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Intrinsically knotted graphs and connected domination

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Abstract

We classify all the maximal linklessly embeddable graphs of order 12 and show that their complements are all intrinsically knotted. We derive results about the connected domination numbers of a graph and its complement. We provide an answer to an open question about the minimal order of a 3-non-compliant graph. We prove that the complements of knotlessly embeddable graphs of order at least 15 are all intrinsically knotted. We provide results on general k-non-compliant graphs and leave a set of open questions for further exploration of the subject.

1 Introduction

Sixty years ago, Battle, Harary, and Kodama [2] proved that the complement of a planar graph of order nine is not planar. This result was independently proved by Tutte [25]. We express this fact by saying that the complete graph K_9 is not *bi-planar*. In general, for a graph property $\mathcal{P}$, we say that K_n is *bi-*$\mathcal{P}$, if there exists a graph G of order n such that both G and its complement, $\overline{G}$, have the property $\mathcal{P}$.

Since then, many similar results were derived, some of them generalizing the original one. Ichihara and Mattman [9] proved that for all nonnegative integers t, the complete graph K_{2t+9} is not bi-t-apex. A graph is called *t-apex* if there exists a choice of t of its vertices which deleted yield a planar graph. Odeneal, Naimi, and the last two authors [17] proved that K_n is not bi-linklessly embeddable (bi-nIL), for all $n \geq 11$. A graph is *intrinsically linked* (*IL*) if every embedding of it in $\mathbb{R}^3$ (or S^3) contains a nonsplittable 2-component link. A graph is *linklessly embeddable* if it is not intrinsically linked (*nIL*). The $n \geq 11$ bound is sharp.

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In this article, we investigate bi-knotless embeddability. A graph is called *intrinsically knotted* if every embedding of it in S^3 contains a nontrivial knot. We abbreviate intrinsically knotted as "IK", and not intrinsically knotted as "nIK". We try to answer the following question: what is the smallest integer n, such that K_n is not bi-nIK? We employ some of the techniques used by the last two authors in [21], where they proved K_{13} is not bi-nIL. In the process, we find all the 6503 maximal nIL graphs of order 12. We focus on finding a large degree vertex in a minor of either the graph or its complement. This, in turn, leads to the study of the connected domination numbers for both G and $\overline{G}$. We then prove:

Theorem 1.1. *Let G be a simple graph on* 15 *vertices and let $\overline{G}$ denote its complement. If G is nIK, then $\overline{G}$ is IK.*

All the graphs considered in this article are simple: non-oriented, without loops or multiple edges. For a given graph G, we let $V(G)$ denote its vertex set, and $E(G)$ denote its edge set. Given a graph G, the *complement* graph $\overline{G}$ has the same vertex set as G and $E(\overline{G}) = \{(u, v) \mid u, v \in V(G) \text{ and } (u, v) \notin E(G)\}$. The maximum degree among all vertices of G is $\Delta(G)$, and the minimum degree is $\delta(G)$. Let $\delta^*(G) := \min\{\delta(G), \delta(\overline{G})\}$. For a vertex $u \in V(G)$, the set $N_G(u)$ is the set of all vertices of G that are adjacent to u, and $N_G[u] = N_G(u) \sqcup \{u\}$. For a given graph G, we denote the number of edges (*size*) of G by $|G|$.

A set $S \subseteq V(G)$ is a *dominating set* of G if every vertex in $V \backslash S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of dominating sets S of G. The *connected domination number* $\gamma_c(G)$ of G is the minimum cardinality of dominating sets S of G that induce a *connected* subgraph $G[S]$ of G.

Some of the necessary results on connected domination were readily available by the work of Karami, Sheikholeslami, Khodkar, and West [11]. They proved a Nordhaus-Gaddum type relation for the connected domination number.

Theorem 1.2 (Karami et al. [11]). *If G and $\overline{G}$ are both connected graphs on n vertices with $\gamma_c(G), \gamma_c(\overline{G}) \geq 4$, then*

$$\gamma_c(G) + \gamma_c(\overline{G}) \leq \delta^*(G) + 2.$$

They showed that equality can only hold when $\delta^*(G) = 6$ and provided an example of a simple graph G with $\gamma_c(G) = \gamma_c(\overline{G}) = 4$, and $\delta^*(G) = 6$. However, this example has nearly 15,000 vertices. They asked what is the minimum order among all such examples. We prove that the Paley graph QR_{13} of order 13 is the minimal example.

A *minor* of a graph G is any graph that can be obtained from G by a sequence of vertex deletions, edge deletions, and edge contractions. An edge contraction is performed by identifying the endpoints of an edge in $E(G)$ and then deleting any loops and double edges thus created.

For $k \geq 1$, we say that G with n vertices is *k-compliant* when G or $\overline{G}$ contains a minor H with $\Delta(H) \geq n - k$. Otherwise, G is *k-non-compliant*. It follows that for $k \geq 1$, if G is k-compliant then G is $(k+1)$-compliant. Notice that a graph G is 1-compliant if and only if G has a dominating vertex (*cone*) or an isolated vertex. A graph is 2-compliant if and only if, in either the graph or its complement, one edge contraction yields a dominant vertex. We also note that a disconnected graph is 2-compliant. The results about general k-non-compliant graphs, proved in Section 5, are independent of the results in previous sections. Among them, we prove Theorem 5.1 which states G is k-compliant if and only if $\min\{\gamma_c(G), \gamma_c(\overline{G})\} \leq k$. We will use the result of Theorem 5.1 throughout this article. In this setting, the question posed by Karami et al. [11] is equivalent to finding the minimum order of a 3-non-compliant graph. This minimum order is 13 and we have Thorem 1.3.

Theorem 1.3. *Up to isomorphism, the Paley* 13*-graph* QR_{13} *is the only* 3*-non-compliant graph of order* 13*.*

We also find all 3-non-compliant graphs of order 14.

Theorem 1.4. *Up to isomorphism, there are only two* 3*-non-compliant graphs of order* 14*: the graph obtained by adding a new vertex to* QR_{13} *connected to all the vertices of the open neighborhood of an existing vertex, and the graph obtained by adding a new vertex to* QR_{13} *connected to all the vertices of the closed neighborhood of an existing vertex.*

2 Three-Non-Compliant Graphs

The following result follows directly from the definition of a 3-non-compliant graph.

Lemma 2.1. *Let G be a graph that is* 3*-non-compliant, and let $S \subseteq V(G)$ have cardinality* 3*. Then*

- *if $G[S]$ is connected, there exists $v \in V \setminus S$ such that $(u,v) \notin E(G)$ for all $u \in S$.*
- *if $\overline{G}[S]$ is connected, there exists $v \in V \setminus S$ such that $(u,v) \in E(G)$ for all $u \in S$.*

Lemma 2.2. *For G a* 3*-non-compliant graph,*

$$|N_G(u) \cap N_G(v)| \geq 3,$$

for any $(u,v) \notin E(G)$.

Proof. Let us assume on the contrary G is 3-non-compliant and $|N_G(u) \cap N_G(v)| \leq 2$ for $(u,v) \notin E(G)$. If there exists a vertex $w \in N_{\overline{G}}(u) \cup N_{\overline{G}}(v)$ such that $(w, v_i) \in E(\overline{G})$ for each $v_i \in N_G(u) \cap N_G(v)$, then we see $\gamma_c(\overline{G}) \leq 3$ by taking $S = \{u, v, w\}$.

Otherwise, for each vertex $w \in N_{\overline{G}}(u) \cup N_{\overline{G}}(v)$, we have $(w, v_i) \in E(G)$ for some $v_i \in N_G(u) \cap N_G(v)$. By taking $S = (N_G(u) \cap N_G(v)) \sqcup \{u\}$ or $(N_G(u) \cap N_G(v)) \sqcup \{v\}$, we see $\gamma_c(G) \leq 3$.

In either case, if $|N_G(u) \cap N_G(v)| \leq 2$, there is a contradiction, and we conclude $|N_G(u) \cap N_G(v)| \geq 3$. □

Lemma 2.3. *For G a 3-non-compliant graph,*

$$|N_G(u) \cap N_G(v)| \geq 2,$$

for any $(u, v) \in E(G)$.

Proof. Let us assume on the contrary G is 3-non-compliant and $|N_G(u) \cap N_G(v)| \leq 1$ for $(u, v) \in E(G)$. If $|N_G(u) \cap N_G(v)| = 0$, since G is 3-non-compliant, it must be that there exists $w \in V(G) \setminus \{u, v\}$ such that $(v, w), (u, w) \notin E(G)$. By Lemma 2.1, it follows that u, v, and w must share a neighbor in $V(G)$, thus $|N_G(u) \cap N_G(v)| > 0$. Thus $|N_G(u) \cap N_G(v)| = 1$, that is, there is a vertex $w_0 \in V(G)$ such that $N_G(u) \cap N_G(v) = \{w_0\}$. In $\overline{G}$, by Lemma 2.1, the set $\{u, v, w_0\}$ must share a common neighbor, say x. Then $\{u, v, x\}$ forms a connected dominating set in $\overline{G}$, a contradiction. □

Corollary 2.4. If a graph G of order n is 3-non-compliant, then $n \geq 13$.

