam}(G \square H) = \text{diam}(G) + \text{diam}(H).$$

Corollary 2.2 ([11]). *If G' is an isometric subgraph of G and H' is an isometric subgraph of H , then $G' \square H'$ is an isometric subgraph of $G \square H$.*

Corollary 2.2 is particularly useful, as it is often used to find a rainbow structure in the graph product by only considering a small subgraph that preserves distances. Lemma 2.3 takes this a step further when we have exact 3-colorings. Since we are applying the lemma to products of trees, we know that C_3 subgraphs do not exist so an isometric path exists that has all three colors. In practice, we take a shortest such path which implies that endpoints are uniquely colored and all interior vertices colored the same. This allows us to find structure within an arbitrary coloring.

Lemma 2.3 ([13]). *If G is a connected graph on at least three vertices with an exact r -coloring c where $r \geq 3$, then there exists a subgraph G' in G with at least three colors where G' is either an isometric path or $G' = C_3$.*

Lemma 2.4 ([11]). *If G and H are connected, $|G|, |H| \geq 2$ and c is an exact r -coloring of $G \square H$, $3 \leq r$, that avoids rainbow 3-APs, then $|c(V(G_i))| \leq 2$ for $1 \leq i \leq |H|$.*

The following results are used to derive structural properties of rainbow-free colorings.

Corollary 2.5 ([11]). *If G and H are connected graphs, $|G| \geq 2$, $|H| \geq 3$, c is an exact, rainbow-free r -coloring of $G \square H$ with $r \geq 3$, and $v_i v_j \in E(H)$, then*

$$|c(V(G_i) \cup V(G_j))| \leq 2.$$

Proposition 2.6 ([11]). *If G and H are connected graphs, $|G| \geq 2$, $|H| \geq 3$, c is an exact, rainbow-free r -coloring of $G \square H$ with $r \geq 3$, then there is a color in $c(G \square H)$ that appears in every copy of G .*

The following proposition guarantees existence of a dominating color in a rainbow-free coloring. This observation is helpful in constructing such coloring, when it exists.

Proposition 2.7. *If G and H are connected graphs, $|G| \geq 2$, $|H| \geq 2$, c is an exact, rainbow-free r -coloring of $G \square H$ with $r \geq 3$, then there is a color in $c(G \square H)$ that appears in every copy of G .*

Proof. The case of $|H| \geq 3$ is handled by Proposition 2.6. So, suppose $|H| = 2$, say $V(H) = \{w_1, w_2\}$. Assume G_1 and G_2 do not share a color. The pigeonhole principle implies that some copy of G must have at least two colors, say G_1 has at least two colors. Then, up to relabeling the vertices of G , there must exist adjacent vertices $v_{1,\ell-1}, v_{1,\ell} \in V(G_1)$ such that $c(v_{1,\ell-1}) \neq c(v_{1,\ell})$. Since G_1 and G_2 do not share a color, the 3-AP $(v_{1,\ell-1}, v_{1,\ell}, v_{2,\ell})$ is rainbow, a contradiction. Thus, G_1 and G_2 share a color, as desired. $\square$

Many conditions in this paper are about peripheral vertices, which we define next. For common graph theory terminology, see [17]. For a vertex v in a connected graph G the *eccentricity* of v , denoted $\epsilon(v)$, is the distance between v and a vertex furthest from v in G . If a vertex has minimum eccentricity, we call it a *central vertex*. The *radius* of G , denoted $\text{rad}(G)$, is the eccentricity of any central vertex. The collection of all central vertices in G is the *center* of G . If a vertex v has $\epsilon(v) = \text{diam}(G)$ we call v a *peripheral vertex*. If a graph G contains vertices $u_1, \dots, u_n$ such that $d(u_i, u_j) = \text{diam}(G)$ for all distinct $i, j \in \{1, \dots, n\}$, then we call G *n-peripheral*. A graph is *non-n-peripheral* if we cannot find n vertices that are pairwise diameter away from each other. Specifically, we focus on graphs that are 3-peripheral and graphs that are non-3-peripheral.

Theorem 2.8 ([4]). *If T is a 3-peripheral tree and G is connected with $2 \leq |G|$, then*

$$\text{aw}(T \square G, 3) = 3.$$

Theorem 2.9 ([4]). *If T and T' are trees which are non-3-peripheral with $|T|, |T'| \geq 2$ and $\text{diam}(T \square T')$ is odd, then $\text{aw}(T \square T', 3) = 4$.*

Corollary 2.10 says that the eccentricity of any vertex in a tree is realized by a peripheral vertex. However, it is stated in a way that is easier to use in practice.

Corollary 2.10 ([4]). *If u is not a peripheral vertex of T and $v \in V(T)$, then there exists a vertex $w \in V(T)$ such that $d(w, v) = d(u, v) + 1$.*

Theorem 2.11 (Jordan 1869). *The center of a tree consists of one vertex or two vertices.*

In general, the center of a graph need not be connected, and in fact, components of the center can be arbitrarily far apart. Consider, for example, a C_6 with a leaf on every other vertex. For trees, this is not the case. We include the following useful observation and its proof for completeness.

Observation 2.12. *All diameter paths in a tree T intersect every central vertex of T . In particular, all diameter paths in a tree T intersect.*

Proof. Suppose we have a diameter path P with peripheral vertices u and v . Define C_T to be the center of T and C_P to be the center of P as a subgraph of T . Let $w \in C_P$ and assume $w \notin C_T$. Since $\text{rad}(T) \in \left\{ \frac{\text{diam}(T)}{2}, \frac{\text{diam}(T)+1}{2} \right\}$, there exists some $x \in V(T)$ such that $d(w, x) > \frac{\text{diam}(T)+1}{2}$. But now, either $d(u, x) = d(u, w) + d(w, x)$ or $d(v, x) = d(v, w) + d(w, x)$. Since $d(v, w), d(u, w) \geq \frac{\text{diam}(T)-1}{2}$, either case gives a distance larger than the diameter, a contradiction.

Thus, $C_P \subseteq C_T$. Theorem 2.11 implies the center of our tree is one or two vertices depending on the parity of the diameter. This means $|C_P| = |C_T|$ in either case. So, $C_P = C_T$, our desired result. $\square$

The structure given by Observation 2.12 allows us to give the proof of the following straightforward result.

Lemma 2.13. *Suppose T is a tree with $u, v \in V(T)$ which realize the diameter. If x is a peripheral vertex of T , then x is diameter away from u or diameter away from v .*

Proof. If x is diameter away from u or v we are done so assume $d(x, u) \neq \text{diam}(T)$ and $d(x, v) \neq \text{diam}(T)$. Since x is peripheral, there exists some $y \notin \{u, v\}$ such that $d(x, y) = \text{diam}(T)$. Define P and P' to be the $u-v$ and $x-y$ paths in T , respectively. By Observation 2.12 we know that P and P' intersect so define s and t to be the vertices in $V(P) \cap V(P')$ closest to x and y , respectively.

