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Power domination with random sensor failure

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Abstract

The power domination problem seeks to determine the minimum number of phasor measurement units (PMUs) needed to monitor an electric power network. We introduce random sensor failure before the power domination process occurs and call this the fragile power domination process. For a given graph, PMU placement, and probability of PMU failure q, we study the expected number of observed vertices at the termination of the fragile power domination process. This expected value is a polynomial in q, which we relate to fault-tolerant and PMU-defect-robust power domination. We also study the probability that the entire graph becomes observed and give results for some graph families.

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1 Introduction

The power domination problem, defined by Haynes et al. in [2], seeks to find the placement of the minimum number of phasor measurement units (PMUs) needed to monitor an electric power network. This process was simplified by Brueni and Heath in [3]. PMUs are placed on an initial set of vertices of a graph, and vertex observation rules define a propagation process on the graph.

Pai, Chang, and Wang [1] provided a variation to power domination which looked for the minimum number of PMUs needed to monitor a power network in the case where k of the PMUs will fail, called *k-fault-tolerant power domination*. Their work allows the placement of only one PMU per vertex. A generalization of this problem is *PMU-defect-robust power domination* which allows for multiple PMUs to be placed at a given vertex, defined by Bjorkman, Conrad, and Flagg [4].

Fault-tolerant and PMU-defect-robust power domination study a fixed number of PMU failures. However, actual sensor failures occur randomly. Thus, we assign a probability of failure to the PMUs to create the *fragile power domination process*. By studying a model in which PMUs have a random chance of failure, the primary question is no longer to find the placement of a minimum number of sensors. Instead, the *expected observability* of the network is studied.

In particular, we define the *expected value polynomial* for a graph G and PMU placement S as a function of the PMU failure probability. This polynomial is shown to have a direct connection to k-fault-tolerant and k-PMU-defect-robust power domination. We use this polynomial to determine the probability that the entire graph will be observed. Methods of comparing this polynomial for different PMU placements are presented, and the polynomial is calculated for families of graphs.

In Section 2, we give definitions, establish useful binomial properties, and formally define relevant power domination variations. We then define the fragile power domination process and the associated expected value polynomial, and in Section 3 explore properties of this polynomial. In Section 4, we consider the probability of observing the entire graph. Graph families are studied in Section 5. Finally, in Section 6 we give ideas for future work.

2 Preliminaries

To begin, necessary graph theoretic definitions and some useful properties of binomial coefficients are established. The relevant notions of power domination, failed power domination, fault-tolerant power domination, and PMU-defect-robust power domination are given. Finally, we introduce fragile power domination and the expected value polynomial.

2.1 Graph theory

A simple, undirected graph G is a set $V(G)$ of vertices and a set $E(G)$ of edges. Each edge is an unordered pair of distinct vertices $\{x, y\}$, usually written xy. When xy is an edge, x and y are said to be *adjacent* or *neighbors*. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph H is an *induced subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) = \{xy \in E(G) : x, y \in V(H)\}$ and is denoted by $G[V(H)]$.

A *path* from v_1 to $v_{\ell+1}$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_{\ell+1}$ such that $v_i v_{i+1}$ is an edge for all $i = 1, 2, \ldots, \ell$. A graph G is *connected* if, for all $u, v \in V(G)$, there exists a path from u to v. For a graph G that is not connected, the subgraphs $H_1, \ldots, H_k$ are the *connected components* of G when $H_1, \ldots, H_k$ are connected and have no edges between them.

A *cycle* from v_1 to v_ℓ is a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$ such that $v_i v_{i+1}$ is an edge for all $i = 1, 2, \ldots, \ell$ and $v_1 v_\ell$ is an edge. The *neighborhood* of $u \in V(G)$, denoted by $N(u)$, is the set containing all neighbors of u. The *closed neighborhood* of u is $N[u] = N(u) \cup \{u\}$. Given an initial set S of vertices of a graph G, S is a *dominating set* of G if $\bigcup_{v \in S} N[v] = V(G)$. A *universal vertex* v has $N[v] = V(G)$. The *degree* of a vertex u is $\deg(u) = |N(u)|$. A *leaf* is a vertex v with $\deg(v) = 1$.

A *subdivision* of an edge xy creates a new vertex w and replaces the edge xy with the edges xw and wy. If the edge xy is subdivided a times, vertices $w_1, w_2, \ldots, w_a$ are added to the vertex set and the edges $xw_1, w_1w_2, \ldots, w_{a-1}w_a, w_a y$ replace the edge xy.