Proof. Let G be a 3-non-compliant graph. Then $\gamma_c(G), \gamma_c(\overline{G}) \geq 4$, and both G and $\overline{G}$ are connected. Theorem 1.2 implies $8 \leq \delta^*(G) + 2$, that is $\delta^*(G) \geq 6$. The smallest possible order for G with $\delta^*(G) \geq 6$ is $n = 13$. □

Remark 2.5. By the argument of Corollary 2.4, for a 3-non-compliant graph G of order $n \geq 13$, the degree of every vertex of G and the degree of every vertex of $\overline{G}$ must range between 6 and $n - 7$. This implies that QR_{13} is also the minimal 3-non-compliant graph by size.

If G is a 3-non-compliant graph of order 13, by the proof of Corollary 2.4, G must be 6-regular. There are 367,860 non-isomorphic 6-regular graphs of order 13. However, only one of them is 3-non-compliant. In [21], the last two authors gave another proof for the following theorem. We present a shorter proof which uses Lemmas 2.2 and 2.3.

Theorem 1.3. *Up to isomorphism, the Paley 13-graph QR_{13} is the only 3-non-compliant graph of order 13.*

Proof. By Corollary 2.4, if G is a 3-non-compliant graph of order 13, it must be 6-regular. Let $v \in V(G)$, let N denote the subgraph of G induced by $N_G(v)$, let $H = G \setminus N_G[v]$ be the subgraph of G induced by the non-neighbors of v, and let L be the bipartite graph whose edges have one endpoint in N and one endpoint in H. By Lemma 2.2, each vertex of H must be adjacent to at least 3 vertices of N, thus $|L| \geq 6 \cdot 3 = 18$. By Lemma 2.3, $\delta(N) \geq 2$, thus, by the handshaking lemma, $|N| \geq 6$. Adding the degrees in G of the vertices of N, we get

$$36 = \sum_{v \in N} \deg_G(v) = 6 + 2|N| + |L| \geq 6 + 12 + 18 = 36,$$

which shows that $|N| = 6$ and N is 2-regular, and that $|L| = 18$ and L is 3-regular. Since the same must hold for every vertex of G, lest G be 3-compliant, it follows that G is a strongly regular $(13, 6, 2, 3)$ graph. This means G is a 6-regular graph of order

13, where any two adjacent vertices have exactly two common neighbors, and any two non-adjacent vertices have exactly 3 common neighbors. Thus G is isomorphic to the Paley 13-graph [1]. See Figure 1.

On the other hand, if G is isomorphic to the Paley 13-graph, up to isomorphism, there are exactly 13 non-isomorphic minors obtained by contracting two edges of G, and each of them has maximum degree at most 9. Since the Paley 13-graph is self-complementary, it follows that G is 3-non-compliant. □

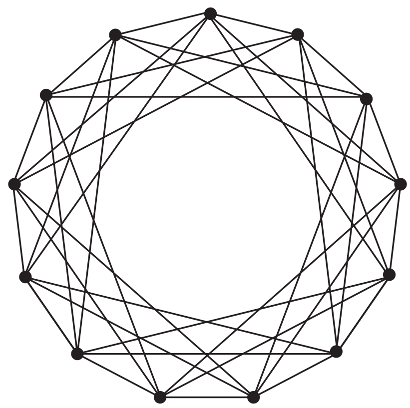

Figure 1: The Paley graph of order 13.

Proposition 2.6. *Let u and v be twin vertices of a graph G ($N_G(u) = N_G(v)$). For $k \geq 2$, G is k-non-compliant if and only if $G' := G - v$ is k-non-compliant.*

Proof. Note that if u and v are twin vertices of G, then they are also twin vertices of $\overline{G}$. So, without loss of generality, we may assume $(u, v) \in E(G)$.

Assume that G is k-non-compliant and that G' is k-compliant. If $\gamma_c(G') \leq k$, let $S = \{u_1, ..., u_p\}$, with $p \leq k$, be a connected dominating set in G'. Since S is dominating, there exist $1 \leq i \leq p$ such that $(u, u_i) \in E(G')$. Since u and v were twin vertices in G, we have the adjacency $(v, u_i) \in E(G)$, so S is a connected dominating set of G, a contradiction.

If $\gamma_c(\overline{G'}) \leq k$, then let $S = \{u_1, ..., u_p\}$, with $p \leq k$, be a connected dominating set of $\overline{G'}$. Since S is dominating, there is some $1 \leq i \leq p$ such that $(u, u_i) \in E(\overline{G'})$. As u and v are twins, we have $(v, u_i) \in E(\overline{G})$, so S dominates $\overline{G}$.

Assume that G is k-compliant and that G' is k-non-compliant. If $\gamma_c(G) \leq k$, let $S = \{u_1, ..., u_p\}$, with $p \leq k$, be a connected dominating set in G. As G' is k-non-compliant, we have $v \in S$. Since u and v have the same neighbors in G, $(S - v) \cup \{u\}$ is a connected dominating set of G' of size p. This is a contradiction, since G' is k-non-compliant.

If $\gamma_c(\overline{G}) \leq k$, let $S = \{u_1, ..., u_p\}$, with $p \leq k$, be a connected dominating set in $\overline{G}$. If $v \in S$, then $(S - v) \cup \{u\}$ is a connected dominating set of $\overline{G'}$. If $v \notin S$, then S is a connected dominating set of $\overline{G'}$. In either case, G' must be k-compliant, a contradiction. □

Note that deleting a vertex from the Paley 13 graph yields a subgraph of order 12 which, by Corollary 2.4, must be 3-compliant, but 2-non-compliant. We pose the following question:

Question 1: Under what condition does adding a vertex to a k-compliant graph yield a k-non-compliant graph?

Proposition 2.6 allows one to build 3-non-compliant graphs of order at least 13 by repeatedly adding twin vertices. For order 14, one obtains two non-isomorphic structures: adding a new vertex v to the open/closed neighborhood of a vertex in the QR_{13} graph. Theorem 1.4 shows that these two are the only 3-non-compliant graphs of order 14.

Lemmas 2.2 and 2.3 imply that if G is 3-non-compliant, then every pair of adjacent vertices has at least two common neighbors and every pair of non-adjacent vertices has at least three common neighbors. For order $n = 14$, the possible degrees for a vertex of a 3-non-compliant graph are 6 and 7. This implies the number of edges for a 3-non-compliant graph of order 14 is $42 \leq m \leq 49$. To find all 3-non-compliant graphs, it suffices to find those graphs with size $42 \leq m \leq 45$, and then check their complements. More restrictions can be derived by inspecting the possible configurations of the neighborhoods of vertices of degree 6.

Lemma 2.7. *Let G be a graph with 14 vertices and $u \in V(G)$ with $\deg_G(u) = 6$. If G is 3-non-compliant, then*

$$\sum_{v \in N_G(u)} \deg_G(v) \geq 39.$$

Proof. Let N be the subgraph induced by $N_G(u)$, let H be the subgraph induced by $V(G) \setminus N_G[u]$, and let L be the bipartite graph whose edges have one endpoint on N and one endpoint in H. By Lemma 2.2, each vertex of H shares at least three neighbors with u, thus $|L| \geq 21$. By Lemma 2.3, u shares at least two neighbors with each of of its neighbors, thus $\deg_N(v) \geq 2$, for each $v \in N$ and thus, by the handshaking lemma, $|N| \geq 6$. We have

$$\sum_{v \in N_G(u)} \deg_G(v) = 6 + 2|N| + |L| \geq 6 + 12 + 21 = 39.$$

□

The next two lemmas partially strengthen the results of Lemmas 2.2 and 2.3 for graphs of order 14.

Lemma 2.8. *Let G be a 3-non-compliant graph of order 14. Let $u, v \in V(G)$, such that $deg_G(u) = deg_G(v) = 7$, and $(u, v) \notin E(G)$. Then $|N_G(u) \cap N_G(v)| \geq 4$.*

Proof. By Lemma 2.3, u and v share at least two neighbors in $\overline{G}$. As u and v are not adjacent in G, we have

$$10 \geq |N_G(u) \cup N_G(v)| = |N_G(u)| + |N_G(v)| - |N_G(u) \cap N_G(v)| = 7 + 7 - |N_G(u) \cap N_G(v)|,$$

which implies

$$|N_G(u) \cap N_G(v)| \geq 14 - 10 = 4.$$

□

Lemma 2.9. *Let G be a 3-non-compliant graph of order 14. Let $u, v \in V(G)$, such that $\deg_G(u) = \deg_G(v) = 7$, and $(u, v) \in E(G)$. Then $|N_G(u) \cap N_G(v)| \geq 3$.*

Proof. By Lemma 2.2, u and v share at least three neighbors in $\overline{G}$. As u and v are adjacent in G, we have

$$11 \geq |N_G(u) \cup N_G(v)| = |N_G(u)| + |N_G(v)| - |N_G(u) \cap N_G(v)| = 7 + 7 - |N_G(u) \cap N_G(v)|,$$

which implies

$$|N_G(u) \cap N_G(v)| \geq 14 - 11 = 3.$$

□

Lemma 2.10. *Graphs G of order 14 with 42, 43, or 44 edges are 3-compliant.*

Proof. For graphs G of size 42, 43, consider a vertex u with $\deg(u) = 6$. Since there are at most two vertices of degree 7 in G, the sum of degrees for the neighbors of u is strictly less than 39. By Lemma 2.7, the graph G is 3-compliant.
For graphs G of size 44, we only need to consider graphs with four vertices of degree 7 and ten vertices of degree 6. Using the pigeonhole principle, we see that there exists a vertex of degree 6 which neighbors at most two vertices of degree 7. By Lemma 2.7, the graph G is 3-compliant. □