Without loss of generality, suppose $d(u, s) \leq d(u, t)$ (see Figure 2). If $d(x, s) < d(u, s)$, then a contradiction follows from

$$\begin{aligned} \text{diam}(T) &= d(x, y) \\ &= d(x, s) + d(s, t) + d(t, y) \\ &< d(u, s) + d(s, t) + d(t, y) \\ &= d(u, y). \end{aligned}$$

Similarly, if $d(u, s) < d(x, s)$ we get the contradiction that $\text{diam}(T) < d(x, v)$. Thus, $d(u, s) = d(x, s)$ which implies that $d(x, v) = \text{diam}(T)$. $\square$

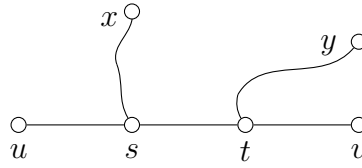


Figure 2: Relationship between peripheral vertices u, x, y and v when $d(u, s) \leq d(u, t)$, as in Lemma 2.13.

Lemma 2.14 ([4]). *Suppose T is a non-3-peripheral tree with and $u_i, u_j \in V(T)$ realize the diameter of T . If there exist $u_x, u_y \in V(T)$ such that $d(u_x, u_j) = \text{diam}(T)$ and $d(u_i, u_y) = \text{diam}(T)$, then $d(u_x, u_y) = \text{diam}(T)$.*

Note that the four vertices in Lemma 2.14 need not be distinct to apply the result, a fact that is used regularly in this paper.

Lemma 2.15 ([4]). *If T is 3-peripheral, then $\text{diam}(T)$ is even. Further, for any three vertices that are pairwise distance $\text{diam}(T)$ apart, there is some vertex that is equidistant from all three of them.*

In Section 5 of [4], it was found that when classifying $\text{aw}(T \square T', 3)$ when $\text{diam}(T \square T')$ is even, the partition of trees into 3-peripheral and non-3-peripheral was insufficient. To refine the partition further some new definitions are needed.

If T is a tree with peripheral vertex v , we define T_{v-} to be the tree obtained from T by removing all vertices which realize the diameter of T with v . If u is any vertex of T , we define T_{u+} to be the tree T with an additional leaf adjacent to u .

One motivation for the T_{v-} definition is that when v is peripheral we change the parity of the diameter of T . In particular, we want to use Lemma 2.14 with leaves that are $\text{diam}(T) - 1$ away from each other. By moving to the subgraph T_{v-} , Observation 2.16 allows us to apply the desired lemma.

Observation 2.16. *For any peripheral vertex v in a tree T , we have*

$$\text{diam}(T_{v-}) = \text{diam}(T) - 1.$$

Proof. Since $\epsilon_{T_{v-}}(v) = \text{diam}(T) - 1$, we certainly have $\text{diam}(T_{v-}) \geq \text{diam}(T) - 1$. To show the other inequality, we will show that all diameter paths in T lose a vertex in T_{v-} . If x and y realize the diameter of T , Lemma 2.13 implies that either x or y is diameter away from v meaning that one of them will not appear in T_{v-} . Thus, T_{v-} has no geodesics of length $\text{diam}(T)$, our desired result. $\square$

Since T_{v-} is an isometric subgraph of T and T is an isometric subgraph of T_{v+} , we will often use the notation $d_T(x, y)$ interchangeably with $d_{T_{v-}}(x, y)$ or $d_{T_{v+}}(x, y)$ since these quantities are equal provided x and y are in each of the necessary trees.

Definition 2.17. Let T be a nontrivial tree.

- (i) Let T be non-3-peripheral with odd diameter. We say T is *strongly non-3-peripheral* if there exists a peripheral vertex v such that T_{v-} is non-3-peripheral. Otherwise, we say T is *weakly non-3-peripheral*. That is, for all peripheral vertices v of T , T_{v-} is 3-peripheral.
- (ii) Let T be non-3-peripheral with even diameter. We say T is *strongly non-3-peripheral* if for all $v \in V(T)$, T_{v+} is non-3-peripheral. Otherwise, we say it is *weakly non-3-peripheral*. That is, there exists some $v \in V(T)$ such that T_{v+} is 3-peripheral.

Example 2.18. Recall from Figure 1 that the graph H is weakly non-3-peripheral with odd diameter. Using the same vertex labeling as the figure, this can be seen because T_{v_1-} is 3-peripheral. However, P_4 is strongly non-3-peripheral of odd diameter because the removal of any peripheral vertex yields a P_3 which is not 3-peripheral.

An example of a weakly non-3-peripheral tree with even diameter is P_3 . This can be seen because if c is the central vertex of P_3 , then T_{c+} is isomorphic to the star $K_{1,3}$ which is 3-peripheral. However, P_5 is strongly non-3-peripheral with even diameter via Lemma 2.19. Specifically, P_5 has no vertex which is $\text{diam}(P_5) - 1$ away from both its peripheral vertices.

Recall that Observation 2.16 states that the T_{v-} operation lowers the diameter by 1 when applied to a peripheral vertex. This is important because we would like to be able to achieve a 3-peripheral graph with this. Since this can only be done if T has even diameter (as seen in Lemma 2.15), it is important to alter the diameter in some fashion. However, when using the T_{v+} operation to an even diameter vertex, it is important that we do not change the diameter. Since T_{v+} only changes the diameter when applied to a peripheral vertex, we never use it on a peripheral vertex. As we will find in Lemma 2.19, it is useful to apply T_{v+} to a vertex whose eccentricity is one less than the diameter of our tree.

Lemma 2.19. *If T is a weakly non-3-peripheral tree with even diameter, say T_{u+} is 3-peripheral, then for any peripheral vertex v of T , $d(u, v) = \text{diam}(T) - 1$.*

Proof. Let u' be the added leaf to u in T_{u+} . Since T_{u+} is 3-peripheral while T is not, there exist peripheral vertices v_j and v_k in T which realize the diameter of T with each other and with u' . If v_i and v_j are the only peripheral vertices of T , then we are done. So suppose v is any other peripheral vertex of T . Since T is non-3-peripheral, either $d(v, v_j)$ or $d(v, v_k)$ is less than $\text{diam}(T)$. Without loss of generality, suppose $d(v, v_j) < \text{diam}(T)$. Since u' and v_j realize the diameter in T_{u+} , applying Lemma 2.13 gives $d_{T_{u+}}(v, u') = \text{diam}(T_{u+}) = \text{diam}(T)$. This means $d(v, u) = \text{diam}(T) - 1$. $\square$

3 Strongly and weakly peripheral trees

In this section, we prove our main theorem below which classifies the anti-van der Waerden number of all products of trees when the diameter is even.

Theorem 1.3. Let T and T' be nontrivial trees, where $\text{diam}(T \square T')$ is even. Then

$$\text{aw}(T \square T', 3) = \begin{cases} 3 & \text{if } T \text{ or } T' \text{ is weakly non-3-peripheral or isomorphic to } P_2, \\ 4 & \text{if } T \text{ and } T' \text{ are both strongly non-3-peripheral and not } P_2. \end{cases}$$

Recall that Theorem 1.2 says this number will be either three or four. In [4], this has already been done when one of the trees is 3-peripheral in which we get an anti-van der Waerden number of three. As for the remaining cases, the language developed allows us to succinctly categorize the two different possibilities. If both trees are strongly non-3-peripheral, then the anti-van der Waerden number is 4, and if either tree is weakly non-3-peripheral, then the anti-van der Waerden number is 3. These two cases will be split further into the cases of whether our trees both have odd diameter or both have even diameter. These four cases can be seen in Propositions 3.2, 3.3, 3.5, and 3.6.

