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Descents in the Grand Dyck paths and the Chung-Feller property

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Abstract

A Grand Dyck path of semilength n with m flaws is a path in the integer lattice which starts at the origin and consists of n up steps $U = (1, 1)$ and n down steps $D = (1, -1)$, and that has exactly m up steps below the line $y = 0$. The classical Chung-Feller theorem asserts that the number of grand Dyck paths of semilength n with m flaws is the nth Catalan number and is independent of m. In this paper, by using a bijection and generating functions, we prove a refinement of the Chung-Feller theorem: the number of Grand Dyck paths of semilength n having m flaws and k descents is the Narayana number $N_{n,k}$, and is independent of m. We also enumerate the Grand Dyck paths ending with a down step or an up step, and obtain some interesting results related to the Narayana numbers or Catalan numbers.

1 Introduction

A Grand Dyck path of semilength $n \geq 0$ is a lattice path from $(0, 0)$ to $(2n, 0)$, using exactly n up steps $U = (1, 1)$ and n down steps $D = (1, -1)$, and that has exactly m up steps below the line $y = 0$. Let $\mathcal{G}_n$ be the set of all Grand Dyck paths of semilength n, and $\mathcal{G} = \bigcup_{n=0}^{\infty} \mathcal{G}_n$, where $\mathcal{G}_0 = \{\epsilon\}$ and ϵ is the empty path. The cardinality of $\mathcal{G}_n$ is given by the nth central binomial coefficient $\binom{2n}{n}$ (sequence A000984 in OEIS [20]). A Dyck path is a Grand Dyck path that never goes below the x-axis. We denote by $\mathcal{D}_n$, $n \geq 0$, the set of all Dyck paths of semilength n, and $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}_n$. Every up step of a Grand Dyck path that lies below the x-axis is called a flaw. The set of Grand Dyck paths of semilength n with m flaws is denoted $\mathcal{G}_{n,m}$. In particular, $\mathcal{G}_{n,0} = \mathcal{D}_n$ is the set of Dyck paths of semilength n, and $\mathcal{G}_{n,n} = \overline{\mathcal{D}}_n$ is the set of reflected Dyck paths of semilength n.

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It is well known that the set $\mathcal{D}_n$ is counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ [4, 20, 21]. The classical Chung-Feller theorem states that $|\mathcal{G}_{n,m}| = C_n$, for all $0 \le m \le n$. The proof of the Chung-Feller theorem in [3] is based on an analytic method. A combinatorial proof is given by Narayana [18] and rediscovered by Chen [2]. Eu, Fu and Yeh obtained a refinement of this theorem in [8]. Liu, Wang and Yeh [14] use a unified algebra approach to prove the Chung-Feller theorems for Dyck paths and Motzkin paths and develop a new method for finding some combinatorial structures which have the Chung-Feller property. Some other interesting proofs and generalizations are given in [7, 12, 13, 16, 17, 19, 26, 28, 29, 32].

In a Grand Dyck path, a peak (respectively, valley) is an occurrence of UD (respectively, DU). A double rise (respectively, double fall) is an occurrence of UU (respectively, DD). An ascent (respectively, descent) is a maximal sequence of consecutive up steps U (respectively, down steps D) in a Dyck path, that is, a sequence that is not preceded or followed immediately by another U (respectively, D).

The Narayana polynomials are defined as [1, 23, 27]

$$N_0(x) = 1, N_n(x) = \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1}x^k \quad \text{for } n \ge 1,$$

and $N_{n,k} = [x^k]N_n(x) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is known as a Narayana number. Equivalently, the Narayana polynomials are determined by the generating function

$$\sum_{n=0}^{\infty} N_n(x)t^n = \frac{1 + t - xt - \sqrt{(1 + t - xt)^2 - 4t}}{2t}.$$

The Narayana numbers are widely used in enumeration of Dyck paths [5, 6, 22, 23, 24, 25, 31]. For example, the number of Dyck paths of semilength n with k peaks equals the number of Dyck paths of semilength n with k ascents (or descents), and is given by the Narayana number $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. Similarly, the number of Dyck paths of semilength n with k valleys equals the number of Dyck paths of semilength n with k double rises (or double falls), and is given by the Narayana number $N_{n,k+1} = \frac{1}{n}\binom{n}{k+1}\binom{n}{k}$.

In [15], Ma and Yeh proved a refined Chung-Feller type theorem for Dyck paths of semilength n with k double rises.

Theorem 1.1 ([15])**.** *Let n be an integer with $n \ge 0$ and $0 \le k \le n-1$. Then, the total number of Grand Dyck paths in $\mathcal{G}_{n,m}$ having k double rises is equal to $N_{n,k+1} = \frac{1}{n}\binom{n}{k+1}\binom{n}{k}$, for any $0 \le m \le n$.*

In this paper, we provide the following Chung-Feller type theorem for the Grand Dyck paths.

Theorem 1.2. *For $1 \le k \le n$, the number of Grand Dyck paths of semilength n having m flaws and k descents is equal to $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, and independent of m for $0 \le m \le n$.*

We will give two proofs of this result. Although the combinatorial proof in Section 2 is short and self-contained, we include a generating-function proof in Section 3 for additional reasons. The functional equations and bivariate generating functions developed there are new and encode several refined statistics (such as descents, flaws, and the type of the last step) in a unified analytic framework. These generating functions allow us to derive further identities and refined enumerative results that are not easily accessible from the bijective argument alone. Thus, the generating-function approach provides an additional perspective that we believe is of independent interest.