Lemma 2.11. *Let G be a graph with 14 vertices, size 45, and $u \in V(G)$ with $\deg_G(u) = 7$. If G is 3-non-compliant, then*

$$\sum_{v \in N_G(u)} \deg_G(v) \geq 44.$$

Proof. Let $N = N_G(u)$ and let $H = N_{\overline{G}}(u)$. By Lemma 2.7,

$$\sum_{v \in H} \deg_{\overline{G}}(v) \geq 39 \Rightarrow \sum_{v \in N_{\overline{G}}[u]} \deg_{\overline{G}}(v) \geq 45.$$

By the handshaking lemma,

$$\sum_{v \in N} \deg_{\overline{G}}(v) \leq 2|\overline{G}| - 45 = 92 - 45 = 47.$$

Taking complements,

$$\sum_{v \in N} \deg_G(v) \geq 7 \cdot 13 - 47 = 91 - 47 = 44.$$

□

Theorem 1.4. *Up to isomorphism, there are only two 3-non-compliant graphs of order 14: the graph obtained by adding a new vertex to QR_{13} connected to all the vertices of the open neighborhood of an existing vertex, and the graph obtained by adding a new vertex to QR_{13} connected to all the vertices of the closed neighborhood of an existing vertex.*

Proof. Since a graph is 3-non-compliant if and only if its complement is 3-non-compliant, by Lemma 2.10 it suffices to consider simple graphs of order 14 and size 45. Let's assume $|G| = 45$. Since $\delta^*(G) = 6$, it follows that the degree sequence of G is $(6, 6, 6, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7)$. Let A denote the subgraph of G induced by the vertices of degree 7, let B denote the subgraph of G induced by the vertices of degree 6, and let P denote the bipartite graph obtained by deleting all the edges of A and all the edges of B from G. By Lemma 2.7, every vertex of degree 6 must connect to at least three vertices of degree 7. It follows that $\Delta(B) \leq 3$ and that $|B| \leq 12$. By Lemma 2.11, it follows that $\delta(A) \geq 2$, and thus $|A| \geq 6$.

$$42 = \sum_{v \in A} \deg_G(v) = 2|A| + |P| \geq 12 + |P| \Rightarrow |P| \leq 30.$$

$$48 = \sum_{v \in B} \deg_G(v) = 2|B| + |P| \leq 24 + |P| \Rightarrow |P| \geq 24.$$

Either relation shows that $|P|$ is even, and thus $|P| \in \{24, 26, 28, 30\}$.

If $|P| = 24$, then $|A| = 9$, $2 \leq \delta(A) \leq 3$, $|B| = 12$, and every vertex of degree 6 is adjacent to exactly three vertices of degree 7. There are 15 non-isomorphic structures possible for A, as seen in Figure 2.

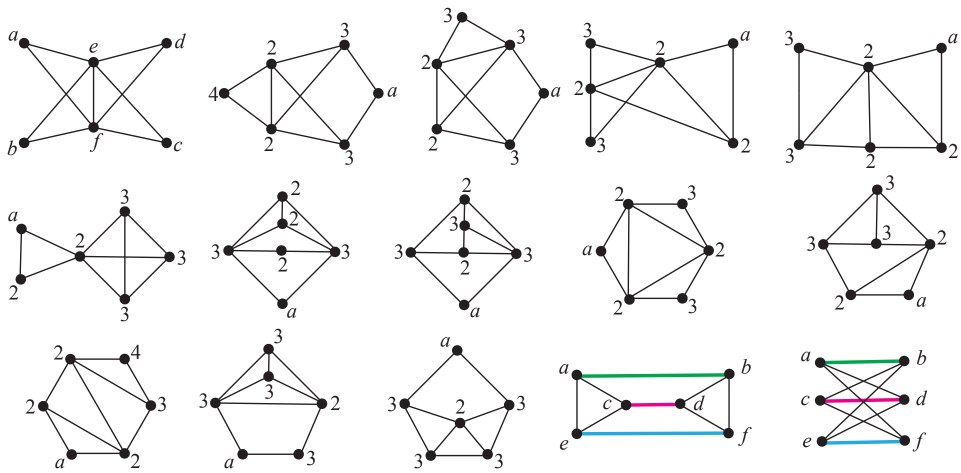

Figure 2: The 15 graphs of order 6, size 9, and minimum degree at least 2.

If $\delta(A) = 3$, then A is 3-regular and is therefore isomorphic to either the complete bipartite graph $K_{3,3}$, or the triangular prism (the last two graphs in Figure 2). In either case, the edges (a, b), (c, d), and (e, f) are non-triangular in A, with both their endpoints vertices of degree 7. By Lemma 2.9, the endpoints of each of these edges must share at least three neighbors among the vertices of B. As B has order 8, the pigeonhole principle implies there is a vertex of degree 6 which is adjacent to at least 4 vertices of A, a contradiction.

If $\delta(A) = 2$, then A is among the first 13 graphs in Figure 2. For a vertex $v \in V(A)$, let $p_2(v)$ denote the number of length-two paths containing v with both endpoints in A and the middle vertex in B. Any vertex of degree 2 in A must be adjacent to exactly 5 vertices of B. Since every vertex of B is adjacent to exactly 3 vertices of A, then $p_2(v) \leq 10$. On the other hand, without exception, there is always a vertex of degree 2 in A, labeled a for all 13 graphs in Figure 2, for which $p_2(a) > 10$.

We provide here the computation for the first graph as an example, and we leave the rest to the reader. Note that in Figures 2, 3, and 4, the numerical values associated to vertices v different from a represent lower bounds for the number of length-two paths with endpoints a and v, and middle vertex in B.

By Lemma 2.9, a and e have at least three common neighbors. Since they only share one neighbor in A, they must have at least two common neighbors among the vertices of B. The same argument works for a and f. By Lemma 2.8, a and b share at least 4 neighbors, with at least two of them among the vertices of B. However, if a and b have exactly two common neighbors among the vertices of B, then $\{a, e, b\}$ is a connected dominating set of G, so G is 3-compliant. This argument shows that a and b share at least three neighbors among the vertices of B. The same must be true for each of the pairs $\{a, c\}$ and $\{a, d\}$. It follows that a must be in at least $2 + 2 + 3 + 3 + 3 = 13 > 10$ pairs of vertices of A with a shared vertex in B, a contradiction.

So, if $|P| = 24$, G must be 3-compliant.

If $|P| = 26$, then $|A| = 8$, $\delta(A) = 2$, $|B| = 11$, and every vertex of degree 6 is adjacent to at least three vertices of degree 7. There are 11 non-isomorphic structures possible for A, depicted in Figure 3.

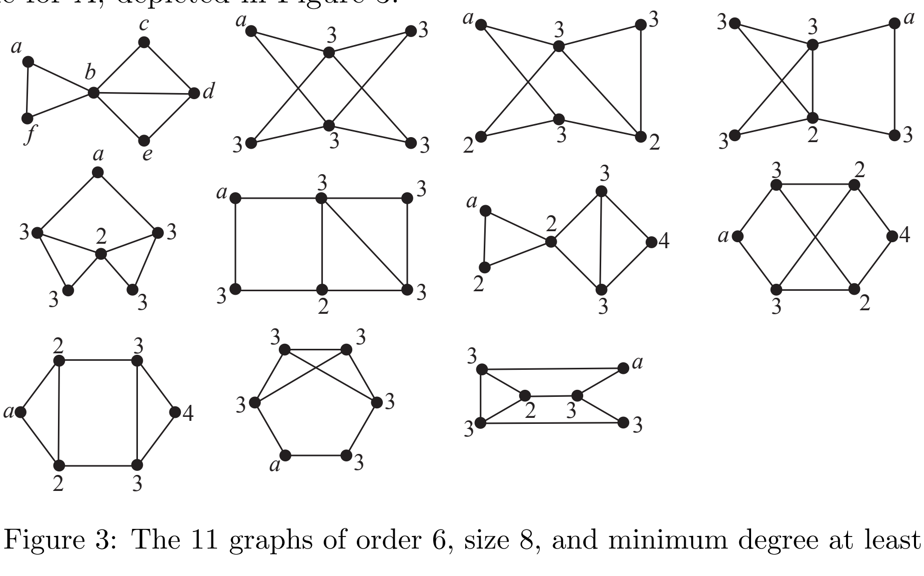

Figure 3: The 11 graphs of order 6, size 8, and minimum degree at least 2.

Every vertex v of A with $\deg_A(v) = q \geq 2$ is adjacent to $7 - q$ vertices of B.

Since every vertex of B is adjacent to at least three vertices of A, it follows that $p_2(v) \leq 2 \cdot (7 - q) + 26 - 24 = 16 - 2q$. In particular, for any vertex v of degree 2 of A, $p_2(v) \leq 12$. However, for each of the graphs in Figure 3, by Lemmas 2.8 and 2.9, there exists a vertex of degree 2, labeled a, with $p_2(a) > 12$. We provide the computations for the first graph in the list, and leave the rest to the reader. We note that for the second graph in Figure 3, the argument is identical to the case covered when $|P| = 24$. By Lemma 2.9, a and b have at least three common neighbors. Since they share a neighbor in A, they must share at least two neighbors in B. The same argument holds for the pair $\{a, f\}$. Each of the pairs $\{a, c\}$, $\{a, d\}$, and $\{a, e\}$ must share at least 4 neighbors in G, by Lemma 2.8. However, each of them has only one common neighbor in A, giving them each at least three common neighbors in B. Then, $p_2(a) \geq 2 + 2 + 3 + 3 + 3 = 13 > 12$.