We denote the *path on n vertices* by P_n, the *cycle on n vertices* by C_n, and the *complete graph on n vertices* by K_n. The *wheel on n vertices*, denoted by W_n, is constructed by adding a universal vertex to C_{n-1}. A *complete multipartite graph*, denoted by $K_{r_1, r_2, \ldots, r_k}$, has its vertex set partitioned into disjoint sets $A_1, A_2, \ldots, A_k$ with $|V_i| = r_i$, and edge set $\{xy : x \in A_i, y \in A_j, i \neq j\}$. The case of $k = 2$ is the *complete bipartite graph*. A special case of the complete bipartite graph is the *star on n vertices*, $S_n = K_{1,n-1}$.

2.2 Probability notation

Let A and B be a statistical events, that is, subsets of a finite sample space. We write $\text{Prob}\left[A\right]$ to denote the probability that A occurs, and $\text{Prob}\left[A : B\right]$ to denote the probability of A given that B occurs. If X is a random variable with possible outcomes $x_1, x_2, \ldots, x_n$, then $\{X = x_i\}$ is the event that X takes the value x_i and occurs with some probability $\text{Prob}\left[X = x_i\right]$. The expected value of the random variable is $\mathbb{E}\left[X\right] = \sum_{i=1}^{n} x_i \,\text{Prob}\left[X = x_i\right]$.

If A_1 and A_2 are events, $A_1 \vee A_2$ denotes the event that at least one of A_1 or A_2 occurs. Additionally, $A_1 \wedge A_2$ denotes the event that both A_1 and A_2 occur.

2.3 Power domination

Formally, the *power domination process* on a graph G with initial set $S \subseteq V(G)$ proceeds as follows:

1. (*Domination*) Initialize $B = \bigcup_{v \in S} N[v]$.
2. (*Zero Forcing*) While there exists $v \in B$ such that exactly one neighbor, say u, of v is *not* in B, add u to B.

During the process, a vertex in B is *observed* and a vertex not in B is *unobserved*. We denote the set of vertices observed at the termination of the power domination process by $\operatorname{Obs}(G; S)$. If v causes u to join B, then v *observes* u. A *power dominating set* of a graph G is an initial set S such that $\operatorname{Obs}(G; S) = V(G)$. The *power domination number* of a graph G is the minimum cardinality of a power dominating set of G and is denoted by $\gamma_P(G)$.

Failed power domination is a related, reversed, notion to power domination introduced by Glasser et al. in [5] and will be very useful throughout this work.

Definition 2.1. The *failed power domination number* of a graph G is the cardinality of a largest set $F \subseteq V(G)$ such that $\operatorname{Obs}(G; F) \neq V(G)$. This maximum cardinality is denoted by $\overline{\gamma}_P(G)$.

Fault-tolerant power domination was introduced in [1] in order to account for PMU failure when determining minimum power dominating sets. We will think of multisets as sets where the multiplicity of an element is the number of times the element is repeated.

Definition 2.2. For a graph G and an integer k with $0 \leq k < |V(G)|$, a set $S \subseteq V(G)$ is called a *k-fault-tolerant power dominating set of* G if $S \setminus F$ is still a power dominating set of G for any subset $F \subset S$ with $|F| \leq k$. The *k-fault-tolerant power domination number* is the minimum cardinality of a k-fault-tolerant power dominating set of G.

PMU-defect-robust power domination was introduced in [4] to extend vertex-fault tolerant power domination into multiset PMU placements.

Definition 2.3. Let G be a graph, $k \geq 0$ be an integer, and S be a set or multiset whose elements are in $V(G)$. The set or multiset S is a *k-PMU-defect-robust power dominating set* (k-rPDS) if for any submultiset F with $|F| = k$, $S \setminus F$ contains a power dominating set of vertices. The minimum cardinality of a k-rPDS is the k-PMU-defect-robust power domination number.

2.4 Fragile power domination

The fragile power domination process is a variation of the power domination process with the addition that before the domination step, each PMU fails independently with probability q. We utilize the same notation and terminology as the power domination process, given in the previous section, with suitable modifications.

Definition 2.4. The *fragile power domination process* on a graph G with initial set or multiset of vertices S is as follows:

0. *(Sensor Failure)* Let $S^* = \varnothing$. For each $v \in S$, add v to S^* with probability $1 - q$. Note that each $v \in S$ is added to S^* as independent random events.
1. *(Domination)* Initialize $B = \bigcup_{v \in S^*} N[v]$.
2. *(Zero Forcing)* While there exists $v \in B$ such that exactly one neighbor, say u, of v is *not* in B, add u to B.

