AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **94(1)** (2026), Pages 195–200

The design spectrum of the Shrikhande graph

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Abstract

The design spectrum of a simple graph G is the set of positive integers n such that there exists an edgewise decomposition of the complete graph K_n into $n(n-1)/(2|E(G)|)$ copies of G. The purpose of this short paper is to prove that the Shrikhande graph and the line graph of $K_{4,4}$ have the design spectrum $\{96t+1 : t = 0, 1, 2, \dots\}$.

1 Introduction

If F and G are simple graphs, an *edgewise decomposition* of F into G, which we also refer to as a *G-decomposition of F*, is a partition $\mathcal{E}$ of the edges of F such that each $E \in \mathcal{E}$ is the edge set of a graph isomorphic to G. If F is the complete graph K_n, we usually refer to the decomposition as a *G-design* of order n. The *design spectrum* of G is the set of positive integers n for which a G-design of order n exists. A straightforward analysis confirms that when G is d-regular the necessary conditions for the existence of a G-design are

$$\begin{aligned} n &\geq |V(G)| \text{ or } n = 1, \\ n(n-1) &\equiv 0 \pmod{2|E(G)|}, \\ n-1 &\equiv 0 \pmod{d}. \end{aligned} \tag{1}$$

Given a d-regular graph G, by a theorem of Wilson [9], the conditions (1) are sufficient for all sufficiently large n, and hence the determination of G's design spectrum is actually a finite problem. However, it is usually impossible to resolve all of the cases not covered by 'sufficiently large' whenever d or the chromatic number is large. Nevertheless, design spectra have been computed for many graphs, including some infinite classes. See [2] for the most recent comprehensive survey of graph designs. In particular, the design spectrum has been resolved for the Petersen graph [1].

In our paper we address the design spectrum problem for the Shrikhande graph as well as the line graph of $K_{4,4}$; see Figure 1. Our objective is to prove in Theorems 2.1 and 3.1 that in the conditions (1) are sufficient in each case. Incidentally, we mention that there exists a decomposition into 5 Shrikhande graphs of $2K_{16}$; see [4].

ISSN: 2202-3518

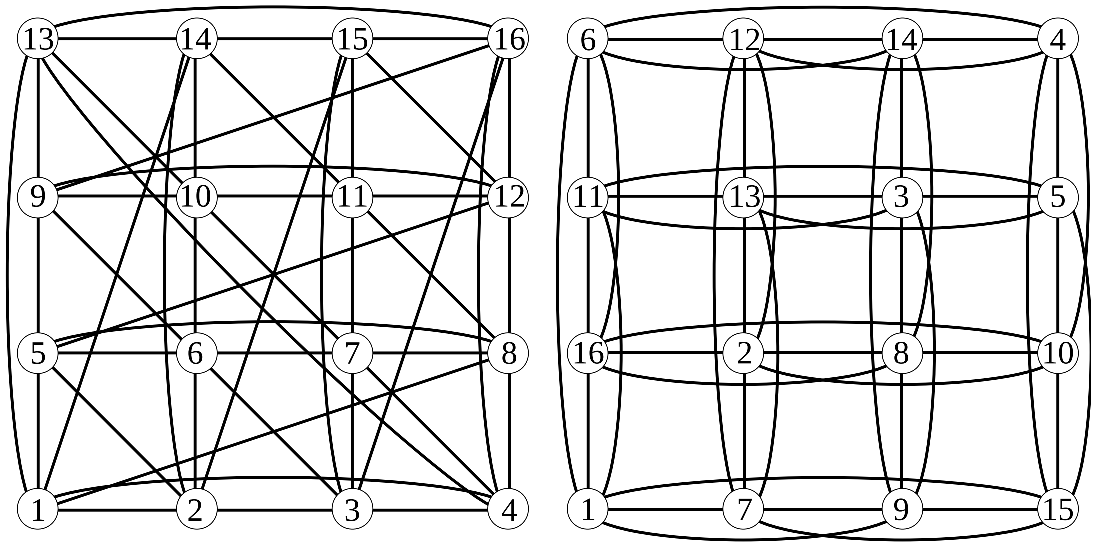

Figure 1: The Shrikhande graph (left) and $L(K_{4,4})$ (right)

The Shrikhande graph is strongly regular with parameters srg(16,6,2,2) and eigenvalues $6^1, 2^6, (-2)^9$, it is 4-chromatic, and its complement is 6-chromatic. It is named after Sharad–Chandra S. Shrikhande, who showed that the graph is identified by these properties [7]. For a biographical account of Shrikhande, various constructions of the Shrikhande graph, and a textbook account of how the graph relates to various areas of discrete mathematics, we recommend the excellent book by Cameron, Lakshmanan and Vijayakumar [5].

2 The design spectrum of the Shrikhande graph

For our proof of Theorem 2.1, we employ a technique of design theory known as Wilson's fundamental construction [8]. The method uses group divisible designs to build large graph decompositions from small ones.