So, if $|P| = 26$, G must be 3-compliant.

If $|P| = 28$, then $|A| = 7$, $\delta(A) = 2$, $|B| = 10$, and every vertex of degree 6 is adjacent to at least three vertices of degree 7. There are 5 non-isomorphic structures possible for A, depicted in Figure 4.

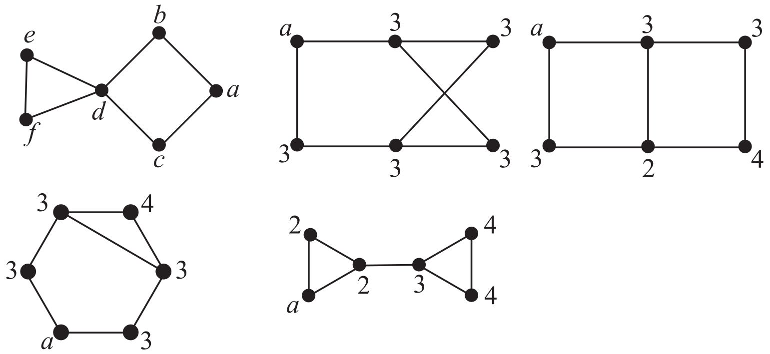

Figure 4: The 5 graphs of order 6, size 7, and minimum degree at least 2.

Every vertex v of A with $\deg_A(v) = q \geq 2$ is adjacent to $7 - q$ vertices of B. Since every vertex of B is adjacent to at least three vertices of A, it follows that $p_2(v) \leq 2 \cdot (7 - q) + 28 - 24 = 18 - 2q$. In particular, every vertex v of degree 2 in any of the graphs in Figure 4 must have $p_2(v) \leq 14$. However, each of the graphs in Figure 4, has a vertex of degree 2, labeled a, with $p_2(a) > 14$. We provide the computations for the first graph in the list, and leave the rest to the reader.

By Lemma 2.9, each pair $\{a, b\}$ and $\{a, c\}$ must have at least 3 common neighbors in B, since neither shares neighbors in A. By Lemma 2.8, a and d must have at least 4 common neighbors, at least 2 of which must be in B. By Lemma 2.8, each pair $\{a, e\}$ and $\{a, f\}$ must have at least 4 common neighbors in B, since neither shares neighbors in A. Thus $p_2(a) \geq 3 + 3 + 2 + 4 + 4 = 16 > 14$, a contradiction.

So, if $|P| = 28$, G must be 3-compliant.

If $|P| = 30$, then $|A| = 6$, A is 2-regular, $|B| = 9$, and every vertex of degree 6 is adjacent to at least three vertices of degree 7. There are only two structures possible for A: the disjoint union $K_3 \sqcup K_3$ and the cycle graph C_6, as in Figure 5.

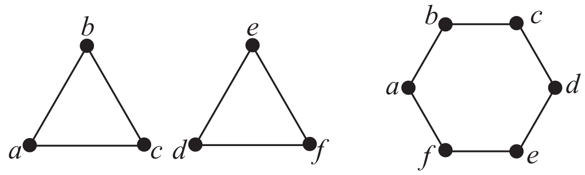

Figure 5: The 2-regular graphs of order 6.

Since A is 2-regular, each vertex of A is adjacent to exactly 5 vertices of B. As each vertex of B is adjacent to at least three vertices of A, each vertex of A can be in at most $2 \cdot 5 + 30 - 24 = 16$ pairs of vertices of A which share a neighbor in B.

If $A \simeq K_3 \sqcup K_3$, then by Lemma 2.9, each pair $\{a, b\}$ and $\{a, c\}$ must have at least 2 neighbors in B, since they each share exactly one neighbor in A. By Lemma 2.8, each of the pairs $\{a, d\}$, $\{a, e\}$, and $\{a, f\}$ must share at least 4 neighbors in B, since they share no neighbors in A. As A is vertex-transitive, it follows that each vertex of A must be in at least $2 + 2 + 4 + 4 + 4 = 16$ pairs of vertices in A with a shared vertex in B. This can only happen if exactly two vertices of B are adjacent to all the vertices of A, and each of the remaining vertices of B is adjacent to exactly 3 vertices of A. Let v be a vertex of B which is adjacent to all vertices of A. If a and b share only two neighbors in B, then, as a set, $\{a, b\}$ is incident to every vertex of B, and thus $\{a, v, b\}$ is a connected dominating set and G is 3-compliant. Else, a and b share at least three neighbors in B and thus a must be part of at least $3 + 2 + 4 + 4 + 4 = 17 > 16$ pairs of vertices of A with a shared neighbor in B, a contradiction.

If $A \simeq C_6$, then by Lemma 2.9, each pair $\{a, b\}$ and $\{a, f\}$ must have at least 3 common neighbors in B, since they share no neighbors in A. By Lemma 2.8, each of the pairs $\{a, c\}$ and $\{a, e\}$ must share at least 3 neighbors in B, since they share exactly one neighbor in A. By Lemma 2.8, a and d must have at least 4 common neighbors in B, as they have no shared neighbors in A. Since A is vertex-transitive, it follows that each vertex of A must be in at least $3 + 3 + 3 + 3 + 4 = 16$ pairs of vertices in A with a shared vertex in B. This can only happen if exactly two vertices of B are adjacent to all the vertices of A, and each of the remaining vertices of B is adjacent to exactly 3 vertices of A. Then G has a pair of twin vertices u and v of degree 6, adjacent to all the vertices of A. By Proposition 2.6, $G - v$ is a 3-non-compliant graph of order 13. But Theorem 1.3 shows that $G - v \simeq QR_{13}$, thus G is obtained by adding a twin vertex which connects to the open neighborhood of a vertex in QR_{13}. Theorem 1.3 guarantees that G is 3-non-compliant along with its complement $\overline{G}$, which is the graph obtained by adding a twin vertex which connects to the closed neighborhood of a vertex in QR_{13}. □

Note that the only 3-non-compliant graphs of order 14 have the graph QR_{13} as an induced subgraph. We also found fourteen 3-non-compliant graphs of order 15 (seven pairs) which all have a subgraph isomorphic to QR_{13}. We provide their edge lists in Appendix A. On the other hand, the graph QR_{17} is 3-non-compliant, yet it does not have any subgraph isomorphic to QR_{13}. However, QR_{13} is a minor of QR_{17}. We would like to pose the following questions.

Question 2: What is the complete list of 3-non-compliant graphs of order 15?

Question 3: Is there an example of a 3-non-compliant graph which does not have QR_{13} as a minor?

3 Three-non-compliant graphs of order 15

Lemma 3.1. *Let G be a graph with 15 vertices and $u \in V(G)$ with $\deg_G(u) = 6$. If G is 3-non-compliant, then*

$$\sum_{v \in N(u)} \deg_G(v) \geq 42.$$

Proof. Let N be the subgraph induced by $N_G(u)$, let $H = G \setminus N_G[u]$, and let L be the bipartite graph whose edges have one endpoint in N and one endpoint in H. By Lemma 2.2, each vertex of H shares at least three neighbors with u, thus $|L| \geq 24$. By Lemma 2.3, u shares at least two neighbors with each of of its neighbors, thus $\deg_N(v) \geq 2$, for each $v \in V(N)$ and $|N| \geq 6$. We have

$$\sum_{v \in V(N)} \deg_G(v) = 6 + 2|N| + |L| \geq 6 + 12 + 24 = 42.$$

□

Lemma 3.2. *For $n = 15$, graphs G with $m = 45, 46, 47$ or 48 edges are 3-compliant.*

Proof. By Remark 2.5, the degrees of the vertices of G range between 6 and 8. If there is no vertex of degree 6, by the handshaking lemma, the size of the graph is at least $7 \times 15/2 = 52.5$, a contradiction. For $m = 45, 46, 47$, consider a vertex $u \in V(G)$ with $\deg(u) = 6$. The sum of degrees for the neighbors of u is strictly less than 42. By Lemma 3.1, the graph G is 3-compliant.
For $m = 48$, independent of its degree sequence, the graph G has at least nine vertices of degree 6. There is at least one vertex u with $\deg(u) = 6$ such that u does not neighbor all vertices of degree 7 or 8. Then the sum of degrees for the neighbors of u is strictly less than 42. By Lemma 3.1, the graph G is 3-compliant. □

Lemma 3.3. *Let G be a graph with 15 vertices and $v \in V(G)$ with $\deg_G(v) = 6$. If*

$$\sum_{u \in N(v)} \deg_G(u) \leq 45,$$

then at least one of the following three statements is true: G is 3-compliant; G has a K_7 minor; $\overline{G}$ has a K_7 minor.

Proof. Let N be the subgraph induced by $N_G(v)$, let $H = G \setminus N_G[v]$, and let L be the bipartite graph whose edges have one endpoint in N and one endpoint in H. We denote by $v_1, v_2, \ldots, v_8$ the vertices of H.

Assume G is 3-non-compliant. By Lemmas 2.2 and 2.3, v shares three neighbors with each vertex of H and two neighbors which each vertex of N. This means $\deg_L(u) \geq 3$ for all $u \in H$, and $\deg_N(u) \geq 2$ for each $u \in N$. This means $|L| \geq 24$ and $|N| \geq 6$, and since

$$\sum_{u \in N(v)} \deg_G(u) = 6 + 2|N| + |L| \leq 45,$$

$|N| = 6$ or $|N| = 7$. With these restrictions, N can be one of the seven graphs in Figure 6. The respective complements of these seven graphs can be seen in Figure 7.