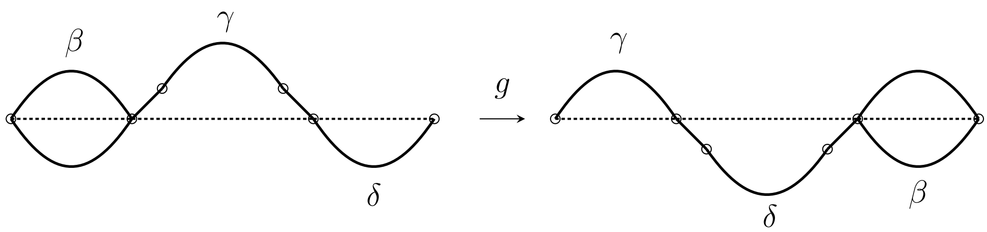
This paper is organized as follows. In Section 2, we provide a combinatorial proof of Theorem 1.2. In Section 3, we give a generating function proof. In Section 4, we enumerate the Grand Dyck paths that end with a down step or an up step, and obtain some interesting results related to the Narayana numbers. Specifically, we show that the number of the Grand Dyck paths of semilength n that end with a down step and having k descents and exactly 1 flaw is equal to $N_{n,k} - N_{n-1,k}$, and that the total number of the Grand Dyck paths of semilength n ending with a down step and having k descents is $kN_{n,k}$.

2 A combinatorial proof of Theorem 1.2

Now we give a bijective proof of Theorem 1.2. To do this, we establish a bijection from $\mathcal{G}_{n,m}$ to $\mathcal{G}_{n,m+1}$, for each $0 \le m \le n-1$. Every $\alpha \in \mathcal{G}_{n,m}$ can be uniquely decomposed into the form $\alpha = \beta U \gamma D \delta$, where $\beta \in \mathcal{G}$, $\gamma \in \mathcal{D}$ and $\delta \in \overline{\mathcal{D}}$, where $\overline{\mathcal{D}} = \bigcup_{n=0}^{\infty} \overline{\mathcal{D}}_n$ is the set of all reflected Dyck paths. This is the last-non-flaw decomposition. We define a path $g(\alpha)$ as

$$g(\alpha) = \gamma D \delta U \beta.$$

Clearly, the number of descents in $g(\alpha)$ is equal to the number of descents in α, and the number of flaws in $g(\alpha)$ is 1 more than the number of flaws in α (see Figure 1).



The last-non-flaw decomposition of $\mathcal{G} \setminus \overline{\mathcal{D}}$ The first-flaw decomposition of $\mathcal{G} \setminus \mathcal{D}$

Figure 1: The bijection g.

To prove that the mapping g is a bijection, we describe the inverse g^{-1} of the mapping g as follows. Every $\alpha' \in \mathcal{G}_{n,m+1}$ can be uniquely decomposed into the form

$\alpha' = \gamma D\delta U\beta$, where $\gamma \in \mathcal{D}$, $\delta \in \overline{\mathcal{D}}$, $\beta \in \mathcal{G}$. This is the first-flaw decomposition. We define a path $g^{-1}(\alpha')$ as

$$g^{-1}(\alpha') = \beta U\gamma D\delta.$$

Clearly, the number of descents in $g^{-1}(\alpha')$ is equal to the number of descents in α', and the number of flaws in $g^{-1}(\alpha')$ is 1 less than the number of flaws in α'.

The bijection g preserves the statistic "number of descents". In the set $\mathcal{G}_{n,0}$, the set of Dyck paths of semilength n, we have (see [6])

$$|\{\alpha; \alpha \in \mathcal{G}_{n,0}, |\alpha|_{des} = k\}| = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Therefore, for any $0 \le m \le n$,

$$|\{\alpha; \alpha \in \mathcal{G}_{n,m}, |\alpha|_{des} = k\}| = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},$$

where $|\alpha|_{des}$ denotes the number of descents in the path α.

In Figure 2, we illustrate the bijection for $n = 3$, where the blue segments indicate descents and the red segments indicate flaws.

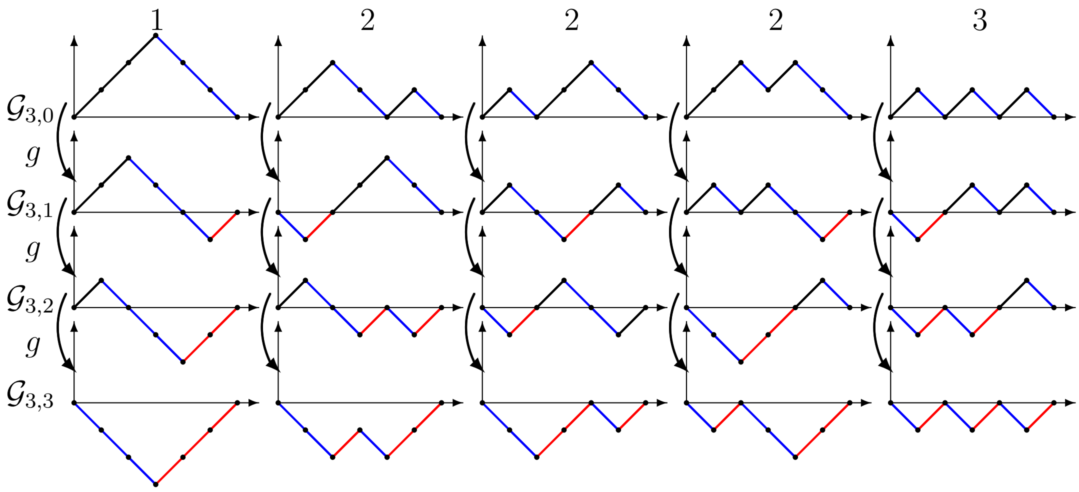


Figure 2: Illustration of the bijection g used to prove Theorem 1.2.