For the purpose of this paper, a *group divisible design*, k-GDD, of type g^u is an ordered triple $(V, \mathcal{G}, \mathcal{B})$ where

(i) V is a set of *gu points*,

(ii) $\mathcal{G}$ is a partition of V into u subsets of size g, called *groups*, and

(iii) $\mathcal{B}$ is a collection of k-subsets V, called *blocks*, which has the property that each pair of points from distinct groups occurs in precisely one block but a pair of distinct points from the same group does not occur in any block.

Our first lemma asserts the existence of the group divisible designs that we require for our constructions.

Lemma 2.1 [3, Theorem 6.3] *There exists a* 4-GDD *of type* 24^t *for* $t \geq 4$.

Next, we give the direct constructions of decompositions into the Shrikhande graph that we require for our proof of Theorem 2.1. The Shrikhande graph, S, that we use for our computations is defined on vertex set $\{1, 2, \ldots, 16\}$ by the edge set $E(S) = \{\{1,2\}, \{1,4\}, \{1,5\}, \{1,8\}, \{1,13\}, \{1,14\}, \{2,3\}, \{2,5\}, \{2,6\}, \{2,14\}, \{2,15\}, \{3,4\}, \{3,6\}, \{3,7\}, \{3,15\}, \{3,16\}, \{4,7\}, \{4,8\}, \{4,13\}, \{4,16\}, \{5,6\}, \{5,8\}, \{5,9\}, \{5,12\}, \{6,7\}, \{6,9\}, \{6,10\}, \{7,8\}, \{7,10\}, \{7,11\}, \{8,11\}, \{8,12\}, \{9,10\}, \{9,12\}, \{9,13\}, \{9,16\}, \{10,11\}, \{10,13\}, \{10,14\}, \{11,12\}, \{11,14\}, \{11,15\}, \{12,15\}, \{12,16\}, \{13,14\}, \{13,16\}, \{14,15\}, \{15,16\}\}$ (Figure 1, left). The sets of labelled graphs that form the decompositions are created from base blocks. A *base block* is an ordered 16-tuple $(\ell_1, \ell_2, \ldots, \ell_{16})$ that defines S relabelled such that vertex i has label ℓ_i, $i \in \{1, 2, \ldots, 16\}$.

For the benefit of non-specialists, we explain in detail how to construct an S-decomposition of order 97 from the base block of Lemma 2.3,

$$B_0 = (0, 4, 6, 62, 1, 11, 19, 45, 69, 80, 59, 78, 32, 74, 28, 44),$$

and the set of mappings $M = \{x \mapsto x + d\colon\ 0 \le d < 97\}$ with arithmetic in $\mathbb{Z}_{97}$. Suppose the vertices $(1, 2, \ldots, 16)$ of S, as defined above or as illustrated on the left of Figure 1, are labelled with B_0. Applying the elements of M creates 97 labelled versions of S:

$$\begin{aligned}\mathcal{B} = \{&(0, 4, 6, 62, 1, 11, 19, 45, 69, 80, 59, 78, 32, 74, 28, 44),\\ &(1, 5, 7, 63, 2, 12, 20, 46, 70, 81, 60, 79, 33, 75, 29, 45),\\ &(2, 6, 8, 64, 3, 13, 21, 47, 71, 82, 61, 80, 34, 76, 30, 46),\\ &\ldots,\\ &(96, 3, 5, 61, 0, 10, 18, 44, 68, 79, 58, 77, 31, 73, 27, 43)\},\end{aligned}$$

which form an edgewise decomposition of K_{97} into 97 relabelled versions of S. Indeed, a straightforward computation confirms that the edges of these 97 6-regular 16-vertex graphs generate $97 \cdot 48 = 4656$ distinct unordered pairs that correspond precisely to the edges of K_{97} with its vertices labelled $0, 1, \ldots, 96$. That is,

$$\{\{b_i, b_j\} : b_i, b_j \in B \in \mathcal{B}, \{i, j\} \in E(S)\} = \{\{x, y\} : 0 \le x < y < 97\}.$$

The technique is also explained in [6, Section 1.1].

Lemma 2.2 *There exists an edgewise decomposition of the complete* 4*-partite graph* $K_{4,4,4,4}$ *into* 2 *copies of the Shrikhande graph.*

Proof. The point set is $\mathbb{Z}_{16}$ partitioned by residue class modulo 4. The decomposition consists of just the two base blocks:

$(0, 1, 2, 5, 3, 4, 7, 6, 10, 9, 8, 13, 11, 14, 15, 12)$,

$(0, 2, 8, 10, 9, 3, 5, 15, 12, 14, 4, 6, 7, 13, 11, 1)$. □

Lemma 2.3 *There exist Shrikhande-graph-designs of orders* 97*,* 193 *and* 289*.*

Proof. The graphs for the design of order n are developed from a single base block by $x \mapsto \omega^e x + d$, $0 \le e < (n-1)/96$, $0 \le d < n$, where ω is a specified parameter. For order $n \in \{97, 193\}$, the arithmetic is performed in the field $\mathbb{Z}_n$. For order 289, we use $\mathrm{GF}(17^2)$, where element $az + b$, $a, b \in \mathbb{Z}_{17}$, is represented by the number $17a + b$. The polynomial for multiplication in $\mathrm{GF}(17^2)$ is $z^2 + 3z + 1$.