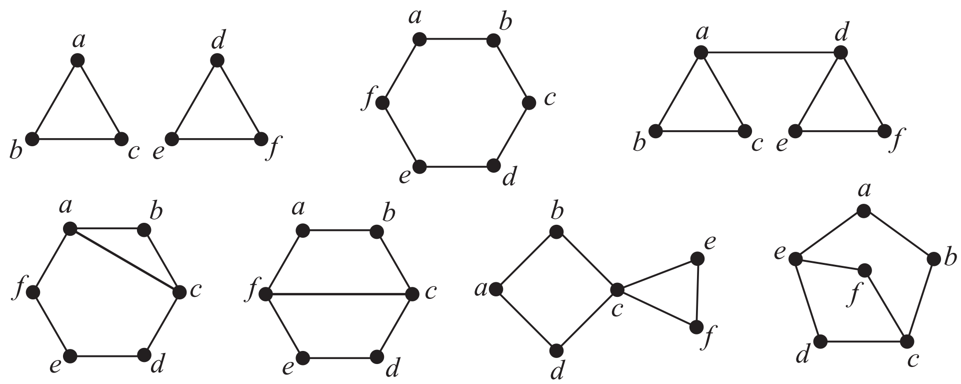

Figure 6: Graphs of order 6, size 6 or 7, and minimal degree 2.

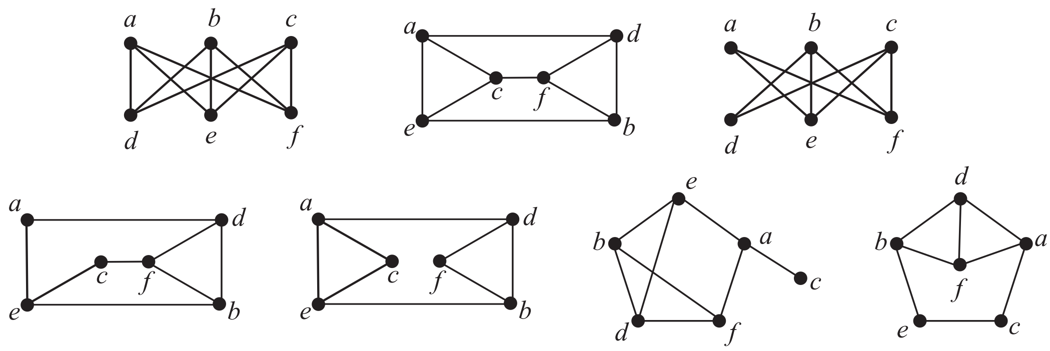

Figure 7: Complements of the seven graphs of order 6, size 6 or 7, and minimal degree 2.

In each of these seven cases, we show that under the assumption that G is 3-non-compliant, either G or $\overline{G}$ has a K_7 minor. We primarily work in the complement $\overline{G}$. We note that regardless of N, in the complement $\overline{G}$, each pair of vertices of N have a common neighbor in $\overline{H}$. Otherwise, this pair of vertices together with v form a connected domination set of cardinality 3 in G, contradicting the assumption that G is 3-non-compliant. We also note that each vertex of $\overline{H}$ neighbors at most three vertices of $\overline{N}$. In each case below we consider a graph N as labeled in Figure 6.

Case 1. N is a disjoint union of two triangles. A vertex of H, say v_1, cannot neighbor all three vertices a, b, c in $\overline{G}$. Otherwise $\{v, v_1, b\}$ dominates $\overline{G}$. Similarly, a vertex of H cannot neighbor all vertices d, e, f. It follows that the edges $(a, b), (b, c), (a, c), (d, e), (d, f)$, and (e, f) are all covered by a different vertex of $\overline{H}$. We say a vertex *covers* an edge if the vertex is adjacent to both endpoints of that edge. Six edge contractions give a graph which contains a K_7 subgraph induced by $\{a, b, c, d, e, f, v\}$

Case 2. N is a 6-cycle. Each edge $(a, b), (c, d)$, and (e, f) is covered by a different vertex of $\overline{H}$. Also, each edge $(b, c), (d, e)$, and (a, f) is covered by a different vertex of $\overline{H}$. If a single vertex of $\overline{H}$, say v_1 covers two edges, say (a, b) and (b, c), then $\{v, v_1, b\}$ dominates $\overline{G}$, a contradiction. It follows that the edges $(a, b), (c, d), (e, f), (b, c), (d, e)$, and (a, f) are all covered by a different vertex of $\overline{H}$. Six edge contractions give a graph which contains a K_7 subgraph induced by $\{a, b, c, d, e, f, v\}$.

Case 3. N is a union of two triangles and an edge. As in Case 1, the edges $(a, b), (b, c), (a, c), (d, e), (d, f)$, and (e, f) are all covered by a different vertex of $\overline{H}$. Six edge contractions give a graph which contains a K_7 subgraph induced by $\{a, b, c, d, e, f, v\}$.

If the edge (a, d) is covered by same vertex that covers (a, b) or (a, c), say v_2, then edge (a, v_2) is contracted. If the edge (a, d) is covered by same vertex that covers (d, e) or (d, f), say v_3, then edge (d, v_3) is contracted.

Case 4. N is a 6-cycle together with a chord of length 2. Since $\{d, e, f\}$ does not dominate G, the vertices d, e, and f have a common neighbor in $\overline{H}$, say v_1. Then $\{v, v_1, e\}$ dominate $\overline{G}$, a contradiction.

Case 5. N is a 6-cycle together with a chord of length 3. Since $\{a, e, f\}$ does not dominate G, the vertices a, e, and f have a common neighbor in $\overline{H}$, say v_1. Since $\{b, c, d\}$ does not dominate G, the vertices a, e, and f have a common neighbor in $\overline{H}$, say v_2. Then $\{v, v_1, v_2\}$ dominate $\overline{G}$, a contradiction.

Case 6. N is a 4-cycle and a 3-cycle that share a vertex. Since $\{a, c, d\}$ does not dominate G, the vertices a, c, and d have a common neighbor in $\overline{H}$, say v_1. Then $\{v, v_1, d\}$ dominate $\overline{G}$, a contradiction.

Case 7. N is a 5-cycle together with a vertex that neighbors two non-adjacent vertices of the 5-cycle. Since $\{a, e, f\}$ does not dominate G, the vertices a, e, and f have a common neighbor in $\overline{H}$, say v_1. Since $\{b, c, d\}$ does not dominate G, the vertices b, c, and d have a common neighbor in $\overline{H}$, say v_2. Then $\{v, v_1, v_2\}$ dominate $\overline{G}$, a contradiction. □

Lemma 3.4. *Let G be a 3-non-compliant graph with 15 vertices and $m = 49, 50, 51$, or 52 edges. Then there exists $H \in \{G, \overline{G}\}$ and $v \in V(H)$ such that $deg_H(v) = 6$ and*

$$\sum_{u \in N_H(v)} \deg_H(u) \leq 45.$$

Proof. By Remark 2.5, the degrees of the vertices of both G and $\overline{G}$ range between 6 and 8. By the handshaking lemma, G has at least one vertex w of degree 6. If $\overline{G}$ has no vertices of degree 6, it implies that G has maximum degree at most 7, so the sum of the degrees of the neighbors of w in G is at most 42, and the lemma holds. So we may assume that both G and $\overline{G}$ have vertices of degree 6. Let α, β, and γ represent the number of vertices of degree 6, 7, and 8, respectively in G. Then α, β, and γ represent the number of vertices of degree 8, 7, and 6, respectively in $\overline{G}$. Under our working assumptions, $\alpha > 0$ and $\gamma > 0$. Assume by contradiction that for both G and $\overline{G}$, for all vertices of degree 6, the sum of the degrees of their neighbors is at least 46. Then each vertex of degree 6 must neighbor at least four vertices of degree 8, in both G and $\overline{G}$. Counting the number of edges with one endpoint a vertex of degree 6 and the other endpoint a vertex of degree 8, in both G and $\overline{G}$, we have $4\alpha + 4\gamma \leq \alpha\gamma$. This gives

$$(\alpha - 4)(\gamma - 4) \geq 16. \tag{3.1}$$

Since $\alpha + \gamma \leq 15$, we have $\alpha - 4 \leq 11 - \gamma$ and Equation (3.1) implies

$$(11 - \gamma)(\gamma - 4) \geq 16. \tag{3.2}$$

No value of γ satisfies equation (3.2), since $f(x) = (11-x)(x-4)$ attains a maximum of 12.25 at $x = 7.5$. We conclude the existence of a vertex v as in the statement of the lemma. □

Theorem 3.5. *If G is graph with 15 vertices, then at least one of the following three statements is true: G is 3-compliant; G has a K_7 minor; $\overline{G}$ has a K_7 minor.*

4 K_{15} is not bi-nIK

The properties of being nIL or nIK are *hereditary*, which means that any minor of a graph which is nIL (nIK), is itself nIL (nIK). Considering complements, any graph which has an IL minor is itself IL, and any graph which has an IK minor is itself IK. Conway and Gordon [4] proved that K_6 is IL and that K_7 is IK. The combined work of Conway and Gordon [4], Sachs [23], and Robertson, Seymour and Thomas [22] fully characterize IL graphs: a graph is IL if and only if it contains a graph in the Petersen family as a minor. The Petersen family consists of seven graphs obtained from K_6 by $\nabla Y-$moves and $Y\nabla-$moves, as presented in Figure 8.