Given initial PMU placement S, the set or multiset S^* is the random collection of vertices which did not fail with probability $1 - q$. We denote the set of vertices observed at the termination of the fragile power domination process by $\mathrm{Obs}(G; S, q)$. Observe that $|\mathrm{Obs}(G; S, q)|$ is a random variable and $\mathrm{Obs}(G; S) = \mathrm{Obs}(G; S, 0)$. Moreover, for a fixed set or multiset of vertices S, the expected value of $|\mathrm{Obs}(G; S, q)|$ is a polynomial in q.

Definition 2.5. For a given graph G, set or multiset of vertices S, and probability of PMU failure q, we define the *expected value polynomial* to be

$$\mathcal{E}(G; S, q) = \mathbb{E}\left[|\mathrm{Obs}(G; S, q)|\right],$$

the expected value of the random variable $|\mathrm{Obs}(G; S, q)|$.

This polynomial will serve as a central tool for investigating fragile power domination.

Observation 2.6. *$\mathcal{E}(G; S, q)$, as a polynomial in q with degree at most $|S|$, can be calculated via:*

$$\mathcal{E}(G; S, q) = \sum_{W \subseteq S} \left|\mathrm{Obs}(G; W)\right| q^{|S \setminus W|}(1 - q)^{|W|}.$$

In the next example we explore $\mathcal{E}(G; S, q)$ for all 2-multisets of a specific graph.

Example 2.7. Let G be the graph G in Fig. 1. Note that $\gamma_P(G) = 2$. The 28 unique placements of 2 PMUs, corresponding to the 28 multisets of 2 vertices, result in 10 distinct expected value functions, which are plotted in Fig. 1:

- $\mathcal{E}(G; S, q) = 4(1 - q)q + 2(1 - q)^2$ when $S = \{2, 2\}, \{3, 3\}, \{6, 6\}$ or $\{7, 7\}$,
- $\mathcal{E}(G; S, q) = 4(1 - q)q + 3(1 - q)^2$ when $S = \{2, 3\}$ or $\{6, 7\}$,
- $\mathcal{E}(G; S, q) = 4(1 - q)q + 4(1 - q)^2$ when $S = \{2, 6\}, \{2, 7\}, \{3, 6\}$, or $\{3, 7\}$,
- $\mathcal{E}(G; S, q) = 6(1 - q)q + 3(1 - q)^2$ when $S = \{1, 1\}$,

- $\mathcal{E}(G; S, q) = 5(1-q)q + 5(1-q)^2$ when $S = \{1,2\}, \{1,3\}, \{1,6\}$, or $\{1,7\}$,
- $\mathcal{E}(G; S, q) = 7(1-q)q + 5(1-q)^2$ when $S = \{2,4\}, \{3,4\}, \{5,6\}$, or $\{5,7\}$,
- $\mathcal{E}(G; S, q) = 8(1-q)q + 5(1-q)^2$ when $S = \{1,4\}$ or $\{1,5\}$,
- $\mathcal{E}(G; S, q) = 10(1-q)q + 5(1-q)^2$ when $S = \{4,4\}$ or $\{5,5\}$,
- $\mathcal{E}(G; S, q) = 7(1-q)q + 7(1-q)^2$ when $S = \{2,5\}, \{3,5\}, \{4,6\}$, or $\{4,7\}$, and
- $\mathcal{E}(G; S, q) = 10(1-q)q + 7(1-q)^2$ when $S = \{4,5\}$.

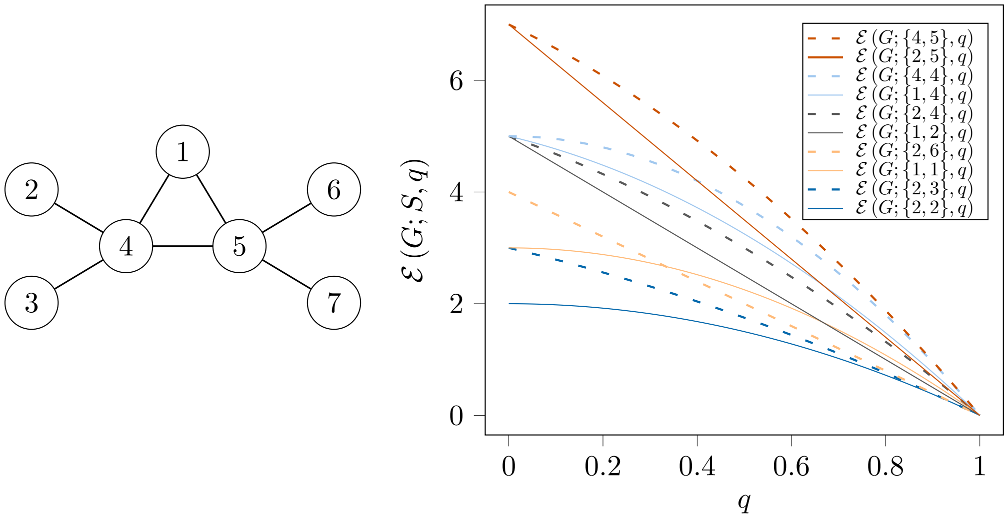

Figure 1: A graph G and all distinct $\mathcal{E}(G; S, q)$ for 2-multisets S containing vertices of G.