3 Proof of Theorem 1.2 using the generating functions

In this section, we give a generating function proof of Theorem 1.2 by using the symbolic method [11] and the linear algebra method [9, 10].

3.1 The generating function of Dyck paths with respect to peaks or valleys

Although the results of Lemma 3.1 and Lemma 3.2 have been previously given in [15], we provide a new proof using a combinatorial method here.

Lemma 3.1 ([15]). *Let $P_{n,k}$ denote the number of Dyck paths of semilength n with exactly k peaks. Define the generating function*

$$P(t,x) = \sum_{n\geq 0}\sum_{k\geq 0} P_{n,k}\, t^n x^k,$$

where t marks the semilength and x marks the number of peaks. Then

$$P(t,x) = \frac{1+t-xt-\sqrt{(1+t-xt)^2-4t}}{2t}. \tag{3.1}$$

Proof. Any non-empty Dyck path α decomposes uniquely as $\alpha = UD\beta$, or as $\alpha = U\beta' D\beta''$ with $\beta, \beta', \beta'' \in \mathcal{D}$ and $\beta' \neq \epsilon$ (see Figure 3). Hence, we have the identity

$$P(t,x) = 1 + txP(t,x) + t(P(t,x)-1)P(t,x)$$

whose solution is (3.1). □

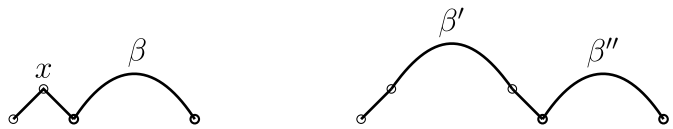


Figure 3: Decomposition of Dyck paths according to the occurrence of peaks.

Applying the Lagrange inversion formula, we obtain, for $n \geq 1$

$$[t^n]P(t,x) \;=\; \sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} x^k = N_n(x),$$

which is the Narayana polynomial $N_n(x)$.

Lemma 3.2 ([15]). *Let $V_{n,k}$ denote the number of Dyck paths of semilength n with exactly k valleys. Define the generating function*

$$V(t,x) = \sum_{n\geq 0}\sum_{k\geq 0} V_{n,k}\, t^n x^k,$$

where t marks the semilength and x marks the number of valleys. Then,

$$V(t,x) = \frac{1-t+xt-\sqrt{(1-t+xt)^2-4xt}}{2xt}. \tag{3.2}$$

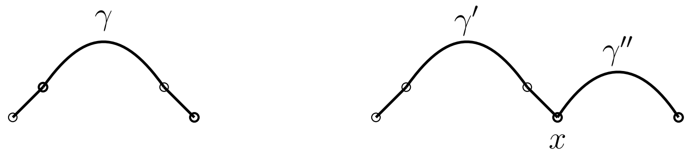


Figure 4: Decomposition of the Dyck paths according to the occurrence of valleys.

Proof. Any non-empty Dyck path α decomposes uniquely as $\alpha = U\gamma D$ (with $\gamma \in \mathcal{D}$) or as $\alpha = U\gamma' D\gamma''$ (with $\gamma', \gamma'' \in \mathcal{D}$, $\gamma'' \neq \epsilon$) (see Figure 4). Hence, we have

$$V(t,x) \;=\; 1 + tV(t,x) + txV(t,x)(V(t,x)-1)$$

whose solution is series (3.2). □

By the Lagrange inversion formula, we obtain, for $n \geq 1$

$$[x^k t^n]V(t,x) \;=\; [x^k]\sum_{k=0}^{n-1}\frac{1}{n}\binom{n}{k+1}\binom{n}{k}x^k = \frac{1}{n}\binom{n}{k+1}\binom{n}{k} = N_{n,k+1}.$$

From the formulas (3.1) and (3.2) we derive successively

$$P(t,x) - 1 \;=\; xV(t,x) - x, \tag{3.3}$$
$$P(t,x) - 1 \;=\; txP(t,x)V(t,x). \tag{3.4}$$

Equation (3.3) also admits a direct combinatorial interpretation. The generating function $P(t,x)-1$ enumerates nonempty Dyck paths by semilength t and number of peaks x, while $V(t,x)-1$ enumerates nonempty Dyck paths by semilength t and number of valleys x. The factor of x in $x(V(t,x)-1)$ accounts for the fact that a nonempty Dyck path with k valleys always has $k+1$ peaks. This one-to-one correspondence increases the statistic marked by x by one, and thus establishes the generating function identity (3.3).

To derive the generating function for Grand Dyck paths, we first establish two determinant identities relating $P(t,x)$ and $V(t,x)$.

Lemma 3.3. *The generating functions $P(t,x)$ and $V(t,x)$ satisfy the following determinant identity*

$$\begin{vmatrix} xtV(t,x)-1 & xtV(t,x) \\ ytV(ty,x) & xytV(ty,x)-1 \end{vmatrix} \;=\; \frac{y-1}{yP(ty,x)-P(t,x)}, \tag{3.5}$$

$$\begin{vmatrix} tP(t,x)-1 & tP(t,x) \\ txyP(ty,x) & tyP(ty,x)-1 \end{vmatrix} \;=\; \frac{y-1}{yV(ty,x)-V(t,x)}. \tag{3.6}$$