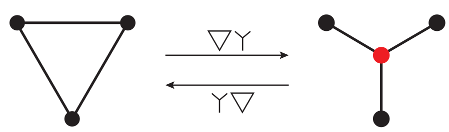

Figure 8: $\nabla Y-$ and $Y\nabla-$moves

In the absence of a known IK minor, it is relatively difficult to prove a given graph is IK. However, under certain linking conditions on the graph, it follows that the graph is IK. The D_4 graph is the (multi)graph shown in Figure 9. A *double-linked* D_4 is a D_4 graph embedded in S^3 such that each pair of opposite 2-cycles

($C_1 \cup C_3$, and $C_2 \cup C_4$) has odd linking number. The following lemma was proved by Foisy [6]; a more general version of it was proved independently by Taniyama and Yasuhara [24].

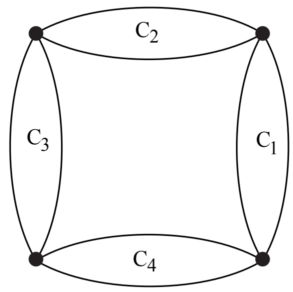

Figure 9: The D_4 graph.

Lemma 4.1. *Every double-linked D_4 contains a nontrivial knot.*

A graph which contains a double-linked D_4-minor in every embedding is called *intrinsically D4* (*ID4*). Lemma 4.1 remains the gold standard in showing a graph is IK, since any intrinsically D4 graph is intrinsically knotted. For this reason, Miller and Naimi [15] developed an algorithm, implemented as a Mathematica program, to determine whether a graph is intrinsically D4 .

We used their program and other Mathematica code developed with Naimi [16] to prove the following theorem.

Theorem 4.2. *Let G be a nIL graph of order* 12*. Then $\overline{G}$ is intrinsically knotted (IK).*

Proof. There are a total of 6503 maximal linklessly embeddable (*maxnIL*) graphs of order 12 [20]. The search was organized according to these observations:

- If $\delta(G)$ is the minimal degree of a maxnIL graph G, then $2 \leq \delta(G)$. It is straightforward to check that any nIL graph with a vertex of degree 0 or 1 is a proper subgraph of another nIL graph of the same order.
- If G is a maxnIL graph of order 12, then $\delta(G) \leq 5$. This follows from [18], where the second author and his student proved that every graph of order 12 with minimal degree at least 6 has a K_6-minor and is therefore intrinsically linked.
- When deleting a vertex from a maxnIL graph of order 12, one obtains a nIL graph of order 11.

The previous observations show that any maxnIL graph of order 12 is an edge-deletion subgraph of a graph obtained by adding a vertex of degree at most 5 to a maxniIL graph of order 11. By the work of the last two authors, Ryan Odeneal, and Naimi [17], any maxnIL graph which has a vertex of degree 2 or 3 can be obtained from a maxnIL of order one less by performing a clique sum with a K_3 over a K_2 induced subgraph, or a clique sum with a K_4 over an induced K_3-subgraph. We took the list of 710 maxnIL graphs of order 11, obtained the graphs of order 12 through the clique sum operation, and sifted out the 2273 maxnIL graphs with minimal degree at most 3. The remaining 2230 were found by adding a vertex of degree 4 or 5, respectively, and checking whether any nIL graphs were obtained, as well as which ones were maximal. This was done using Naimi's Mathematica program [16] which decides whether a given graph is intrinsically linked by solving linear systems over $\mathbb{Z}$ and $\mathbb{Z}_2$. Then, we took the ones that were intrinsically linked, used McKay and Piperno's nauty program [14] to obtain all their 1-edge deletion subgraphs up to isomorphism, selected those which have minimum degree at least 4, and then sorted them as IL or nIL. We repeated the process until no graphs with minimal degree 4, respectively 5 were left.

We then used Miller and Naimi's program [15] and verified that all the complements of the 6503 maxnIL graphs of order 12 were intrinsically D4. Since any nIL graph of order 12 can be completed, by adding edges, to a maxnIL of order 12, it follows that the complement of a nIL graph of order 12 contains the complement of a maxnIL graph of order 12, and it is therefore intrinsically knotted. □

Remark 7, from the article the last two authors wrote with Odeneal and Naimi, claims that all the maxnIL graphs of order 11 are 2-apex. The authors asked whether all the maxnIL graphs of order 12 are 2-apex. Our search for all the maxnIL graphs of order 12 yielded the one example of a maxnIL graph of order 12 which is not 2-apex. This graph is depicted in Figure 10.

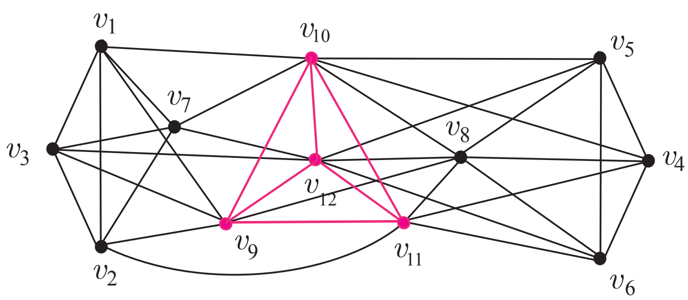

Figure 10: The only maxnIL of order 12 which is not 2-apex.

Observation 4.3. Let G be an IL graph. Then $G * K_1$ is IK.

Proof. Since the statement holds for all the members of the Petersen family, the result follows. □

In 1990, de Verdiére [5] introduced the graph invariant μ, based on spectral properties of matrices associated with a graph G. He showed that μ is monotone under taking minors and that planarity is characterized by the inequality $\mu \leq 3$. Lovász and Schrijver showed that linkless embeddability is characterized by the inequality $\mu \leq 4$ [12]. Kotlov, Lovász, and Vempala [10] conjectured that for any simple graph G of order n, $\mu(G) + \mu(\overline{G}) \geq n - 2$. The conjecture was known to hold for planar graphs, graphs of order at most 10, and graphs with $\mu(G) \geq n-6$, see [8]. In Remark 8 of [17], it is argued that the conjecture is true for graphs of order 11 as well. Given the list of maxnIL graphs of order 12, with computer assistance, we verified that the complements of each one of these graphs has a minor in either the K_7-family, the $K_{3,3,1,1}$-family, or the $E_9 + e$-family. The graphs in these families, described by Goldberg, Mattman, and Naimi in [7], have all $\mu = 6$. As μ is monotone under taking minors, the μ value for each complement of a nIL graph of order 12 is at least 6. This shows that $\mu(G) + \mu(\overline{G}) \geq n$ holds true for graphs of order $n \leq 12$.

Theorem 1.1. *Let G be a simple graph on* 15 *vertices and let $\overline{G}$ denote its complement. If G is nIK, then $\overline{G}$ is IK.*

Proof. If G is 3-compliant, then either G or $\overline{G}$ has a minor with a vertex of degree 12. We assume this minor has order 13. If G has a minor H with a vertex v of degree 12, since G is nIK, by Theorem 4.3, $H - v$ is nIL. By Theorem 4.2, $\overline{H - v}$ is IK, thus $\overline{G}$ is IK. If $\overline{G}$ has a minor H with a vertex v of degree 12 and $\overline{G}$ is nIK, $H - v$ is nIL. By Theorem 4.2, $\overline{H - v}$ is IK, thus G is IK.

If G is 3-non-compliant, by Theorem 3.5, at least one of G or $\overline{G}$ has a K_7 minor. Since K_7 is IK, one of G or $\overline{G}$ is intrinsically knotted. □

Note that the previous result proves that K_n is not bi-nIK, for all $n \geq 15$. On the other hand K_{12} is bi-nIK, as the following example will demonstrate. The graph depicted on the left in Figure 11 is self-complementary (isomorphic to its complement). It is also 2-apex, as the deletion of the "empty" vertices yields the planar graph on the right. As 2-apex graphs are knotlessly embeddable by results of Blain et al. [3] and Ozawa and Tsutsumi [19], the graph on the left in Figure 11 is nIK. It follows K_{12} is bi-nIK.

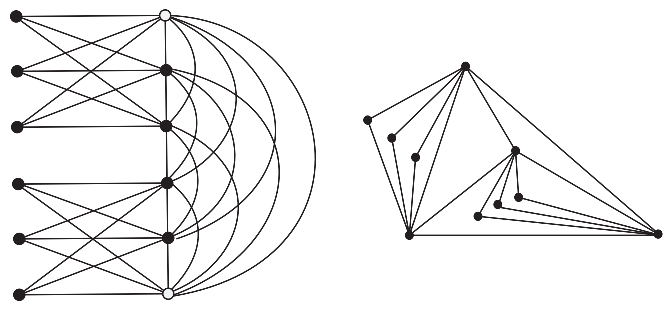

Figure 11: A self-complementary, 2-apex graph of order 12, and the planar graph obtained by deleting two of its vertices.

The following is a natural question:

Question 4: Are either of K_{13} or K_{14} bi-nIK?

Note that in the arguments leading to the proof of Theorem 3.5 and Theorem 1.1, we focused mainly on finding a K_7 minor in either the graph or its complement. Since not having a K_7(resp. K_6) minor is also a hereditary property, we pose the following questions:

Question 5: What is the smallest integer $12 < n \leq 15$, such that K_n is not bi-K_6-free?

Question 6: What is the smallest integer $13 < n \leq 18$, such that K_n is not bi-K_7-free?