Overall, the power dominating set $\{4,5\}$ is expected to observe more vertices than any other placement of two PMUs. The only other power dominating sets of size two are $\{2,5\}$, $\{3,5\}$, $\{4,6\}$, and $\{4,7\}$, all of which have the same expected value polynomial. Notice that for $q \in (2/5, 1)$ the placement of $\{4,4\}$, a failed power dominating set, is expected to observe more vertices than the placement of $\{2,5\}$, a power dominating set. We will explore the intersection of the expected value polynomials for power dominating sets and failed power dominating sets in Section 3.3.

Example 2.7 demonstrates $\mathcal{E}(G; S, q)$ for different PMU placements S and a range of q values. A natural baseline comparison for these polynomials is a placement of PMUs for which the observed vertices are roughly evenly distributed. For instance, let G be a graph on n vertices and let S be a power dominating set of G such that for any $W \subseteq S$, it holds that $|\operatorname{Obs}(G; W)| = |W| \frac{n}{\gamma_P(G)}$. We calculate

$$\begin{aligned}\mathcal{E}(G; S, q) &= \sum_{W \subseteq S} |\operatorname{Obs}(G; W)| q^{|S \setminus W|}(1-q)^{|W|} \\ &= \sum_{W \subseteq S} \left(|W| \frac{n}{\gamma_P(G)}\right) q^{\gamma_P(G) - |W|}(1-q)^{|W|}.\end{aligned}$$

The $\frac{n}{\gamma_P(G)}$ term is independent of the set W and can be factored out of the sum. As the specific set W does not impact the sum, we re-index the sum over all possible sizes of $W \subseteq S$. There are $\binom{\gamma_P(G)}{|W|}$ such sets for each possible $|W|$. Then we use the expected value of a binomial random variable to find

$$\begin{aligned}\mathcal{E}(G; S, q) &= \frac{n}{\gamma_P(G)} \sum_{|W|=0}^{\gamma_P(G)} |W| \binom{\gamma_P(G)}{|W|} q^{\gamma_P(G)-|W|}(1-q)^{|W|} \\ &= \frac{n}{\gamma_P(G)} \gamma_P(G)(1-q) \\ &= n(1-q).\end{aligned}$$

In this sense, $\mathcal{E}(G; S, q)$ being linear in q corresponds to a roughly evenly distributed PMU covering of G. This notion is formalized in Proposition 3.2.

3 Properties of the expected value polynomial

As in power domination, fragile power domination on a disconnected graph G can be observed as the sub-problems on the connected components of G. Particularly, given a graph G with connected components $H_1, \ldots, H_k$ and initial set or multiset of vertices S, $\mathcal{E}(G; S, q) = \sum_{i=1}^{k} \mathcal{E}(H_i; V(H_i) \cap S, q)$. Thus, from now on, all graphs are assumed to be connected.

3.1 Linearity

With $\mathcal{E}(G; S, q)$ having degree at most $|S|$, it is trivial to show that when $|S| = 1$ then $\mathcal{E}(G; S, q)$ is a linear function of q.

Observation 3.1. *For a graph G on n vertices with $\gamma_P(G) \geq 1$, a power dominating set or multiset S, and probability of PMU failure q, $\mathcal{E}(G; S, q) \geq n(1-q)^{|S|}$. Equality holds when $|S| = 1$, resulting in a linear $\mathcal{E}(G; S, q)$.*

To see this, notice

$$\begin{aligned}\mathcal{E}(G; S, q) &= \sum_{W \subseteq S} |\operatorname{Obs}(G; W)| q^{|S \setminus W|}(1-q)^{|W|} \\ &= n(1-q)^{|S|} + \sum_{W \subsetneq S} q^{|S \setminus W|}(1-q)^{|W|}.\end{aligned}$$

The following yields another condition for when $\mathcal{E}(G; S, q)$ is linear.