The first step in this process is handling when the anti-van der Waerden number is four, which as we stated is when both trees are strongly non-3-peripheral. To achieve this, Theorem 1.2 implies that it suffices to provide a 3-coloring that is rainbow 3-AP free. Lemma 3.1 will provide such coloring, however, this classification differs depending on the parity of the trees' diameters and the argument is separated into Propositions 3.2 and 3.3.

The intuition of why these definitions are important can be seen by exploring past papers. In [3, 4, 7, 11, 13, 18], rainbow-free colorings are often constructed using *red, blue, green* where one color is dominantly used (*green* in Section 3) and a small number of vertices are colored with *red* and *blue*. The intuition is that in order to avoid a rainbow 3-AP, we would like the distance from any *red* vertex to any *blue* vertex to be odd and large. If the distance were even, then it is possible for a path with *red* and *blue* endpoints to have a midpoint colored *green* giving us a rainbow 3-AP of the form $(\text{red}, \text{green}, \text{blue})$, and if the distance were small enough, then it may be possible to construct a rainbow 3-AP of the form $(\text{red}, \text{blue}, \text{green})$ or $(\text{blue}, \text{red}, \text{green})$. Therefore, our rainbow-3AP-free coloring in Lemma 3.1 was constructed by making the distance between any *red* and *blue* vertices as large as possible, i.e. $\text{diam}(T \square T') - 1$ since $\text{diam}(T \square T')$ is even. This was the motivation to turn to subgraphs of T and T' whose diameter is exactly 1 less than diameter of T or diameter of T' . While not obvious, it turns out that the correct way to achieve this is with T_{v-} when the diameter of T is odd and T_{v+} when the diameter of T is even. Observation 2.16 explains that performing the T_{v-} operation on a tree will lower the diameter by 1. If *red* and *blue* vertices have distance $\text{diam}(T \square T') - 1$, then one must be peripheral. Lemma 2.19 now gives a specific and exploitable vertex which is $\text{diam}(T) - 1$ from this *red* or *blue* vertex.

Lemma 3.1. *Suppose T, T' are nontrivial trees which are non-3-peripheral such that $\text{diam}(T \square T')$ is even. Suppose $v_{1,1}$ and $v_{j,k}$ realize the diameter of $T \square T'$ such that $T_{u_1^-}$ and $T'_{w_1^-}$ are non-3-peripheral. Define $c : V(T \square T') \rightarrow \{\text{red}, \text{blue}, \text{green}\}$ as follows*

$$c(v_{a,b}) = \begin{cases} \text{blue} & \text{if } d(v_{a,b}, v_{1,1}) = \text{diam}(T \square T') - 1, \\ \text{red} & \text{if } d(v_{a,b}, v_{j,k}) = \text{diam}(T \square T'), \\ \text{green} & \text{otherwise.} \end{cases}$$

Then we have the following:

- (i) c is well-defined;
- (ii) if $c(x) = \text{red}$, then $c(y) = \text{blue}$ if and only if $d(x, y) = \text{diam}(T \square T') - 1$;
- (iii) if (x, y, z) is a rainbow 3-AP, then $c(y) = \text{blue}$.

Proof. To show (i), note that by the definition of c , no vertex will be *blue* and *green*, and no vertex will be *red* and *green*. Thus, we only need to check if a vertex will be colored both *red* and *blue*. For the sake of contradiction, assume $v_{a,b}$ is such a vertex, that is, $d(v_{a,b}, v_{1,1}) = \text{diam}(T \square T') - 1$ and $d(v_{a,b}, v_{j,k}) = \text{diam}(T \square T')$. Since $d(v_{a,b}, v_{1,1}) = \text{diam}(T \square T') - 1$, either $d_T(u_a, u_1) = \text{diam}(T)$ or $d_{T'}(w_b, w_1) = \text{diam}(T')$. Further, $d(v_{a,b}, v_{j,k}) = \text{diam}(T \square T')$ implies $d_T(u_a, u_j) = \text{diam}(T)$ and $d_{T'}(w_b, w_k) = \text{diam}(T')$. In either case, either T or T' is 3-peripheral, a contradiction. Thus, c is well-defined.

To prove the forward direction of (ii), suppose $c(x) = \text{red}$ with $x = v_{x_1, x_2}$ and $c(y) = \text{blue}$ with $y = v_{y_1, y_2}$. Since $d(v_{y_1, y_2}, v_{1,1}) = \text{diam}(T \square T') - 1$, it follows that $d_T(u_{y_1}, u_1) = \text{diam}(T)$ and $d_{T'}(w_1, w_{y_2}) = \text{diam}(T') - 1$, or that $d_T(u_{y_1}, u_1) = \text{diam}(T) - 1$ and $d_{T'}(w_1, w_{y_2}) = \text{diam}(T')$.

First, suppose that

$$d_T(u_{y_1}, u_1) = \text{diam}(T) \text{ and } d_{T'}(w_1, w_{y_2}) = \text{diam}(T') - 1.$$

Since $d(v_{x_1, x_2}, v_{j,k}) = \text{diam}(T \square T')$, it follows that $d_T(u_{x_1}, u_j) = \text{diam}(T)$ and $d_{T'}(w_{x_2}, w_k) = \text{diam}(T')$. So, Lemma 2.14 implies that $d_T(u_{x_1}, u_{y_1}) = \text{diam}(T)$.

It remains to show $d(w_{x_2}, w_{y_2}) = \text{diam}(T') - 1$. We first rule out some trivial cases. If $w_{x_2} = w_1$, then we immediately have our desired result. Notice $w_{x_2} \neq w_k$ and $w_1 \neq w_k$ since both pairs realize the diameter. That is, we can suppose w_1, w_k and w_{x_2} are distinct. Let w_{k-1} be the unique neighbor of w_k , and note that Observation 2.16 implies that $\text{diam}(T'_{w_1^-}) = \text{diam}(T') - 1 = d_{T'}(w_{x_2}, w_{k-1})$. Additionally, $d_{T'}(w_{x_2}, w_1) \neq \text{diam}(T')$; otherwise w_1, w_k, w_{x_2} are pairwise diameter apart, contradicting that T' is non-3-peripheral. This implies $w_{x_2} \in V(T'_{w_1^-})$. Recall that $d_{T'}(w_1, w_{y_2}) = \text{diam}(T') - 1 = \text{diam}(T'_{w_1^-})$ implying that $w_{y_2} \in V(T'_{w_1^-})$. Since $T'_{w_1^-}$ is non-3-peripheral, Lemma 2.14 can be applied to vertices w_1, w_{k-1}, w_{x_2} and w_{y_2} in $T'_{w_1^-}$ to get

$$d_{T'}(w_{x_2}, w_{y_2}) = d_{T'_{w_1^-}}(w_{x_2}, w_{y_2}) = \text{diam}(T'_{w_1^-}) = \text{diam}(T') - 1.$$

Thus,

$$d(x, y) = d_T(u_{x_1}, u_{y_1}) + d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T) + \text{diam}(T') - 1 = \text{diam}(T \square T') - 1.$$

Second, if we instead suppose

$$d_T(u_{y_1}, u_1) = \text{diam}(T) - 1 \text{ and } d_{T'}(w_1, w_{y_2}) = \text{diam}(T'),$$

then a similar argument shows that u_1, u_j and u_{x_1} are distinct and that applying Lemma 2.14 to w_1, w_k, w_{x_2} and w_{y_2} in T' and to u_1, u_{j-1}, u_{x_1} and u_{y_1} in $T_{u_1}^-$ yields the same result.