The upper bounds in the previous two questions are derived from a theorem of Mader [13], which implies that any graph of order n with at least $4n - 9$ edges has a K_6 minor, and any graph of order n with at least $5n - 14$ edges has a K_7 minor. The lower bounds stem from the example in Figure 12(Left). This order 12 graph has no K_6 minor, and neither does its complement.

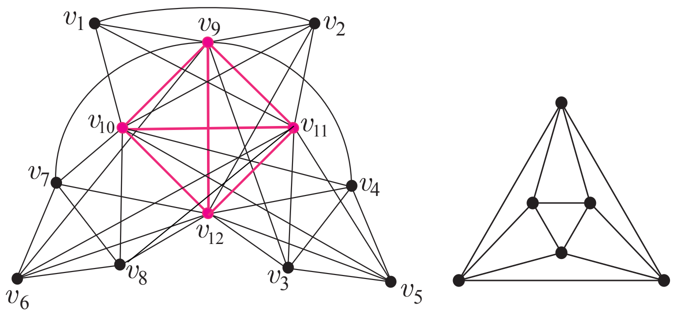

Figure 12: (Left) A K_6-free graph whose complement is also K_6-free. (Right) A maximal planar graph.

The chief argument for both the graph in Figure 12 (Left) and its complement being K_6 minor-free is the following observation:

Observation 4.4. Let G_1 and G_2 be K_6 minor-free graphs. Then any clique sum over K_p of G_1 and G_2, for $1 \leq p \leq 4$, is K_6 minor-free.

Let G denote the graph in Figure 12(Left). Notice that the vertices in $S = \{v_9, v_{10}, v_{11}, v_{12}\}$ form a clique (complete subgraph) in G, and that G is the clique sum over the K_4 subgraph induced by the vertices in S of the subgraphs H_1, induced by $\{v_1, v_2, v_9, v_{10}, v_{11}, v_{12}\}$, H_2, induced by $\{v_3, v_4, v_5, v_9, v_{10}, v_{11}, v_{12}\}$, and H_3, induced by $\{v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. All of H_1, H_2, and H_3 are K_6 minor-free. The graph H_1 is isomorphic to K_6^- (the complete graph on six vertices with one edge removed). The graphs H_2 and H_3 are both isomorphic to the graph $K_1 * H$, the graph obtained by adding one vertex connected to all the vertices of the planar graph in Figure 12 (Right). By Observation 4.4, G is K_6 minor-free.

Consider now the complement $\overline{G}$. The vertices v_9, v_{10}, v_{11}, and v_{12} have degrees $2, 2, 2$, and 1, respectively. Moreover, their neighborhoods are complete subgraphs in $\overline{G}$, thus $\overline{G}$ is obtained from the subgraph M of order 8 induced by the vertices

$\{v_1, v_2, ..., v_8\}$, by repeated clique sums over K_2, and K_1. Thus, by Observation 4.4, $\overline{G}$ is K_6 minor-free, if and only if M is K_6 minor-free. The graph M is isomorphic to the highly symmetry graph $K_{3,3} * E_2$, with E_2 denoting the graph of order 2 with no edges. Up to isomorphism, this graph has 4 different order 6 edge-contraction minors, none of which is isomorphic to K_6. Thus M is K_6 minor-free, and so is $\overline{G}$.

5 k-Non-Compliant Graphs

Theorem 5.1. *A graph G or its complement $\overline{G}$ has a minor H where $\Delta(H) \geq |V(G)| - k = n - k$ if and only if $\min(\gamma_c(G), \gamma_c(\overline{G})) \leq k$.*

Proof. The statement follows from the definitions if $k = 1$, or G is disconnected. For the direct implication, without loss of generality, let H be a minor of G and $s \in V(H)$ be such that $\deg_H(s) = n-k$. Let $S \subset V(G)$ be exactly the set of vertices contracted through edges in G to produce $s \in V(H)$. We note that $G[S]$ is connected. Define

$$N_S = \{u \in V(G) \setminus S \mid \exists\, v \in S : (u, v) \in E(G)\}.$$

By assumption, $|N_S| \geq n - k$. If $N_S = V(G) \setminus S$, then S is a connected dominating set of size at most k, implying $\gamma_c(G) \leq k$.

If $N_S \subsetneq V(G) \setminus S$, then $T = V(G) \setminus (S \sqcup N_S)$ is non-empty such that (in G) no vertex in S is adjacent to a vertex in T. This means $\overline{G}[S \sqcup T]$ is connected. Moreover, if every vertex $u \in N_S = V(G) \setminus (S \sqcup T)$ is adjacent in the complement $\overline{G}$ to some $v \in S \sqcup T$, then $S \sqcup T$ is a connected dominating set and $\gamma_c(\overline{G}) \leq k$.

So, say there exists some $u_0 \in N_S$ such that $(u_0, v) \in E(G)$ for every $v \in S \sqcup T$. Let $S' := S \sqcup \{u_0\}$. Note that $G[S']$ is connected, and since T was nonempty, we have $|S'| \leq k$. Because $u_0 \in S'$, we find S' is a connected dominating set of G. We conclude, in this third case $\gamma_c(G) \leq k$.

Towards the reverse implication, without loss of generality, let $\gamma_c(G) \leq k$. Let $S \subset V(G)$ be a connected dominating set of G such that $|S| \leq k$. A minor H with $\Delta(H) \geq n - k$ is obtained from G by contracting the edges of any spanning tree of the connected subgraph $G[S]$. □

Define $f : \mathbb{N} \to \mathbb{N}$ so that $f(n)$ is the largest number such that for all graphs G of order n, G or $\overline{G}$ has a minor H with $\Delta(H) \geq f(n)$. The first few values of $f(n)$ are $f(1) = 0, f(2) = 1, f(3) = 2, f(4) = 2, f(5) = 2$. Note that $n - f(n)$ corresponds to the minimum value of k such that all graphs G of order n are k-compliant. The construction in Proposition 2.6 shows that $\{n - f(n)\}$ is a non-decreasing sequence.

Theorem 5.2. *For $n \geq 15$,*

$$n - \left\lfloor \frac{n+1}{4} \right\rfloor \leq f(n).$$

Proof. If G is 3-compliant, the inequality holds since

$$n - \left\lfloor \frac{n+1}{4} \right\rfloor \leq n - 3 \leq f(n), \text{for all } n \geq 15.$$

Assume G is 3-non-compliant. Then $\gamma_c(G)+\gamma_c(\overline{G}) \le \delta^*(G)+2$ by Theorem 5 in [11]. If $\delta^*(G) \neq 6$, by Theorem 7 in [11], $\gamma_c(G)+\gamma_c(\overline{G}) < \delta^*(G)+2$. Since $\delta^*(G) \le \lfloor \frac{n-1}{2} \rfloor$, then

$$\gamma_c(G)+\gamma_c(\overline{G}) < \left\lfloor \frac{n-1}{2} \right\rfloor + 2,$$

or equivalently

$$\gamma_c(G)+\gamma_c(\overline{G}) \le \left\lfloor \frac{n-1}{2} \right\rfloor + 1.$$

It follows that

$$\min(\gamma_c(G),\gamma_c(\overline{G})) \le \left\lfloor \frac{\lfloor \frac{n-1}{2} \rfloor + 1}{2} \right\rfloor = \left\lfloor \frac{n+1}{4} \right\rfloor.$$

By Theorem 5.1,

$$n - \left\lfloor \frac{n+1}{4} \right\rfloor \le f(n).$$

If $\delta^*(G) = 6$, since G is 3-non-compliant, then $\gamma_c(G) = \gamma_c(\overline{G}) = 4$. By Theorem 5.1, $f(n) \ge n-4$. For $n \ge 15$, $\lfloor \frac{n+1}{4} \rfloor \ge 4$ and it follows that $f(n) \ge n - \lfloor \frac{n+1}{4} \rfloor$. □

Remark 5.3. Note that $n - \lfloor \frac{n+1}{4} \rfloor = n-3$, for $n = 13, 14$. So the bound $n \ge 15$ is necessary, by Theorems 1.3 and 1.4.

Note that Theorem 5.2 shows that all graphs of order $n \le 18$ are 4-compliant. On the other hand, QR_{61} is 4-non-compliant. This begs the question:

Question 7: What is the smallest order of a k-non-compliant graph, for $k \ge 4$?

One can check that four is the smallest order of a 1-non-compliant graph (C_4, the 4-cycle), and five is the smallest order of a 2-non-compliant graph (C_5, the 5-cycle). Corollary 2.4 shows that 13 is the smallest order for a 3-non-compliant graph.

Theorem 5.4 (Karami et al. [11]). *If G and $\overline{G}$ are both connected of order $n \ge 7$, then*

$$\gamma_c(G) \cdot \gamma_c(\overline{G}) \le 2n-4,$$

with equality if and only if G or $\overline{G}$ is a path or a cycle.