Proposition 3.2. *Let G be a graph, $S \subseteq V(G)$ be a set, and q be a probability of PMU failure. If for every $X \subseteq S$,*

$$|\operatorname{Obs}(G; X)| = \sum_{v \in X} |\operatorname{Obs}(G; \{v\})|,$$

then $\mathcal{E}(G; S, q) = |\operatorname{Obs}(G; S)|(1-q)$.

Proof. We show this by induction. Observe that $|S| = 1$ is immediate.

Now suppose the claim holds for vertex sets of size $k-1$ and let $S \subseteq V(G)$ be of size k such that

$$\left| \operatorname{Obs}(G; X) \right| = \sum_{v \in X} \left| \operatorname{Obs}(G; \{v\}) \right|$$

for every $X \subseteq S$. Fix $v \in S$ and let $Y = S \setminus \{v\}$. Then $\mathcal{E}(G; S, q)$ is equal to

$$q \sum_{W \subseteq Y} \left| \operatorname{Obs}(G; W) \right| q^{|Y \setminus W|}(1-q)^{|W|} + (1-q) \sum_{W \subseteq Y} \left| \operatorname{Obs}(G; W \cup \{v\}) \right| q^{|Y \setminus W|}(1-q)^{|W|}$$

and using the assumption on S we can split $\left| \operatorname{Obs}(G; W \cup \{v\}) \right| = \left| \operatorname{Obs}(G; W) \right| + \left| \operatorname{Obs}(G; \{v\}) \right|$, resulting in

$$\sum_{W \subseteq Y} \left| \operatorname{Obs}(G; W) \right| q^{|Y \setminus W|}(1-q)^{|W|} + (1-q) \sum_{W \subseteq Y} \left| \operatorname{Obs}(G; \{v\}) \right| q^{|Y \setminus W|}(1-q)^{|W|}$$

by combining the q and $1-q$ sums. This is now equal to

$$\mathcal{E}(G; Y, q) + (1-q) \left| \operatorname{Obs}(G; \{v\}) \right| \sum_{i=0}^{|Y|} \binom{|Y|}{i} q^{|Y|-i}(1-q)^{i}.$$

Then by the inductive hypothesis $\mathcal{E}(G; Y, q) = (1-q) \left| \operatorname{Obs}(G; S \setminus \{v\}) \right|$ and so applying the binomial theorem yields

$$(1-q) \left| \operatorname{Obs}(G; S \setminus \{v\}) \right| + (1-q) \left| \operatorname{Obs}(G; \{v\}) \right| = \left| \operatorname{Obs}(G; S) \right| (1-q). \qquad \square$$

We now construct a family of graphs for which the unique minimum power dominating set satisfies the assumption of Proposition 3.2. Start with the complete multipartite graph $K_{r_1, r_2, \ldots, r_k}$ with $k \geq 2$ and $r_i \geq 2$ for all i, and vertex partitions $A_1, A_2, \ldots, A_k$. For each partition A_i, add a vertex a_i adjacent to all vertices in A_i. Then for each a_i add two adjacent leaves. If desired, subdivide any edge incident to a_i any number of times. The set $\{a_1, a_2, \ldots, a_k\}$ is the unique minimum power dominating set and satisfies the hypothesis of Proposition 3.2. The family of graphs constructed from $K_{2,2}$ is shown in Fig. 2. This family of graphs demonstrates Proposition 3.2, where for all $v \in S$, the set $\operatorname{Obs}(G; \{v\})$ is disjoint from $\operatorname{Obs}(G; \{w\})$ for all $w \in S \setminus v$.

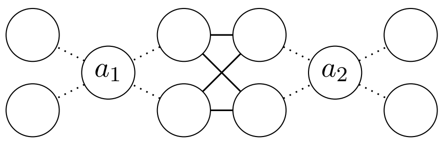

Figure 2: The family of graphs built from $K_{2,2}$ where the minimum power dominating set $\{a_1, a_2\}$ satisfies Proposition 3.2. The dotted lines indicate edges that may be subdivided.

Proposition 3.2 does not only hold when the vertices in S observe disjoint sets of vertices. Let $G = K_{1,3}$ and $S = \{x, y\}$ where x and y have degree 1. We calculate

$\mathcal{E}(G; S, q) = 4(1-q)$, and notice that $|\operatorname{Obs}(G; \{x\})| + |\operatorname{Obs}(G; \{y\})| = 2 + 2 = 4 = |\operatorname{Obs}(G; S)|$ but $\operatorname{Obs}(G; \{x\}) \cup \operatorname{Obs}(G; \{y\}) \neq \operatorname{Obs}(G; \{x, y\})$. In fact, we can construct a family of graphs for which the unique minimum power dominating sets satisfies the assumption of Proposition 3.2 and the sets of observed vertices associated with each PMU overlap. Start with the graph shown in Fig. 3. Add k leaves adjacent to vertex v. Notice that $S = \{v, w\}$ is the unique minimum power dominating set, and that $\mathcal{E}(G; S, q) = (8 + k)(1 - q)$.