To prove the reverse direction of **(ii)**, suppose that $c(x) = \text{red}$ and $d(x, y) = \text{diam}(T \square T') - 1$. Since $c(x) = \text{red}$, we have $d(x, v_{j,k}) = \text{diam}(T \square T')$ implying that $d_T(u_{x_1}, u_j) = \text{diam}(T)$ and $d_{T'}(w_{x_2}, w_k) = \text{diam}(T')$. Since $d(x, y) = \text{diam}(T \square T') - 1$, we have that $d_T(u_{x_1}, u_{y_1}) = \text{diam}(T)$ and $d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T') - 1$ or that $d_T(u_{x_1}, u_{y_1}) = \text{diam}(T) - 1$ and $d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T')$. First, suppose $d_T(u_{x_1}, u_{y_1}) = \text{diam}(T)$ and $d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T') - 1$. Notice that since

$$d_T(u_1, u_j) = d_T(u_{x_1}, u_j) = d_T(u_{x_1}, u_{y_1}) = \text{diam}(T),$$

Lemma 2.14 implies that $d_T(u_1, u_{y_1}) = \text{diam}(T)$.

It remains to show $d(w_1, w_{y_2}) = \text{diam}(T') - 1$. If $w_{x_2} = w_1$, then we immediately have our desired result. Notice $w_{x_2} \neq w_k$ and $w_1 \neq w_k$ since both pairs realize the diameter. Thus we can suppose w_1, w_k and w_{x_2} are distinct. It now follows that $d_{T'}(w_{x_2}, w_{k-1}) = \text{diam}(T') - 1$. Notice that

$$d_{T'}(w_1, w_{k-1}) = d_{T'}(w_{x_2}, w_{k-1}) = d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T') - 1 = \text{diam}(T'_{w_1}).$$

Since T'_{w_1} is non-3-peripheral, we now show $w_{k-1}, w_{x_2}, w_1, w_{y_2} \in V(T'_{w_1})$ so that Lemma 2.14 applied to T'_{w_1} gives $d_{T'}(w_1, w_{y_2}) = \text{diam}(T') - 1$. Since $d_{T'}(w_1, w_k) = \text{diam}(T')$, we have $w_1, w_{k-1} \in V(T'_{w_1})$. If $w_{x_2} \notin V(T'_{w_1})$, then $d(w_1, w_{x_2}) = \text{diam}(T')$ and w_1, w_k and w_{x_2} pairwise realize the diameter, contradicting that T' is non-3-peripheral. If $w_{y_2} \notin V(T'_{w_1})$, then $d_{T'}(w_1, w_{y_2}) = \text{diam}(T')$, then combining Lemma 2.14 and that

$$d_{T'}(w_1, w_{y_2}) = d_{T'}(w_1, w_k) = d_{T'}(w_{x_2}, w_k) = \text{diam}(T')$$

shows that $d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T')$, a contradiction. Thus,

$$d_{T'_{w_1}}(w_1, w_{y_2}) = d_{T'}(w_1, w_{y_2}) = \text{diam}(T') - 1,$$

as desired. Finally,

$$d(v_{1,1}, y) = d_T(u_1, u_{y_1}) + d_{T'}(w_1, w_{y_2}) = \text{diam}(T \square T') - 1,$$

showing that $c(y) = \text{blue}$.

Alternatively, if $d_T(u_{x_1}, u_{y_1}) = \text{diam}(T) - 1$ and $d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T')$, then a similar argument shows that u_1, u_j and u_{x_1} are distinct and that applying Lemma 2.14 to w_1, w_k, w_{x_2} and w_{y_2} in T' and to u_1, u_{j-1}, u_{x_2} and u_{y_2} in $T_{u_1}^-$ yields the same result.

To show **(iii)**, note that because of the symmetry of 3-APs, any rainbow 3-AP can be classified by the color of the middle vertex. Suppose (x, y, z) is a rainbow 3-AP. First, assume $c(y) = \text{green}$. Then x and z are colored *red* and *blue* in some order. Recall that $T \square T'$ is bipartite. Since $\text{diam}(T \square T')$ is even, the forward direction of (ii) implies that x and z are in different partite sets. Thus, no such y can be an equal distance from x and z , contradicting that (x, y, z) is a 3-AP. Second, assume $c(y) = \text{red}$. Since one of x or z is *blue*, the forward direction of (ii) implies the common distance of our 3-AP is $\text{diam}(T \square T') - 1$. But the reverse direction of (ii) implies any vertex distance $\text{diam}(T \square T') - 1$ from y has color *blue*, contradicting that one of x or z is *green*. Thus, $c(y) = \text{blue}$. $\square$

The remainder of this section completes the proof of the main theorem, showing that when $\text{diam}(T \square T')$ is even, we have

$$\text{aw}(T \square T', 3) = \begin{cases} 3 & \text{if } T \text{ or } T' \text{ is weakly non-3-peripheral or isomorphic to } P_2, \\ 4 & \text{if } T \text{ and } T' \text{ are both strongly non-3-peripheral and not } P_2. \end{cases}$$

This will be broken into four cases depending on whether the trees have odd or even diameter and on weakly and strongly non-3-peripheral properties of trees. The case when one of the trees is P_2 is quite different, since $(P_2)_{u-}$ is an isolated vertex. Therefore, we separate this case from the remaining trees in Lemma 3.4.

We provide the motivation for two of the four cases below. First, if T is a strongly non-3-peripheral tree of odd diameter, then for some vertex u , T_{u-} is non-3-peripheral and Lemma 3.1 gives a rainbow-free 3-coloring of $T_{u-} \square T'$. By carefully extending this coloring to $T \square T'$, we can avoid rainbow 3-APs. Applying Theorem 1.2, this means the anti-van der Waerden number is four. Second, if T is a weakly non-3-peripheral tree of even diameter, consider an arbitrary 3-coloring of $T \square T'$. This can be extended to a 3-coloring of $T_{v+} \square T'$ where T_{v+} is 3-peripheral. Then Theorem 2.8 says this coloring admits a 3-AP. If we choose our extension coloring carefully, we can guarantee that this 3-AP is in $T \square T'$, too. The formal proof the authors provide for Proposition 3.6 is a slight variation of this idea. It is based on the fact that any 3-peripheral tree with all its leaves removed will remain 3-peripheral or will be a single vertex. The proof looks for a 3-AP in this subgraph rather than in $T_{v+} \square T'$. That way, we are guaranteed that any 3-AP we find will also appear in the parent graph. While the remaining two cases are less intuitive, they work out as desired (see Propositions 3.3 and 3.5).