Corollary 5.5. For $n \ge 7$, we have

$$\min\{\gamma_c(G),\gamma_c(\overline{G})\} \le \lceil \sqrt{2n-4} \rceil - 1, \text{ and equivalently } n - \lceil \sqrt{2n-4} \rceil + 1 \le f(n).$$

Proof. If either G or $\overline{G}$ are disconnected, then G is 2-compliant and the corollary holds. If $n \ge 7$ and G is a path or a cycle, then $\gamma_c(G) = n-2$ and $\gamma_c(\overline{G}) = 2$. Then $\min\{\gamma_c(G),\gamma_c(\overline{G})\} = 2 \le \lceil \sqrt{2n-4} \rceil - 1$. For G not a path or a cycle, by Theorem 5.4, $\gamma_c(G)\cdot\gamma_c(\overline{G}) < 2n-4$. Assume $\gamma_c(G), \gamma_c(\overline{G}) > \lceil \sqrt{2n-4} \rceil - 1$. Then $\gamma_c(G), \gamma_c(\overline{G}) \ge \lceil \sqrt{2n-4} \rceil$, and $\gamma_c(G)\cdot\gamma_c(\overline{G}) \ge \left(\lceil \sqrt{2n-4} \rceil \right)^2 \ge 2n-4$, a contradiction. □

Remark 5.6. This bound is a strict improvement over Theorem 5.2 for $n \ge 31$.

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A Three-non-compliant graphs of order 15

$$\begin{aligned}E(G_1) =\{&(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(3,5),(3,8),(3,9),(3,10),\\&(3,11),(3,14),(3,15),(4,6),(4,9),(4,10),(4,11),(4,12),(5,7),(5,10),(5,11),(5,12),(5,13),(5,14),\\&(5,15),(6,8),(6,11),(6,12),(6,13),(7,9),(7,12),(7,13),(7,14),(7,15),(8,10),(8,13),(9,11),(10,12),\\&(10,14),(10,15),(11,13),(11,14),(11,15),(12,14),(12,15),(13,14),(13,15),(14,15)\}\end{aligned}$$

$$\begin{aligned}E(G_2) =\{&(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(3,5),(3,8),(3,9),(3,10),\\&(3,11),(3,14),(3,15),(4,6),(4,9),(4,10),(4,11),(4,12),(5,7),(5,10),(5,11),(5,12),(5,13),(5,14),(5,15),\\&(6,8),(6,11),(6,12),(6,13),(7,9),(7,12),(7,13),(7,14),(7,15),(8,10),(8,13),(9,11),(10,12),(10,14),\\&(10,15),(11,13),(11,14),(11,15),(12,14),(12,15),(13,14),(13,15)\}\end{aligned}$$

$$\begin{aligned}E(G_3) =\{&(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(2,15),(3,5),(3,8),\\&(3,9),(3,10),(3,11),(4,6),(4,9),(4,10),(4,11),(4,12),(4,14),(4,15),(5,7),(5,10),(5,11),(5,12),(5,13),\\&(6,8),(6,11),(6,12),(6,13),(7,9),(7,12),(7,13),(7,14),(7,15),(8,10),(8,13),(8,14),(8,15),(9,11),\\&(9,14),(9,15),(10,12),(10,14),(10,15),(11,13),(13,14),(13,15)\}\end{aligned}$$

$$\begin{aligned}E(G_4) =\{&(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(1,15),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),\\&(3,5),(3,8),(3,9),(3,10),(3,11),(4,6),(4,9),(4,10),(4,11),(4,12),(4,14),(4,15),(5,7),(5,10),(5,11),\\&(5,12),(5,13),(5,15),(6,8),(6,11),(6,12),(6,13),(6,15),(7,9),(7,12),(7,13),(7,14),(7,15),(8,10),\\&(8,13),(8,14),(9,11),(9,14),(10,12),(10,14),(10,15),(11,13),(13,14)\}\end{aligned}$$

$$\begin{aligned}E(G_5) =\{&(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(3,5),(3,8),(3,9),\\&(3,10),(3,11),(4,6),(4,9),(4,10),(4,11),(4,12),(4,14),(4,15),(5,7),(5,10),(5,11),(5,12),(5,13),\\&(6,8),(6,11),(6,12),(6,13),(7,9),(7,12),(7,13),(7,14),(7,15),(8,10),(8,13),(8,14),(8,15),(9,11),\\&(9,14),(9,15),(10,12),(10,14),(10,15),(11,13),(13,14),(13,15)\}\end{aligned}$$

$$\begin{aligned}E(G_6) = \{&(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(1,15),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(2,15),(3,5),\\&(3,8),(3,9),(3,10),(3,11),(3,14),(4,6),(4,9),(4,10),(4,11),(4,12),(5,7),(5,10),(5,11),(5,12),\\&(5,13),(5,14),(5,15),(6,8),(6,11),(6,12),(6,13),(7,9),(7,12),(7,13),(7,14),(7,15),(8,10),\\&(8,13),(9,11),(9,15),(10,12),(10,14),(11,13),(11,14),(12,14),(12,15),(13,14),(13,15),(14,15)\}\end{aligned}$$

$$\begin{aligned}E(G_7) = \{&(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(1,15),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(2,15),(3,5),\\&(3,8),(3,9),(3,10),(3,11),(4,6),(4,9),(4,10),(4,11),(4,12),(4,14),(4,15),(5,7),(5,10),(5,11),\\&(5,12),(5,13),(5,15),(6,8),(6,11),6,12),(6,13),(6,15),(7,9),(7,12),(7,13),(7,14),(7,15),\\&(8,10),(8,13),(8,14),(9,11),(9,14),(10,12),(10,14),(10,15),(11,13),(13,14)\}\end{aligned}$$

$$\begin{aligned}E(G_8) = \{&(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(1,15),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(2,15),(3,5),\\&(3,8),(3,9),(3,10),(3,11),(3,14),(4,6),(4,9),(4,10),(4,11),(4,12),(5,7),(5,10),(5,11),(5,12),\\&(5,13),(5,14),(5,15),(6,8),(6,11),(6,12),(6,13),(7,9),(7,12),(7,13),(7,14),(7,15),(8,10),(8,13),\\&(9,11),(9,15),(10,12),(10,14),(11,13),(11,14),(12,14),(12,15),(13,14),(13,15)\}\end{aligned}$$

$$E(G_9) = \{(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(1,15),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(2,15),(3,5),(3,8),(3,9),(3,10),(3,11),(3,14),(4,6),(4,9),(4,10),(4,11),(4,12),(5,7),(5,10),(5,11),(5,12),(5,13),(5,14),(5,15),(6,8),(6,11),(6,12),(6,13),(7,9),(7,12),(7,13),(7,14),(8,10),(8,13),(9,11),(9,15),(10,12),(10,14),(11,13),(11,14),(12,14),(12,15),(13,14),(13,15),(14,15)\}$$

$$E(G_{10}) = \{(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(1,15),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(2,15),(3,5),(3,8),(3,9),(3,10),(3,11),(3,15),(4,6),(4,9),(4,10),(4,11),(4,12),(4,14),(4,15),(5,7),(5,10),(5,11),(5,12),(5,13),(6,8),(6,11),(6,12),(6,13),(7,9),(7,12),(7,13),(7,14),(7,15),(8,10),(8,13),(8,14),(9,11),(9,14),(10,12),(10,14),(11,13),(11,15),(13,14),(14,15)\}$$

$$E(G_{11}) = \{(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(1,15),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(2,15),(3,5),(3,8),(3,9),(3,10),(3,11),(3,14),(3,15),(4,6),(4,9),(4,10),(4,11),(4,12),(4,15),(5,7),(5,10),(5,11),(5,12),(5,13),(5,14),(6,8),(6,11),(6,12),(6,13),(7,9),(7,12),(7,13),(7,14),(7,15),(8,10),(8,13),(9,11),(9,15),(10,12),(10,14),(11,13),(11,14),(11,15),(12,14),(13,14)\}$$

$$E(G_{12}) = \{(1,3),(1,6),(1,7),(1,8),(1,9),(1,12),(1,15),(2,4),(2,7),(2,8),(2,9),(2,10),(2,13),(3,5),(3,8),(3,9),(3,10),(3,11),(3,15),(4,6),(4,9),(4,10),(4,11),(4,12),(4,14),(5,7),(5,10),(5,11),(5,12),(5,13),(5,15),(6,8),(6,11),(6,12),(6,13),(7,9),(7,12),(7,13),(7,14),(8,10),(8,13),(8,14),(8,15),(9,11),(9,14),(9,15),(10,12),(10,14),(10,15),(11,13),(11,15),(13,14)\}$$

$$E(G_{13}) = \{(1,2),(1,4),(1,5),(1,10),(1,11),(1,13),(1,14),(2,3),(2,5),(2,6),(2,11),(2,12),(2,14),(3,4),(3,6),(3,7),(3,12),(3,13),(3,14),(3,15),(4,5),(4,7),(4,8),(4,13),(5,6),(5,8),(5,9),(5,14),(6,7),(6,9),(6,10),(6,14),(7,8),(7,10),(7,11),(8,9),(8,11),(8,12),(8,15),(9,10),(9,12),(9,13),(9,15),(10,11),(10,13),(11,12),(11,14),(11,15),(12,13),(12,15),(13,15),(14,15)\}$$

$$E(G_{14}) = \{(1,2),(1,4),(1,5),(1,10),(1,11),(1,13),(1,14),(2,3),(2,5),(2,6),(2,11),(2,12),(2,14),(2,15),(3,4),(3,6),(3,7),(3,12),(3,13),(3,14),(4,5),(4,7),(4,8),(4,13),(4,14),(4,15),(5,6),(5,8),(5,9),(5,15),(6,7),(6,9),(6,10),(6,14),(6,15),(7,8),(7,10),(7,11),(7,15),(8,9),(8,11),(8,12),(8,14),(9,10),(9,12),(9,13),(9,14),(10,11),(10,13),(11,12),(12,13),(12,15),(13,15)\}$$

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