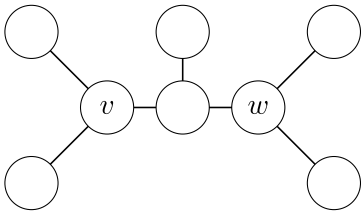

Figure 3: The base graph where the minimum power dominating set $\{v, w\}$ satisfies Proposition 3.2.

A consequence of the proof of Proposition 3.2 is a condition for when the degree of the expected value polynomial does not change after a vertex is added to a given PMU placement.

Corollary 3.3. *Let G be a graph and let q be a probability of PMU failure. Let S be a set or multiset of vertices and $v \in V(G) \setminus S$. If for all $W \subseteq S$, $|\operatorname{Obs}(G; W \cup \{v\})| = |\operatorname{Obs}(G; W)| + |\operatorname{Obs}(G; \{v\})|$, then $\mathcal{E}(G; S, q)$ and $\mathcal{E}(G; S \cup \{v\}, q)$ are the same degree.*

A natural question to ask is whether, for all q, a power dominating set is better or worse than the linear expected value given by Proposition 3.2. When q is large, it is important that each PMU individually observes as many vertices as possible. One way to formalize this idea is through the following definition.

Definition 3.4. Let G be a graph and let S be a set or multiset of the vertices of G. Then we call S a *local cover of G* if for every $x \in V(G)$, there exists a vertex $v \in S$ such that $x \in \operatorname{Obs}(G; \{v\})$.

The notion of a local cover gives an alternative to Proposition 3.2.

Proposition 3.5. *Let G be a graph on n vertices and let q be a probability of PMU failure. If S is a local cover of $G' = G\big[\operatorname{Obs}(G; S)\big]$, then $\mathcal{E}(G; S, q) \geq |\operatorname{Obs}(G; S)|(1-q)$ for all q. In particular, if S is a power dominating set then $\mathcal{E}(G; S, q) \geq n(1-q)$ for all q.*

Proof. Note that $\mathcal{E}(G; S, q) = \sum_{v \in V(G)} \operatorname{Prob}[v \text{ is observed}]$. Since S is a local cover of G', there is at least one $w \in S$ such that $v \in \operatorname{Obs}(G', S)$ for every $v \in G'$. In this way, the following inequality holds:

$$\begin{aligned}\sum_{v\in V(G)} \text{Prob}[v \text{ is observed}] &= \sum_{v\in V(G')} \text{Prob}[v \text{ is observed}]\\ &\geq \sum_{v\in V(G')} (1-q) = |V(G')|(1-q).\end{aligned}$$

When S is a power dominating set of G, then $G' = G$ and thereby $\mathcal{E}(G; S, q) \geq n(1-q)$. □

The following is an immediate corollary for dominating sets.

Corollary 3.6. *For a graph G on n vertices, dominating set D of G, and probability of PMU failure q, $\mathcal{E}(G; D, q) \geq n(1-q)$ for all q.*

Notice that the converse of Proposition 3.5 is not true. This can be seen by the graph G in Fig. 4. For this graph,

$$\mathcal{E}(G; \{a, b, c\}, q) = 16(1-q)^3 + 40q(1-q)^2 + 23q^2(1-q).$$

For all $q \in [0, 1]$, $\mathcal{E}(G; \{a, b, c\}, q) \geq 16(1-q)$, but $\{a, b, c\}$ is not a local cover of G as x is not observed by any individual vertex in $\{a, b, c\}$. Moreover, being a k-rPDS for $k \geq 1$ does not guarantee being a local cover. For example, the set of leaves of G is a 1-rPDS because a set consisting of any six of the seven leaves forms a power dominating set, but vertex z is not in $\text{Obs}(G; \{v\})$ for any leaf v of G. In fact, the only minimum local covers of G are $\{a, b, c, x\}$ and $\{a, b, c, y\}$.