We begin showing that if T and T' are strongly non-3-peripheral trees, where neither are P_2 , then their product $T \square T'$ has anti-van der Waerden number 4. Before we begin Proposition 3.2, as a reminder, a tree T with odd diameter is called strongly non-3-peripheral if the tree T_{v-} is non-3-peripheral for every peripheral $v \in V(T)$.

Proposition 3.2. *Suppose T and T' are strongly non-3-peripheral trees with odd diameter of at least 3. Then, $\text{aw}(T \square T', 3) = 4$.*

Proof. Since T and T' are strongly non-3-peripheral, there exist peripheral vertices $u_1 \in V(T)$ and $w_1 \in V(T')$ such that $T_{u_1^-}$ and $T'_{w_1^-}$ are non-3-peripheral. Since u_1 and w_1 are peripheral, there exist $u_j \in V(T)$ and $w_k \in V(T')$ such that $d_T(u_1, u_j) = \text{diam}(T)$ and $d_{T'}(w_1, w_k) = \text{diam}(T')$. Now color $T \square T'$ using the coloring from Lemma 3.1 with $v_{1,1}$ and $v_{j,k}$ playing the same role as in the lemma. If (x, y, z) is a rainbow 3-AP, then Lemma 3.1(ii) and (iii) give that the common difference is $\text{diam}(T \square T') - 1$ and, without loss of generality, that $c(x) = \text{green}$, $c(y) = \text{blue}$ and $c(z) = \text{red}$.

Suppose that $x = v_{x_1, x_2}$ and $y = v_{y_1, y_2}$. Since $c(y) = \text{blue}$, we have that $d(y, v_{1,1}) = \text{diam}(T \square T') - 1$ implying that either

$$d_T(u_1, u_{y_1}) = \text{diam}(T) \text{ and } d_{T'}(w_1, w_{y_2}) = \text{diam}(T') - 1 \quad (1)$$

or

$$d_T(u_1, u_{y_1}) = \text{diam}(T) - 1 \text{ and } d_{T'}(w_1, w_{y_2}) = \text{diam}(T'). \quad (2)$$

Additionally, since $d(x, y) = \text{diam}(T \square T') - 1$, it follows that

$$d_T(u_{x_1}, u_{y_1}) = \text{diam}(T) - 1 \text{ and } d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T') \quad (3)$$

or

$$d_T(u_{x_1}, u_{y_1}) = \text{diam}(T) \text{ and } d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T') - 1. \quad (4)$$

First suppose that Equations from (1) and (3) hold. Recall that $d_{T'}(w_1, w_k) = \text{diam}(T')$. Since $d_{T'}(w_1, w_{y_2}) < \text{diam}(T')$ and w_{y_2} is a peripheral vertex of T' , Lemma 2.13 implies that $d_{T'}(w_{y_2}, w_k) = \text{diam}(T')$. Let w_{k-1} be the neighbor of w_k in T' and notice that w_{k-1}, w_{y_2} and w_1 are vertices in $T'_{w_1^-}$. Additionally, Observation 2.16 implies $\text{diam}(T'_{w_1^-}) = \text{diam}(T') - 1$. Thus,

$$\text{diam}(T'_{w_1^-}) = d_{T'}(w_1, w_{y_2}) = d_{T'}(w_1, w_{k-1}) = d_{T'}(w_{y_2}, w_{k-1}).$$

Furthermore, these three vertices are distinct since otherwise we have that $\text{diam}(T') = 1 < 3$. Thus, $T'_{w_1^-}$ is 3-peripheral, a contradiction.

Second, suppose that Equations (1) and (4) hold. Since $d_T(u_1, u_j) = \text{diam}(T)$, Lemma 2.14 implies that $d_T(u_{x_1}, u_j) = \text{diam}(T)$.

Consider the case where $d_{T'}(w_1, w_{x_2}) = \text{diam}(T')$. Let P and P' represent the $w_1 - w_{y_2}$ and $w_{x_2} - w_{y_2}$ paths, respectively. Let $w_\ell \in V(T')$ be the vertex in $V(P) \cap V(P')$ nearest to w_1 . Then

$$\text{diam}(T') - 1 = d_{T'}(w_1, w_{y_2}) = d_{T'}(w_1, w_\ell) + d_{T'}(w_\ell, w_{y_2}),$$

$$\begin{aligned} \text{diam}(T') - 1 &= d_{T'}(w_{x_2}, w_{y_2}) = d_{T'}(w_{x_2}, w_\ell) + d_{T'}(w_\ell, w_{y_2}), \text{ and} \\ \text{diam}(T') &= d_{T'}(w_1, w_{x_2}) = d_{T'}(w_1, w_\ell) + d_{T'}(w_\ell, w_{x_2}). \end{aligned}$$

The first two equations imply that $d_{T'}(w_1, w_\ell) = d_{T'}(w_{x_2}, w_\ell)$ and the third implies that both distances equal $\text{diam}(T')/2$. However, $\text{diam}(T')$ is odd, a contradiction. Thus, $d_{T'}(w_1, w_{x_2}) < \text{diam}(T')$. It follows that $w_{x_2} \in V(T'_{w_1-})$ and applying Lemma 2.14 to $w_1, w_{y_2}, w_{x_2}, w_{k-1}$ and graph T'_{w_1-} gives $d_{T'}(w_{x_2}, w_{k-1}) = \text{diam}(T') - 1$. If $w_{x_2} = w_k$, then the previous sentence implies $\text{diam}(T')$ is even, a contradiction. This means $w_{x_2} \neq w_k$ so the $w_{x_2} - w_k$ path contains w_{k-1} and we can conclude $d_{T'}(w_{x_2}, w_k) = \text{diam}(T')$. Finally, we have

$$d(x, v_{j,k}) = d_T(u_{x_1}, u_j) + d_{T'}(w_{x_2}, w_k) = \text{diam}(T) + \text{diam}(T') = \text{diam}(T \square T'),$$

showing that $c(x) = \text{red}$, contradicting the assumption that $c(x) = \text{green}$.

The case where the Equations from (2) and (4) hold has an argument similar to the case where Equations (1) and (3) hold. The case where the Equations from (2) and (3) hold has an argument similar to the case where Equations (1) and (4) hold. Since all cases end in a contradiction, c is rainbow-free and $\text{aw}(T \square T', 3) > 3$. Theorem 1.2 now completes the proof. $\square$

Recall that non-3-peripheral tree T with even diameter is strongly non-3-peripheral if T_{v+} is non-3-peripheral for all $v \in V(T)$.

Proposition 3.3. *Suppose T and T' are nontrivial, strongly non-3-peripheral trees with even diameter. Then $\text{aw}(T \square T', 3) = 4$.*

Proof. First notice that Lemmas 2.15 and 2.16 imply that any peripheral vertices $u_1 \in V(T)$ and $w_1 \in V(T')$ have the property that T_{u_1-} and T'_{w_1-} are non-3-peripheral. Let c be the coloring from Lemma 3.1 and let $v_{1,1}, v_{j,k} \in V(T \square T')$ also be defined as in the statement of Lemma 3.1.