Proof. By using the formulas (3.3) and (3.4) iteratively, we have

$$
\begin{aligned}
&(yP(ty,x)-P(t,x))\begin{vmatrix} xtV(t,x)-1 & xtV(t,x) \\ ytV(ty,x) & xytV(ty,x)-1 \end{vmatrix} \\
=\;& yP(ty,x)\begin{vmatrix} xtV(t,x)-1 & xtV(t,x) \\ ytV(ty,x) & xytV(ty,x)-1 \end{vmatrix} \\
&-P(t,x)\begin{vmatrix} xtV(t,x)-1 & xtV(t,x) \\ ytV(ty,x) & xytV(ty,x) \end{vmatrix} - P(t,x)\begin{vmatrix} xtV(t,x)-1 & xtV(t,x) \\ 0 & -1 \end{vmatrix} \\
=\;& y\begin{vmatrix} xtV(t,x)-1 & xtV(t,x) \\ ytV(ty,x)P(ty,x) & -1 \end{vmatrix} - y\begin{vmatrix} x^2tV(t,x)-x & tV(t,x) \\ xtV(ty,x)P(t,x) & tV(ty,x)P(t,x) \end{vmatrix} - 1 \\
=\;& y\begin{vmatrix} xtV(t,x)-1 & xtV(t,x) \\ -1 & -1 \end{vmatrix} + y\begin{vmatrix} xtV(t,x)-1 & xtV(t,x) \\ ytV(ty,x)P(ty,x)+1 & 0 \end{vmatrix} \\
&-y\begin{vmatrix} x^2tV(t,x)-x & tV(t,x) \\ xtV(ty,x)P(t,x) & tV(ty,x)P(t,x) \end{vmatrix} - 1 \\
=\;& y-1+y\begin{vmatrix} x^2tV(t,x)-x & tV(t,x) \\ xytV(ty,x)P(ty,x)-xtV(ty,x)P(t,x)+x & -tV(ty,x)P(t,x) \end{vmatrix} \\
=\;& y-1+y\cdot 0 = y-1,
\end{aligned}
$$

which is equivalent to (3.5). The proof of (3.6) is obtained similarly. □

We include Formula (3.7) here as an example of how the function $V(t,x)$ derived above can be applied in the enumeration of Grand Dyck paths. To this end, recall that Ma and Yeh [15] proved the generating function for the class of Grand Dyck paths with respect to semilength (marked by t), double rises (marked by x), and flaws (marked by y) is given by

$$A(t,x,y) = \frac{yV(ty,x)-V(t,x)}{y-1}, \tag{3.7}$$

where $V(t,x)$ is the generating function given by (3.2).

In the following subsection, we will provide the generating function for the class of Grand Dyck paths with respect to semilength (marked by t), descents (marked by x) and flaws (marked by y).

3.2 The generating function of Grand Dyck paths with respect to descents

A descent of a Grand Dyck path is a maximal sequence of consecutive down steps. For a Dyck path, the total number of descents is equal to the number of peaks.

Hence, the generating function for the class of Dyck paths with respect to semilength (marked by t) and descents (marked by x) is $P(t,x) = \frac{1+t-xt-\sqrt{(1+t-xt)^2-4t}}{2t}$. Recall that $\overline{\mathcal{D}}_n$ denotes the set of reflected Dyck paths of semilength n, i.e., $\overline{\mathcal{D}}_n = \mathcal{G}_{n,n}$. Accordingly, recall that $\overline{\mathcal{D}}$ denotes the class of all reflected Dyck paths. We study its corresponding generating function $F(t,x,y)$, where t keeps track of the semilength, x keeps track of the descents, and y keeps track of the flaws. Since every nonempty path $\alpha \in \overline{\mathcal{D}}$ uniquely decomposes as $\alpha = DU\beta$ (with $\beta \in \overline{\mathcal{D}}$) or as $\alpha = D\beta' U\beta''$ (with $\beta', \beta'' \in \overline{\mathcal{D}}, \beta' \neq \epsilon$), we have the identity $F(t,x,y) = 1 + txyF(t,x,y) + ty(F(t,x,y) - 1)F(t,x,y)$, whose unique solution is $F(t,x,y) = P(ty,x)$.

Theorem 3.4. *Let $b_{n,i,j}$ denote the number of Grand Dyck paths of semilength n with i descents and j flaws. Define the generating function*

$$B(t,x,y) = \sum_{n\geq 0}\sum_{i\geq 0}\sum_{j\geq 0} b_{n,i,j}\, t^n x^i y^j,$$

where t marks the semilength, x marks the number of descents, and y marks the number of flaws. Then

$$B(t,x,y) = \frac{1}{(1-xtV(t,x))(1-xytV(yt,x)) - xyt^2V(t,x)V(yt,x)}, \tag{3.8}$$

where $V(t,x)$ is the generating function given in (3.2).

Proof. Let $B_1(t,x,y)$ be the generating function of the Grand Dyck paths ending with a down step and let $B_2(t,x,y)$ be the generating function of the Grand Dyck paths ending with an up step.

Since every path $\alpha \in \mathcal{G}$ uniquely decomposes as product of paths of the form $U\beta D$ (with $\beta \in \mathcal{D}$) and $D\gamma U$ (with $\gamma \in \overline{\mathcal{D}}$), we have the linear system (using the decompositions illustrated in Figure 5 and Figure 6)

$$\begin{cases} B = 1 + B_1 + B_2, \\ B_1 = xtB + t(P-1)B, \\ B_2 = xyt + ytB_1 + xytB_2 + yt(F-1) + x^{-1}yt(F-1)B_1 + yt(F-1)B_2, \end{cases}$$

where $P = P(t,x)$ and $F = P(ty,x)$.