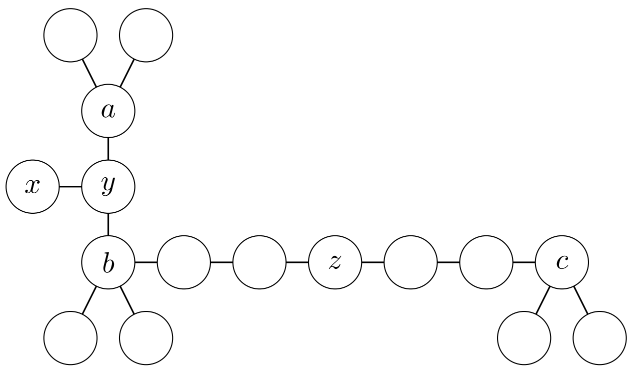

Figure 4: A graph G with a minimum power dominating set $S = \{a, b, c\}$ for which $\mathcal{E}(G; S, q) \geq n(1-q)$ for $q \in [0, 1]$, but S is not a local cover.

3.2 Coefficients

In this section, we will explore the relationship between the structure of the expected value polynomial and and properties of its fixed PMU placements. Specifically, we show a correspondence between the coefficients of $\mathcal{E}(G; S, q)$ and whether the PMU placement S is PMU-defect-robust or fault-tolerant. We also show a connection to the failed power domination number. We begin with the following result which calculates the coefficients of $\mathcal{E}(G; S, q)$ in standard form.

Lemma 3.7. *Let G be a graph and q a probability of PMU failure. Let S be a set or multiset of vertices and $r \in \{0, \ldots, |S|\}$. Then the coefficient of $q^{|S|-r}$ in $\mathcal{E}(G; S, q)$ is*

$$\sum_{\substack{W \subseteq S: \\ |W| \geq r}} (-1)^{|W|-r} \binom{|W|}{r} |\operatorname{Obs}(G; W)|.$$

Proof. Let G be a graph. Notice

$$\begin{aligned}
\mathcal{E}(G; S, q) &= \sum_{W \subseteq S} |\operatorname{Obs}(G; W)| q^{|S \setminus W|}(1-q)^{|W|} \\
&= \sum_{W \subseteq S} |\operatorname{Obs}(G; W)| q^{|S \setminus W|} \sum_{i=1}^{|W|} \binom{|W|}{i} (-q)^i \\
&= \sum_{W \subseteq S} \sum_{i=0}^{|W|} (-1)^i \binom{|W|}{i} |\operatorname{Obs}(G; W)| q^{|S \setminus W|+i}
\end{aligned}$$

and so we contribute to the coefficient of $q^{|S|-r}$ whenever $|W| \geq r$ and $i = |W| - r$. Hence the coefficient of $q^{|S|-r}$ in $\mathcal{E}(G; S, q)$ is

$$\sum_{\substack{W \subseteq S: \\ |W| \geq r}} (-1)^{|W|-r} \binom{|W|}{r} |\operatorname{Obs}(G; W)|. \qquad \square$$

Lemma 3.7 gives a complete characterization of k-PMU-defect-robust power dominating sets or multisets.

Theorem 3.8. *Let G be a graph on n vertices, q be a probability of PMU failure, and the set or multiset of vertices B be an initial placement of PMUs with $|B| = b$. Then B is a k-PMU-defect-robust power dominating set of G if and only if $\mathcal{E}(G; B, q)$ has the form*

$$\mathcal{E}(G; B, q) = n - q^{k+1} h(q)$$

for some polynomial $h(q)$.

Proof. First assume B is a k-rPDS. Consider $\mathcal{E}(G; B, q)$ from the perspective of expected number of vertices *not* observed and see

$$\begin{aligned}
\mathcal{E}(G; B, q) &= \mathbb{E}[|\operatorname{Obs}(G; S, q)|] \\
&= \mathbb{E}[|V(G)| - |V(G) \setminus \operatorname{Obs}(G; S, q)|] \\
&= n - \sum_{W \subseteq S} \big(n - |\operatorname{Obs}(G; W, q)|\big)(1-q)^{|W|} q^{|S|-|W|}.
\end{aligned}$$

Since S is a k-rPDS, $\text{Obs}\,(G; W, q) = V(G)$ for all $W \subseteq S$ with $|W| > |S| - k - 1$. Hence,

$$\begin{aligned}\mathcal{E}\,(G; B, q) &= n - \sum_{\substack{W \subseteq S:\\ |W| \le |S| - k - 1}} \big(n - |\,\text{Obs}\,(G; W, q)\,|\big)(1-q)^{|W|} q^{|S|-|W|} \\ &= n - q^{k+1} \sum_{\substack{W \subseteq S:\\ |W| \le |S| - k - 1}} \big(n - |\,\text{Obs}\,(G; W, q)\,|\big)(1-q)^{|W|} q^{|S|-|W|-k-1}.\end{aligned}$$

Note that $|S| - |W| - k - 1 \ge 0$, so the summation is thereby a polynomial in q.