For the sake of contradiction, assume (x, y, z) is a rainbow 3-AP. Lemma 3.1(ii) and (iii) now give that the common difference is $\text{diam}(T \square T') - 1$ and, without loss of generality, that $c(x) = \text{green}$, $c(y) = \text{blue}$ and $c(z) = \text{red}$. Suppose $x = v_{x_1, x_2}$ and $y = v_{y_1, y_2}$. Since $c(y) = \text{blue}$, we have $d(y, v_{1,1}) = \text{diam}(T \square T') - 1$ implying that

$$d_T(u_1, u_{y_1}) = \text{diam}(T) \text{ and } d_{T'}(w_1, w_{y_2}) = \text{diam}(T') - 1 \quad (5)$$

or that

$$d_T(u_1, u_{y_1}) = \text{diam}(T) - 1 \text{ and } d_{T'}(w_1, w_{y_2}) = \text{diam}(T'). \quad (6)$$

Additionally, since $d(x, y) = \text{diam}(T \square T') - 1$, it follows that

$$d_T(u_{x_1}, u_{y_1}) = \text{diam}(T) \text{ and } d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T') - 1 \quad (7)$$

or that

$$d_T(u_{x_1}, u_{y_1}) = \text{diam}(T) - 1 \text{ and } d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T'). \quad (8)$$

First, suppose that Equations (5) and (7) hold. Lemma 2.14 applied to u_1, u_{y_1}, u_j and u_{x_1} implies that $d_T(u_{x_1}, u_j) = \text{diam}(T)$.

Assume $d(w_1, w_{x_2}) = \text{diam}(T')$ and recall that $d_{T'}(w_1, w_{y_2}) = \text{diam}(T') - 1$ and $d_{T'}(w_{x_2}, w_{y_2}) = \text{diam}(T') - 1$. Notice that $w_{y_2} \notin \{w_1, w_{x_2}\}$ otherwise $\text{diam}(T')$ is odd, a contradiction. Thus, $T'_{w_{y_2}^+}$ is 3-peripheral, contradicting that T' is strongly non-3-peripheral. This means we must have $d(w_1, w_{x_2}) < \text{diam}(T')$. Let w_{k-1} be the neighbor of w_k in T' . Applying Lemma 2.14 to $w_1, w_{y_2}, w_{x_2}, w_{k-1} \in V(T'_{w_1})$ gives $d_{T'}(w_{x_2}, w_{k-1}) = \text{diam}(T') - 1$. If $w_{x_2} = w_k$, then this means $\text{diam}(T') = 2$. So, T' is a star with at least two leaves. However, no such star is strongly non-3-peripheral, a contradiction. Thus, $w_{x_2} \neq w_k$ so the $w_{x_2} - w_k$ path contains w_{k-1} and we can conclude $d_{T'}(w_{x_2}, w_k) = \text{diam}(T')$. Finally, $d(x, v_{j,k}) = \text{diam}(T \square T')$ so $c(x) = \text{red}$ which contradicts that $c(x) = \text{green}$.

Second, assume that Equations (5) and (8) hold. Recall that $d_{T'}(w_1, w_k) = \text{diam}(T')$. Since $d_{T'}(w_1, w_{y_2}) < \text{diam}(T')$ and w_{y_2} is a peripheral vertex, Lemma 2.13 implies that $d_{T'}(w_{y_2}, w_k) = \text{diam}(T')$. Applying Lemma 2.14 to w_k, w_{y_2}, w_1 and w_{x_2} implies that $d(w_{x_2}, w_1) = \text{diam}(T')$. Note that w_1, w_{x_2} and w_{y_2} are distinct since T' is nontrivial with even diameter. Let P be the $w_1 - w_{x_2}$ path, P' be the $w_{y_2} - w_{x_2}$ path and let $w_\ell \in V(P) \cap V(P')$ be the vertex closest to w_{y_2} . Then

$$\begin{aligned} \text{diam}(T') &= d_{T'}(w_1, w_{x_2}) = d_{T'}(w_1, w_\ell) + d_{T'}(w_\ell, w_{x_2}), \\ \text{diam}(T') &= d_{T'}(w_{x_2}, w_{y_2}) = d_{T'}(w_{x_2}, w_\ell) + d_{T'}(w_\ell, w_{y_2}), \text{ and} \\ \text{diam}(T') - 1 &= d_{T'}(w_1, w_{y_2}) = d_{T'}(w_1, w_\ell) + d_{T'}(w_\ell, w_{y_2}). \end{aligned}$$

The first two equations imply that $d_{T'}(w_1, w_\ell) = d_{T'}(w_\ell, w_{y_2})$ and third imply that both equal $(\text{diam}(T') - 1)/2$, contradicting that T' has even diameter.

The case where Equations (6) and (7) hold is similar to when Equations (5) and (8) hold. The case where Equations (6) and (8) hold is similar to when Equations (5) and (7) hold. All cases end in a contradiction so c is rainbow-free and $\text{aw}(T \square T', 3) > 3$. Applying Theorem 1.2 completes the proof. $\square$

The remaining results in this section consider conditions on tree products that give $\text{aw}(T \square T', 3) = 3$. This is done by taking an arbitrary 3-coloring and guaranteeing a rainbow 3-AP. We begin with the smallest case when one of the trees is P_2 , as it behaves quite differently from other trees.

Lemma 3.4. *If T is a tree with odd diameter, then $\text{aw}(P_2 \square T, 3) = 3$.*

Proof. Assume, for the sake of contradiction, that c is an exact rainbow-free 3-coloring of $P_2 \square T$. By Lemma 2.3 we can find either an isometric path or a C_3 that contains all three colors. No such C_3 exists since $P_2 \square T$ is bipartite, so let P be a shortest isometric path that contains all three colors. Further, denote the first vertex of the path by $v_{a,b}$ and the last vertex of the path by $v_{c,d}$ with $a \leq c$ and $b \leq d$.

Without loss of generality, assume that $c(v_{a,b}) = \text{red}$, $c(v_{c,d}) = \text{blue}$ and note that every other vertex on the path is green. Note that $a \neq c$ else we contradict Lemma 2.4 or the minimality of P . Also, P has odd length of at least three, else we can find a rainbow 3-AP. Since P_2 only has two vertices which are u_a and u_c , for the remainder of the proof we will use $u_1 = u_a$ and $u_2 = u_c$. Note that the path $v_{1,b} - v_{2,b} - v_{2,d}$ and the path $v_{1,b} - v_{1,d} - v_{2,d}$ are both of shortest length and both have interior vertices that are all green, thus *green* is in both T_1 and T_2 .