Solving for $B = B(t,x,y)$, using the formulas (3.3) and (3.4), we obtain (3.8). Furthermore, we have

$$\begin{aligned} B_1(t,x,y) &= \frac{xtV(t,x)}{(1-xtV(t,x))(1-xytV(yt,x)) - xyt^2V(t,x)V(yt,x)}, \\ B_2(t,x,y) &= \frac{xytV(yt,x) - (x-1)xyt^2V(t,x)V(yt,x)}{(1-xtV(t,x))(1-xytV(yt,x)) - xyt^2V(t,x)V(yt,x)}. \end{aligned}$$

□

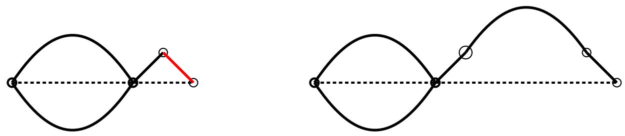


Figure 5: Decomposition of the Grand Dyck paths ending with a down step.

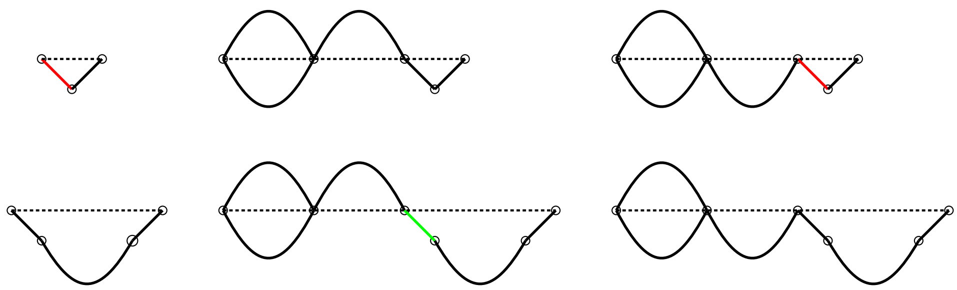


Figure 6: Decomposition of the Grand Dyck paths ending with an up step.

Theorem 3.5. *The generating function* $B(t,x,y)$ *can be written as*

$$B(t,x,y) = \frac{yP(ty,x) - P(t,x)}{y-1},$$

where $P(t,x)$ *is given by* (3.1).

Proof. From the previous theorem,

$$B(t,x,y) \quad = \quad \frac{1}{\begin{vmatrix} xtV(t,x)-1 & xtV(t,x) \\ ytV(ty,x) & xytV(ty,x)-1 \end{vmatrix}}.$$

Hence, the result follows from Lemma 3.3. □

By the previous theorem, we have

$$\begin{aligned} B(t,x,y) &= \frac{yP(ty,x) - P(t,x)}{y-1} \\ &= \frac{1}{y-1}\left(y\sum_{n=0}^{\infty} N_n(x)(yt)^n - \sum_{n=0}^{\infty} N_n(x)t^n\right) \\ &= \frac{1}{y-1}\sum_{n=0}^{\infty} N_n(x)(y^{n+1}-1)t^n \\ &= \sum_{n=0}^{\infty} N_n(x)(y^n + y^{n-1} + \cdots + y + 1)t^n \\ &= \sum_{n=0}^{\infty}\left(N_n(x)\sum_{i=0}^{n} y^i\right)t^n. \end{aligned}$$

Consequently, for $0 \leq m \leq n$, $[y^m t^n]B(t,x,y) = N_n(x)$, and $[x^k y^m t^n]B(t,x,y) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, for $1 \leq k \leq n$.

The formula derived above,

$$[x^k y^m t^n]B(t,x,y) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1},$$

directly yields the enumeration result stated in Theorem 1.2. It confirms that the number of Grand Dyck paths of semilength n with exactly k descents and m flaws is given by the Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, which is independent of m. This completes the proof of Theorem 1.2 via the generating function approach presented in this section.

Since the generating function for the Narayana polynomials $N_n(x)$ is $P(t,x) = \sum_{n=0}^{\infty} N_n(x)t^n$, it follows that

$$\begin{aligned} B(t,x,1) &= \sum_{n=0}^{\infty}(n+1)N_n(x)t^n = \frac{\partial}{\partial t}\left(tP(t,x)\right) \\ &= \frac{1+x-(1-x)^2t+(1-x)\sqrt{(1+t-xt)^2-4t}}{2\sqrt{(1+t-xt)^2-4t}}, \end{aligned} \tag{3.9}$$

and the number of Grand Dyck paths in $\mathcal{G}_n$ having k descents is

$$[x^k t^n]B(t,x,1) = [x^k t^n]\sum_{n=0}^{\infty}(n+1)N_n(x)t^n = (n+1)N_{n,k}.$$

Differentiating with respect to x and evaluating at $x=1$, we then obtain

$$\begin{aligned} \left[\frac{\partial B(t,x,1)}{\partial x}\right]_{x=1} &= \frac{1-2t-(1-4t)\sqrt{1-4t}}{2(1-4t)\sqrt{1-4t}} \\ &= 2t+9t^2+40t^3+175t^4+756t^5+3234t^6+\cdots, \end{aligned}$$

where the coefficient of t^n is equal to $(n+1)\binom{2n-1}{n}$, which is the total number of descents in all paths in $\mathcal{G}_n$. The sequence $2, 9, 40, 175, 756, 3234, \cdots$ is sequence A097070 in OEIS [20].