Design order 97, $\omega = 1$:

$(0, 4, 6, 62, 1, 11, 19, 45, 69, 80, 59, 78, 32, 74, 28, 44)$;

Design order 193, $\omega = 81$:

$(0, 19, 164, 27, 51, 175, 66, 138, 74, 20, 70, 94, 108, 77, 41, 134)$;

Design order 289, $\omega = 139$:

$(0, 136, 232, 11, 176, 180, 89, 159, 288, 257, 90, 42, 45, 260, 37, 19)$. □

For clarification, we explain by an example how multiplication works in our encoding of $\mathrm{GF}(17^2)$. To multiply 136 by 139 we proceed thus: $136 = 8 \cdot 17 \to 8z$; $139 = 8 \cdot 17 + 3 \to 8z + 3$; $(8z)(8z + 3) = 64z^2 + 24z \to -192z - 64 + 24z$ (using $z^2 + 3z + 1 = 0$) $\to 2z + 4 \to 2 \cdot 17 + 4 = 38$.

Theorem 2.1 *There exists a Shrikhande-graph-design of order $n \ge 1$ if and only if $n \equiv 1 \pmod{96}$.*

Proof. A straightforward computation confirms that the necessary conditions (1) for the existence of a Shrikhande-graph-design of order n simplify to $n = 96t + 1$, $t = 0, 1, 2, \ldots$. The case $t = 0$ is trivial—the design is the empty set—and so we assume $t \ge 1$.

Take a 4-GDD of type 24^t from Lemma 2.1, inflate its points by a factor of 4 and replace its blocks by decompositions into Shrikhande graphs of $K_{4,4,4,4}$, which exist by Lemma 2.2. Add a new point and overlay each group plus the new point with a Shrikhande-graph-design of order 97 from Lemma 2.3. The result is a Shrikhande-graph-design of order $96t + 1$ for $t \ge 4$.

The values of n not accounted for by the construction are 1, 97, 193 and 289. For $n = 1$ the design is the empty set. The other designs are provided by Lemma 2.3. □

3 The design spectrum of $L(K_{4,4})$

The line graph of $K_{4,4}$, denoted by $L(K_{4,4})$, has the same eigenvalue spectrum as the Shrikhande graph. Here we show that it has the same design spectrum.

Theorem 3.1 *There exists an $L(K_{4,4})$-design of order $n \ge 1$ if and only if $n \equiv 1 \pmod{96}$.*

Proof. The details are similar to those set out in Section 2 for the Shrikhande graph. The line graph $L(K_{4,4})$, L, that we use for our computations is defined on vertex set $\{1, 2, \ldots, 16\}$ by the edge set {{1,6}, {1,7}, {1,9}, {1,11}, {1,15}, {1,16}, {2,7}, {2,8}, {2,10}, {2,12}, {2,13}, {2,16}, {3,5}, {3,8}, {3,9}, {3,11}, {3,13}, {3,14}, {4,5}, {4,6}, {4,10}, {4,12}, {4,14}, {4,15}, {5,10}, {5,11}, {5,13}, {5,15}, {6,11}, {6,12}, {6,14}, {6,16}, {7,9}, {7,12}, {7,13}, {7,15}, {8,9}, {8,10}, {8,14}, {8,16}, {9,14}, {9,15}, {10,15}, {10,16}, {11,13}, {11,16}, {12,13}, {12,14}} (Figure 1, right). The sets of labelled graphs that form the decompositions are created from base blocks, where now a base block is an ordered 16-tuple $(\ell_1, \ell_2, \ldots, \ell_{16})$ that defines L relabelled such that vertex i has label ℓ_i, $i \in \{1, 2, \ldots, 16\}$. We give just the base blocks for the four decompositions of Lemmas 2.2 and 2.3. The procedures for developing them are the same as in Section 2.

Decomposition of $K_{4,4,4,4}$:

$(0, 1, 2, 3, 4, 6, 10, 8, 7, 14, 5, 12, 11, 9, 13, 15)$,

$(0, 4, 5, 13, 8, 2, 9, 7, 14, 6, 3, 15, 10, 12, 11, 1)$;

Design order 97, $\omega = 1$:

$(0, 49, 34, 94, 1, 2, 6, 76, 68, 58, 47, 80, 59, 83, 38, 75)$;

Design order 193, $\omega = 81$:

$(0, 78, 184, 67, 34, 169, 130, 177, 64, 103, 137, 108, 30, 7, 84, 65)$;

Design order 289, $\omega = 139$:

$(0, 119, 123, 283, 273, 249, 231, 69, 2, 234, 93, 67, 55, 64, 20, 256)$. □

Finally we have the following.

Theorem 3.2 *There exists a decomposition of the complete multipartite graph $K_{4,4,4,4}$ into a Shrikhande graph and a line graph $L(K_{4,4})$.*

Proof. Combine the edges of the Shrikhande graph with the edges of $L(K_{4,4})$, as depicted in Figure 1. The result is a graph that is isomorphic to $K_{4,4,4,4}$. □

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(Received 22 May 2025; revised 29 Nov 2025, 13 Jan 2026)