Since $d_{P_2}(u_1, u_2) = 1$, it follows that $d_T(w_b, w_d) = d(v_{1,b}, v_{2,d}) - 1$ which is even. Additionally, $\text{diam}(T)$ is odd, so we have $d_T(w_b, w_d) < \text{diam}(T)$. Without loss of generality, assume that w_b is peripheral or w_d is not peripheral. If w_b is peripheral, then there is a geodesic $(w_b, w_{b_1}, w_{b_2}, \dots, w_{b_{\text{diam}(T)}})$ such that $d(w_b, w_{b_i}) = i$ for each $1 \leq i \leq \text{diam}(T)$. Since $d_T(w_b, w_d) < \text{diam}(T)$, we must be able to find some $w_\ell \in T$ for which $d_T(w_b, w_\ell) = d_T(w_b, w_d) + 1$. If w_d is not peripheral, then by Corollary 2.10, there exists some $w_\ell \in V(T)$ such that $d_T(w_b, w_\ell) = d_T(w_b, w_d) + 1$. Consider $v_{2,\ell-1}$ where $w_{\ell-1}$ is the unique neighbor of w_ℓ on the $w_b - w_\ell$ path. If $w_d = w_{\ell-1}$, then we have $c(v_{2,\ell-1}) = \text{blue}$. If not, then the 3-AP $(v_{2,d}, v_{1,b}, v_{2,\ell-1})$ implies $c(v_{c,\ell-1}) \neq \text{green}$ and Lemma 2.4 implies that $c(v_{2,\ell-1}) \neq \text{red}$ since $\text{blue}, \text{green} \in c(V(T_2))$. So, $c(v_{2,\ell-1}) = \text{blue}$. Now, the $v_{1,b} - v_{2,\ell}$ path contains all three colors and is a subgraph of some $P_2 \square P_n$ where n is even because $d(w_b, w_\ell) = d(w_b, w_d) + 1$ is odd. This $P_2 \square P_n$ subgraph is isometric, so Theorem 1.1 implies $P_2 \square T$ has a rainbow 3-AP, a contradiction. $\square$

Recall that a tree T with odd diameter is weakly non-3-peripheral if for every peripheral vertex v , T_v^- is 3-peripheral.

Proposition 3.5. *Suppose T is a weakly non-3-peripheral tree with odd diameter. Then $\text{aw}(T \square T', 3) = 3$ for any nontrivial tree T' with odd diameter.*

Proof. Assume, for a contradiction, that c is a rainbow-free, exact 3-coloring of $T \square T'$ using the colors *red*, *blue*, and *green*. By Lemma 2.3, there is either an isometric C_3 or isometric path containing all three colors. Since $T \square T'$ is bipartite, it has no C_3 subgraph so there must be an isometric path containing all three colors. Let P be a shortest such path. Specifically, say P is a $v_{a,b} - v_{c,d}$ path where $v_{a,b}$ is *red*, $v_{c,d}$ is *blue* and all others are *green* and, without loss of generality, $a \leq c$ and $b \leq d$. Note that P must have odd length; otherwise there is a rainbow 3-AP. Note that $b \neq d$ and $a \neq c$ otherwise $T_b = T_d$ and $T'_a = T'_c$, respectively, contain all three colors contradicting Lemma 2.4. Define $u_{c-1} \in V(T)$ as the unique neighbor of u_c on the $u_a - u_c$ path. Note that the path following $v_{a,b} - v_{a,d} - v_{c,d}$ is a $v_{a,b} - v_{c,d}$ geodesic. So, $c(v_{i,d}) = \text{green}$ for $a \leq i \leq c - 1$ otherwise we can construct a path containing all three colors which is shorter than P . In particular, this shows *green* is in every copy of T' which intersects P except for possibly T'_c .

First consider the case where $d_T(u_a, u_c) < \text{diam}(T)$. Then there exists some $u_i \in V(T)$ such that P is contained in $T_{u_i}^- \square T'$. In particular, if either u_a or u_c is peripheral, then choose that to be u_i ; otherwise, if neither is peripheral, any peripheral u_i suffices. Notice that $T_{u_i}^- \square T'$ is an isometric subgraph of $T \square T'$ and

contains three colors. Since T is weakly non-3-peripheral, $T_{u_i^-}$ is 3-peripheral. By Theorem 2.8, there is a rainbow 3-AP in $T_{u_i^-} \square T'$. Since $T_{u_i^-} \square T'$ is an isometric subgraph, this contradicts that c is a rainbow-free coloring.

Now consider when $d_T(u_a, u_c) = \text{diam}(T)$. Since the length of P is odd, it must be less than $\text{diam}(T \square T')$. So, $d_{T'}(w_b, w_d) < \text{diam}(T')$. Without loss of generality, assume that w_b is peripheral or w_d is not peripheral. If w_b is peripheral, then there exists some $w_\ell \in V(T')$ such that $d_{T'}(w_b, w_\ell) = d_{T'}(w_b, w_d) + 1$. If w_d is not peripheral, then by Corollary 2.10, there exists some $w_\ell \in V(T')$ such that $d_{T'}(w_b, w_\ell) = d_{T'}(w_b, w_d) + 1$. Define P' to be a $v_{a,b} - v_{c-1,\ell}$ geodesic in $T \square T'$ and note that P and P' have the same length. We will show $c(v_{c-1,\ell}) = \text{blue}$ and all interior vertices of P' are colored *green* by c . If $c(v_{c-1,\ell}) = \text{green}$, then $(v_{c,d}, v_{a,b}, v_{c-1,\ell})$ is a rainbow 3-AP, a contradiction. If $c(v_{c-1,\ell}) = \text{red}$, then since $\text{green} \in c(V(T'_{c-1}))$, we have $\text{green}, \text{red}, \text{blue} \in c(V(T'_{c-1}) \cup V(T'_c))$, contradicting Corollary 2.5. Thus, $c(v_{c-1,\ell}) = \text{blue}$. Recall that *green* appears in every copy of T'_i for each u_i on the $u_a - u_{c-1}$ path in T . Since we also have $\text{red} \in c(V(T'_a))$, $\text{blue} \in c(V(T'_{c-1}))$, repeated applications of Corollary 2.5 show that there is some $u_i \in V(T)$ on the $u_a - u_{c-1}$ path such that $c(V(T_i)) = \{\text{green}\}$. In particular, there is some interior vertex of P' colored *green*. So, if any other interior vertex of P' were not colored *green*, then there exists a path containing all three colors which is shorter than P , contradicting the minimality of P . Thus, all interior vertices of P' are *green*. Since $d_T(u_a, u_{c-1}) < \text{diam}(T)$ applying the argument in the previous paragraph to P' yields a rainbow 3-AP. $\square$

Recall that a tree T with even diameter is weakly non-3-peripheral if there exists some vertex $v \in V(T)$ such that T_{v+} is 3-peripheral.

Proposition 3.6. *Suppose T is a weakly non-3-peripheral tree with even diameter. Then $\text{aw}(T \square T', 3) = 3$ for any nontrivial even diameter tree T' .*

Proof. Assume, for the sake of contradiction, that c is a rainbow-free, exact 3-coloring of $T \square T'$ using the colors *red*, *blue*, and *green*. By Lemma 2.3, there is either a isometric C_3 or isometric path containing all three colors. Since $T \square T'$ is bipartite no such C_3 exists so let P be a shortest such path. Specifically, say P is a $v_{a,b} - v_{c,d}$ path where $v_{a,b}$ is *red*, $v_{c,d}$ is *blue*, and all others are *green*. We remark that all $v_{a,b} - v_{c,d}$ paths must have the interior vertices be *green*, else we can construct a shorter path containing all three colors. Furthermore, P must have odd length; otherwise there is a rainbow 3-AP.