By setting $x=1$ in $B(t,x,y)$, we obtain

$$B(t,1,y) = \frac{2}{\sqrt{1-4ty}+\sqrt{1-4t}} = \sum_{n=0}^{\infty}\left(C_n\sum_{i=0}^{n} y^i\right)t^n.$$

Differentiating with respect to y and evaluating at $y=1$ we then obtain

$$\begin{aligned} \left[\frac{\partial B(t,1,y)}{\partial y}\right]_{y=1} &= \frac{t}{(1-4t)\sqrt{1-4t}} \\ &= t+6t^2+30t^3+140t^4+630t^5+2772t^6+12012t^7+\cdots, \end{aligned}$$

where the coefficient of t^n is equal to $(2n-1)\binom{2n-2}{n-1}$, which is the total number of flaws in all paths in $\mathcal{G}_n$. The sequence $1, 6, 30, 140, 630, 2772, \cdots$ is sequence A002457 in OEIS [20].

4 Enumerating the Grand Dyck paths ending with a down step or an up step

In this section, we enumerate the Grand Dyck paths ending with a down step or an up step, and we present several counting sequences in terms of the Narayana numbers or the Catalan numbers.

For each refinement parameter, we first derive the corresponding bivariate (or trivariate) generating function. The explicit integer sequences obtained from their expansions, together with their OEIS identifications, are summarized in Table 1 at the end of this section.

Recall that the series $P(t,x)$ and $V(t,x)$ used in this section are the same generating functions introduced in Lemma 3.1 and Lemma 3.2, respectively. In Section 3, the notation $P_{n,k}$ denotes the number of Dyck paths of semilength n with exactly k peaks. To avoid confusion, in this section we use a different symbol for the corresponding quantities of Grand Dyck paths. In particular, we write $G_{n,k}$ for the number of Grand Dyck paths of semilength n ending with a down step and having k descents and exactly one flaw, as stated in Theorem 4.1.

4.1 The Grand Dyck paths ending with a down step

Let $B_1(t,x,y)$ be the generating function for the class of Grand Dyck paths ending with a down step with respect to semilength (marked by t), descents (marked by x) and flaws (marked by y). Then, from the proof of Theorem 3.4, we obtain

$$B_1(t,x,y) \;=\; \frac{xtV(t,x)}{(1-xtV(t,x))\,(1-xytV(yt,x))-xyt^2V(t,x)V(yt,x)}.$$

Upon differentiating with respect to y and evaluating at $y=0$, we obtain another bivariate generating function in t and x,

$$\begin{aligned}\left[\frac{\partial B_1(t,x,y)}{\partial y}\right]_{y=0} &= \frac{1-(2+x)t+(1-x)t^2-(1-t)\sqrt{(1+t-xt)^2-4t}}{2t}\\ &= (P(t,x)-1-xt)-t\,(P(t,x)-1)\\ &= \sum_{n=2}^{\infty}(N_n(x)-N_{n-1}(x))\,t^n.\end{aligned}$$

The first few terms of the series expansion of $\left[\frac{\partial B_1(t,x,y)}{\partial y}\right]_{y=0}$ are

$$\begin{aligned}\left[\frac{\partial B_1(t,x,y)}{\partial y}\right]_{y=0} &= x^2t^2+(2x^2+x^3)t^3+(3x^2+5x^3+x^4)t^4\\ &\quad+(4x^2+14x^3+9x^4+x^5)t^5\\ &\quad+(5x^2+30x^3+40x^4+14x^5+x^6)t^6+\cdots.\end{aligned}$$

Hence, we can state the following result.

Theorem 4.1. *Let $G_{n,k}$ be the number of the Grand Dyck paths of semilength n ending with a down step and having k descents and exactly 1 flaw. Then, for $2 \leq k \leq n$,*

$$G_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} - \frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k-1} = N_{n,k} - N_{n-1,k}, \quad (4.1)$$

and the generating function is given by $\sum_{k=2}^{n} G_{n,k}x^k = N_n(x) - N_{n-1}(x)$.

Using (4.1) and (3.9), we have computed that

$$B_1(t,x,1) = \frac{x - x(x-1)t - x\sqrt{(1+t-xt)^2-4t}}{2\sqrt{(1+t-xt)^2-4t}} = x\frac{\partial}{\partial x}P(t,x).$$

Therefore, we obtain $[t^n]B_1(t,x,1) = x\frac{d}{dx}N_n(x) = \sum_{k=1}^{n}\frac{k}{n}\binom{n}{k}\binom{n}{k-1}x^k$ and $[x^k t^n]B_1(t,x,1) = \frac{k}{n}\binom{n}{k}\binom{n}{k-1}$, for $1 \leq k \leq n$. Hence, we obtain the following result.

Theorem 4.2. *Let $D_{n,k}$ be the total number of the Grand Dyck paths of semilength n ending with a down step and having k descents. Then, for $1 \leq k \leq n$,*

$$D_{n,k} = \frac{k}{n}\binom{n}{k}\binom{n}{k-1} = kN_{n,k},$$

and the generating function is given by $\sum_{k=1}^{n} D_{n,k}x^k = x\frac{d}{dx}N_n(x)$.

Specifically, let P be a standard Dyck path enumerated by $N_{n,k}$ (i.e., with $m = 0$). By definition, P contains exactly k descents. These descents identify k distinct positions in the sequence where a run of down steps concludes. Let us denote the indices of the last steps of these k descents as $t_1, t_2, \ldots, t_k$. For each position t_i, we can decompose the path P into a prefix u (consisting of the first t_i steps) and a suffix v. Note that by this construction, u ends with a down step. We then form a new path $P' = vu$ by swapping the two parts. Since u ends with a down step, the resulting path P' is guaranteed to belong to the set of Grand Dyck paths ending with a down step. Since there are k distinct positions t_i to perform this operation, each single path in $N_{n,k}$ generates exactly k unique paths in $D_{n,k}$.