First consider the case when $d(u_a, u_c) = \text{diam}(T)$. Then $d_T(u_a, u_c)$ is even implying that $d_{T'}(w_b, w_d)$ is odd, and in particular, $d_{T'}(w_b, w_d) < \text{diam}(T')$. Without loss of generality, suppose w_b is peripheral or w_d is not peripheral. If w_b is peripheral, then there is a geodesic $(w_b, w_{b_1}, w_{b_2}, \dots, w_{b_{\text{diam}(T')}})$ such that $d(w_b, w_{b_i}) = i$ for each $1 \leq i \leq \text{diam}(T')$. Since $d_{T'}(w_b, w_d) < \text{diam}(T')$, we must be able to find some $w_{d'} \in T'$ for which $d_{T'}(w_b, w_{d'}) = d_{T'}(w_b, w_d) + 1$. If w_d is not peripheral, Corollary 2.10 yields a vertex $w_{d'}$ in T' for which $d_{T'}(w_b, w_{d'}) = d_{T'}(w_b, w_d) + 1$. Since

T is weakly non-3-peripheral, there exists some vertex u_ℓ in T such that $T_{u_\ell}^+$ is 3-peripheral. Lemma 2.19 implies $d_T(u_a, u_\ell) = d_T(u_c, u_\ell) = \text{diam}(T) - 1$. This implies $(v_{c,d}, v_{a,b}, v_{\ell,d'})$ is a 3-AP. To avoid a rainbow 3-AP, $v_{\ell,d'}$ is not *green*. Since $v_{a,d}$ and $v_{c,b}$ lie on a $v_{a,b} - v_{c,d}$ geodesic, an earlier remark implies they must be *green*. Now, the 3-APs $(v_{a,d}, v_{\ell,d'}, v_{c,d})$ or $(v_{a,b}, v_{\ell,d'}, v_{c,b})$ are rainbow depending on whether $v_{\ell,d'}$ is *red* or *blue*, respectively, contradicting that c is a rainbow free coloring.

Now consider the case when $d(u_a, u_c) < \text{diam}(T)$. Applying the same argument used in the previous case, there exists some $u_{c'}$ in T such that, without loss of generality, $d_T(u_a, u_{c'}) = d(u_a, u_c) + 1$. Let $w_{d-1} \in V(T')$ be the unique neighbor of w_d on the $w_b - w_d$ path. Consider the vertex $v_{c',d-1}$ and the 3-AP $(v_{c,d}, v_{a,b}, v_{c',d-1})$. Notice $v_{c',d-1}$ cannot be *green*; otherwise this 3-AP is rainbow. Since T' is nontrivial and has even diameter, $|V(T')| \geq 3$. So, Corollary 2.5 implies that $v_{c',d-1}$ cannot be *red*; otherwise $|c(V(T_{d-1}) \cup V(T_d))| = 3$ because *green* must appear in T_{d-1} or T_d . Thus, $c(v_{c',d-1}) = \text{blue}$.

Let P' be a $v_{a,b} - v_{c',d-1}$ geodesic. Since the length of P' is the same as the length of P , all internal vertices of P must be colored *green*; otherwise there is a shorter path than P containing all three colors. If $d_T(u_a, u_{c'}) = \text{diam}(T)$, then apply the argument from the previous case. So, suppose $d_T(u_a, u_{c'}) < \text{diam}(T)$. Also, $d(w_b, w_{d-1}) < d(w_b, w_d) \leq \text{diam}(T')$. So, there are some peripheral vertices $u_r \in V(T), w_i \in V(T')$ such that P' is contained in $T_{u_r}^- \square T'_{w_i}^-$. We choose u_r to be equal u_a or $u_{c'}$ if either are peripheral; otherwise we may choose u_r to be any peripheral vertex. Likewise for choosing w_i depending on whether w_b and w_{d-1} are peripheral or not. Recall $T_{u_\ell}^+$ is 3-peripheral. Since u_r is a peripheral vertex of T , there exists some $u_s \in V(T)$ such that $d_T(u_r, u_s) = \text{diam}(T)$. Lemma 2.19 implies that $d(u_\ell, u_r) = d(u_\ell, u_s) = \text{diam}(T) - 1$. Let u_{r-1} and u_{s-1} be the unique neighbors of u_r and u_s , respectively, so that

$$d(u_{r-1}, u_{s-1}) = d(u_\ell, u_{r-1}) = d(u_\ell, u_{s-1}) = \text{diam}(T) - 2 = \text{diam}\left((T_{u_r}^-)_{u_\ell}^-\right),$$

where the last equality holds by Observation 2.16. If $\text{diam}(T) = 2$, then $u_{r-1} = u_{s-1} = u_\ell$, and if $\text{diam}(T) > 2$, then these three vertices are pairwise distinct. So, $(T_{u_r}^-)_{u_\ell}^-$ is a single vertex or 3-peripheral. This implies $T_{u_r}^- \cong P_2$ or $T_{u_r}^-$ is weakly non-3-peripheral. By Observation 2.16, Lemma 3.4 and Proposition 3.5, we have $\text{aw}\left(T_{u_r}^- \square T'_{w_i}^-, 3\right) = 3$. Since P' contains all three colors, it follows that $T_{u_r}^- \square T'_{w_i}^-$ contains all three colors and thus contains a rainbow 3-AP. Since this is an isometric subgraph of $T \square T'$, this rainbow 3-AP also exists in $T \square T'$, a contradiction. $\square$

Now, using Theorems 1.3, 2.8 and 2.9, we have a full classification of the anti-van der Waerden number of the product of two trees. This quickly leads to a full classification of products of two forests.

Corollary 1.4. Let T and T' be trees. Then, $\text{aw}(T \square T', 3) =$

$$\begin{cases} 3 & \text{if } T \text{ or } T' \text{ is 3-peripheral, or} \\ & \text{if } \text{diam}(T \square T') \text{ is even and either } T \text{ or } T' \text{ is } P_2, \text{ or} \\ & \text{if } \text{diam}(T \square T') \text{ is even and either } T \text{ or } T' \text{ is weakly non-3-peripheral;} \\ 4 & \text{otherwise.} \end{cases}$$

The following observation relies on the pigeonhole principle and helps to provide a full classification for products of forests.

Observation 3.7 ([3]). *If G is disconnected with connected components $\{G_i\}_{i=1}^\ell$, then*

$$\text{aw}(G, k) = 1 + \sum_{i=1}^{\ell} (\text{aw}(G_i, k) - 1).$$

Corollary 1.4 and Observation 3.7 give the following corollary. Note that Corollary 3.8 was previously known, but it was not known how to compute $|P|$ and $|S|$ explicitly.

Corollary 3.8. *Let F_1 and F_2 be forests and let P be the set of connected components of $F_1 \square F_2$ whose anti-van der Waerden number is 3 and S be the set of connected components of $F_1 \square F_2$ whose anti-van der Waerden number is 4. Then,*

$$\text{aw}(F_1 \square F_2, 3) = 2|P| + 3|S| + 1.$$

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