Some initial terms of $B_1(t,x,1)$ are

$$\begin{aligned} B_1(t,x,1) &= xt + (x+2x^2)t^2 + (x+6x^2+3x^3)t^3 + (x+12x^2+18x^3+4x^4)t^4 \\ &\quad + (x+20x^2+60x^3+40x^4+5x^5)t^5 + \cdots . \end{aligned}$$

Differentiating with respect to x and evaluating at $x = 1$, we then obtain

$$\begin{aligned} \left[\frac{\partial B_1(t,x,1)}{\partial x}\right]_{x=1} &= \frac{(1-2t)^2 - (1-4t)\sqrt{1-4t}}{2(1-4t)\sqrt{1-4t}} \\ &= t + 5t^2 + 22t^3 + 95t^4 + 406t^5 + 1722t^6 + \cdots , \end{aligned}$$

where the coefficient of t^n is equal to $(n+1)\binom{2n-2}{n}+\binom{2n-2}{n-1}$, which is the total number of descents in all paths in $\mathcal{G}_n$ ending with a down step.

By setting $x = 1$ in $B_1(t, x, 1)$, we obtain the generating function of the number of the Grand Dyck paths that end with a down step,

$$B_1(t,1,1) \;=\; t+3t^2+10t^3+35t^4+126t^5+462t^6+\cdots .$$

By setting $x = 1$ in $B_1(t, x, y)$, we obtain

$$B_1(t,1,y)=\frac{1-\sqrt{1-4t}}{\sqrt{1-4t}+\sqrt{1-4ty}}.$$

Some initial terms of $B_1(t, 1, y)$ are

$$\begin{aligned} B_1(t,1,y) \;=\;& t+(2+y)t^2+(5+3y+2y^2)t^3+(14+9y+7y^2+5y^3)t^4 \\ &+(42+28y+23y^2+19y^3+14y^4)t^5+\cdots . \end{aligned}$$

We let $Q_n(y) = [t^n]B_1(t, 1, y)$, $n \geq 1$. Then

$$\begin{aligned} Q_n(y) &= [t^n]B_1(t,1,y)=[t^n]tC(t)B(t,1,y) \\ &= [t^n]\left(\sum_{n=1}^{\infty} C_{n-1}t^n\right)\left(\sum_{n=0}^{\infty} C_nS_n(y)t^n\right) \\ &= \sum_{i=0}^{n-1} C_{n-i-1}C_iS_i(y), \end{aligned}$$

where $S_i(y) = \sum_{j=0}^{i} y^j$. Therefore, we have

$$[y^m]Q_n(y) \;=\; \sum_{i=m}^{n-1} C_{n-i-1}C_i.$$

Hence we can state the following result.

Theorem 4.3. *Let $Q_{n,m}$ be the number of the Grand Dyck paths of semilength n ending with a down step and having m flaws. Then, for $0 \leq m \leq n-1$,*

$$Q_{n,m} \;=\; \sum_{i=m}^{n-1} C_{n-i-1}C_i.$$

Differentiating with respect to y and evaluating at $y = 1$, we then obtain

$$\begin{aligned} \left[\frac{\partial B_1(t,1,y)}{\partial y}\right]_{y=1} &= \frac{\left(1-\sqrt{1-4t}\right)t}{2(1-4t)\sqrt{1-4t}} \\ &= t^2+7t^3+38t^4+187t^5+874t^6+3958t^7+\cdots , \end{aligned}$$

where the coefficient of t^n is equal to $\frac{1}{2}\left((2n-1)\binom{2n-2}{n-1}-4^{n-1}\right)$, which is the total number of flaws in all paths in $\mathcal{G}_n$ ending with a down step.

4.2 The Grand Dyck paths ending with an up step

In this subsection, we enumerate Grand Dyck paths ending with an up step. Since every Grand Dyck path of semilength n either ends with a down step or an up step, the corresponding generating functions satisfy

$$B(t,x,y) = B_1(t,x,y) + B_2(t,x,y).$$

Consequently, most results in this subsection can be derived directly from the formulas obtained in Subsection 3.2 and Subsection 4.1. In particular,

$$[x^k t^n]B(t,x,1) = (n+1)N_{n,k}, \qquad [x^k t^n]B_1(t,x,1) = kN_{n,k},$$

imply immediately that

$$[x^k t^n]B_2(t,x,1) = (n-k+1)N_{n,k}.$$

Hence we obtain the following result.

Theorem 4.4. *Let $D'_{n,k}$ be the number of the Grand Dyck paths of semilength n ending with an up step and having k descents. Then, for $1 \le k \le n$,*

$$D'_{n,k} \;=\; \frac{n-k+1}{n}\binom{n}{k}\binom{n}{k-1} = (n-k+1)N_{n,k}.$$

Differentiating with respect to x and evaluating at $x = 1$, we then obtain

$$\begin{aligned}\left[\frac{\partial B_2(t,x,1)}{\partial x}\right]_{x=1} &= \frac{(1-2t)t}{(1-4t)\sqrt{1-4t}} \\ &= t + 4t^2 + 18t^3 + 80t^4 + 350t^5 + 1512t^6 + \cdots,\end{aligned}$$

where the coefficient of t^n is equal to $n\binom{2n-2}{n-1}$, which is the total number of descents in all paths in $\mathcal{G}_n$ ending with an up step.

By setting $x = 1$ in $B_2(t,x,y)$, we have

$$B_2(t,1,y) \;=\; \frac{1-\sqrt{1-4ty}}{\sqrt{1-4t}+\sqrt{1-4ty}}.$$

Some initial terms of $B_2(t,1,y)$ are

$$\begin{aligned}B_2(t,1,y) &= yt + (y+2y^2)t^2 + (2y+3y^2+5y^3)t^3 + (5y+7y^2+9y^3+14y^4)t^4 \\ &\quad + (14y+19y^2+23y^3+28y^4+42y^5)t^5 + \cdots.\end{aligned}$$

Differentiating with respect to y and evaluating at $y = 1$, we then obtain

$$\begin{aligned}\left[\frac{\partial B_2(t,1,y)}{\partial y}\right]_{y=1} &= \frac{(1+\sqrt{1-4t})t}{2(1-4t)\sqrt{1-4t}} \\ &= t + 5t^2 + 23t^3 + 102t^4 + 443t^5 + 1898t^6 + 8054t^7 + \cdots,\end{aligned}$$

where the coefficient of t^n is equal to $\frac{1}{2}\left((2n-1)\binom{2n-2}{n-1}+4^{n-1}\right)$, which is the total number of flaws in all paths in $\mathcal{G}_n$ ending with an up step.

We let $Q'_n(y) = [t^n]B_2(t,1,y)$, $n \geq 1$. Then

$$\begin{aligned} Q'_n(y) &= [t^n]B_2(t,1,y) = [t^n]tyC(ty)B(t,1,y) \\ &= [t^n]\left(\sum_{n=1}^{\infty} C_{n-1}y^n t^n\right)\left(\sum_{n=0}^{\infty} C_n\left(\sum_{j=0}^{n} y^j\right)t^n\right) \\ &= \sum_{i=0}^{n-1} C_{n-i-1}C_i y^{n-i}\left(\sum_{j=0}^{i} y^j\right) \\ &= \sum_{i=0}^{n-1} C_{n-i-1}C_i \sum_{j=0}^{i} y^{n-i+j} = \sum_{j=0}^{n-1}\sum_{i=0}^{n+j} C_{n-i-1}C_i y^{n-i+j} \\ &= \sum_{m=1}^{n}\sum_{i=0}^{m-1} C_{n-i-1}C_i y^m. \end{aligned}$$

Therefore, we have

$$[y^m]Q'_n(y) = \sum_{i=0}^{m-1} C_{n-i-1}C_i.$$

Hence we can state the following result.

Theorem 4.5. *Let $Q'_{n,m}$ be the number of the Grand Dyck paths of semilength n ending with an up step and having m flaws. Then, for $1 \leq m \leq n$,*

$$Q'_{n,m} = \sum_{i=0}^{m-1} C_{n-i-1}C_i.$$

5 Conclusion

In this work, we proved a refined Chung-Feller theorem for Grand Dyck paths by showing that the joint enumeration by flaws and descents is given by the Narayana numbers $N_{n,k}$ and, remarkably, is independent of the number of flaws. This refined independence phenomenon appears to be new and highlights the robustness of the Chung-Feller property under additional statistics.

We also obtained explicit enumerations for certain natural subclasses of Grand Dyck paths, illustrating further structural consequences of the refinement.

Several directions naturally follow from our results. One is to investigate whether analogous refined Chung-Feller phenomena occur in other families of lattice paths, such as Motzkin or Schröder paths. Another is to study q-analogues or additional statistics (e.g., peaks or valleys) to see whether similar independence properties persist.

Table 1: Integer sequences for Grand Dyck paths with their OEIS identifications

Sequence	**Description**	**OEIS**	**Initial Terms**
$(G_{n,k})_{2\le k\le n}$	Number of paths in $\mathcal{G}_n$ ending with a down step, with k descents and exactly 1 flaw	A119308[20, 30]	1, 2, 1, 3, 5, 1, 4, 14, ...
$(D_{n,k})_{n\ge 1,k\ge 1}$	Number of paths in $\mathcal{G}_n$ ending with a down step and having k descents	A132813[20]	1, 1, 2, 1, 6, 3, 1, 12, ...
$\left[\frac{\partial B_1(t,x,1)}{\partial x}\right]_{x=1}$	Total number of descents in all paths in $\mathcal{G}_n$ ending with a down step	A141222[20]	1, 5, 22, 95, 406, 1722, 7260, 30459, ...
$B_1(t,1,1)$	Number of paths in $\mathcal{G}_n$ ending with a down step	A001700[20]	1, 3, 10, 35, 126, 462, 1716, 6435, ...
$(Q_{n,m})_{n\ge 1,m\ge 0}$	Number of paths in $\mathcal{G}_n$ ending with a down step and having m flaws	A067323[20]	1, 2, 1, 5, 3, 2, 14, 9, ...
$\left[\frac{\partial B_1(t,1,y)}{\partial y}\right]_{y=1}$	Total number of flaws in all paths in $\mathcal{G}_n$ ending with a down step	A000531[20]	1, 7, 38, 187, 874, 3958, 17548, 76627, ...
$(D'_{n,k})_{n\ge 1,k\ge 1}$	Number of paths in $\mathcal{G}_n$ ending with an up step and having k descents	A103371[20]	1, 2, 1, 3, 6, 1, 4, 18, ...
$\left[\frac{\partial B_2(t,x,1)}{\partial x}\right]_{x=1}$	Total number of descents in all paths in $\mathcal{G}_n$ ending with an up step	A037965[20]	0, 1, 4, 18, 80, 350, 1512, 6468, ...
$\left[\frac{\partial B_2(t,1,y)}{\partial y}\right]_{y=1}$	Total number of flaws in all paths in $\mathcal{G}_n$ ending with an up step	A258431[20]	0, 1, 5, 23, 102, 443, 1898, 8054, ...
$(Q'_{n,m})_{n\ge 1,m\ge 1}$	Number of paths in $\mathcal{G}_n$ ending with an up step and having m flaws	A028364[20]	1, 1, 2, 2, 3, 5, 5, 7, ...

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