Next assume that $\mathcal{E}\,(G; B, q) = n - q^{k+1} h(q)$. Note that

$$n - \sum_{v \in V(G)} \text{Prob}\,[v \text{ is not observed}] = \mathcal{E}\,(G; B, q) = n - q^{k+1} h(q).$$

Thus,

$$\sum_{v \in V(G)} \text{Prob}\,[v \text{ is not observed}] = q^{k+1} h(q).$$

That is, we can factor q^{k+1} from each term in the sum. Thus, for any vertex, we see that at least $k+1$ PMUs must fail in order for the vertex to be unobserved. This means that any k PMUs can fail and the vertex is observed, so S is a k-rPDS. □

Notice that a set which is k-fault-tolerant is also k-PMU-defect-robust, and so Theorem 3.8 gives us the following corollary.

Corollary 3.9. *Let G be a graph on n vertices, q be a probability of PMU failure, and the set $S \subseteq V(G)$ be an initial placement of PMUs. Then S is a k-fault-tolerant power dominating set of G if and only if $\mathcal{E}\,(G; S, q)$ has the form*

$$\mathcal{E}\,(G; S, q) = n - q^{k+1} h(q)$$

for some polynomial $h(q)$.

The largest set that can be used in k-fault-tolerant power domination is the entire vertex set. This means that $V(G)$ determines the largest possible k for which G can be k-fault-tolerant. We can determine this k using the expected value polynomial.

Proposition 3.10. *Let G be a graph on n vertices and let q be a probability of PMU failure. Writing $\mathcal{E}\,(G; V(G), q) = n - q^{k+1} h(q)$ for some polynomial $h(q)$ containing a nonzero constant term, it follows that k is the largest possible k for which G can be k-fault-tolerant.*

Notice that if the entire vertex set is at most k-PMU-defect-robust, then there must exist some failed power dominating set F with $|F| = n - k - 1$. Then by Theorem 3.8, we obtain a similar structural constraint on the expected value polynomial in terms of the failed power domination number.

Corollary 3.11. *For a graph G on n vertices such that $\mathcal{E}\,(G; V(G), q)$ has the form $n - q^{k+1} h(q)$ with $h(q)$ containing a nonzero constant term, we have $\overline{\gamma}_P\,(G) = n - k - 1$.*

3.3 Comparisons

Now we will use the expected value polynomial as a tool to compare PMU placements. We examine what happens when a PMU is added to an existing placement. Then we compare failed power dominating sets to power dominating sets of the same size.

Lemma 3.12. *Let G be a graph, S a set or multiset of vertices of G, and q a probability of PMU failure. Given a vertex $v \in V(G)$, it holds that $\mathcal{E}(G; S, q) \leq \mathcal{E}(G; S \cup \{v\}, q)$.*

Proof. By definition, $\mathcal{E}(G; S \cup \{v\}, q)$ equals

$$\sum_{W \subseteq S} |\operatorname{Obs}(G; W)| q^{|S \setminus W|+1}(1-q)^{|W|} + \sum_{W \subseteq S} |\operatorname{Obs}(G; W \cup \{v\})| q^{|S \setminus W|}(1-q)^{|W|+1}.$$

where the first sum corresponds to subsets or submultisets of S, and the second sum to those which add v. Note that the first sum reduces to $q\,\mathcal{E}(G; S, q)$ and the second sum reduces to $(1-q) \sum_{W \subseteq S} |\operatorname{Obs}(G; W \cup \{v\})| * q^{|S \setminus W|}(1-q)^{|W|}$. The set $W \cup \{v\}$ observes all the vertices observed by the set W, so $|\operatorname{Obs}(G; W \cup \{v\})| \geq |\operatorname{Obs}(G; W)|$. Thus,

$$\begin{aligned} \mathcal{E}(G; S \cup \{v\}, q) &\geq q\mathcal{E}(G; S, q) + (1-q)\mathcal{E}(G; S, q) \\ &= \mathcal{E}(G; S, q). \end{aligned}$$

□

Restricting S to be a set results in the following upper bound for the expected value polynomial.

Corollary 3.13. *Let G be a graph, $S \subseteq V(G)$ a set, and q a probability of PMU failure. Then $\mathcal{E}(G; S, q) \leq \mathcal{E}(G; V(G), q)$.*

We now compare a fixed set or multiset of vertices S to those of size $|S| + 1$, including ones which do not contain S. Consider the graph G in Fig. 5. The best placement of 1 PMU is $\{6\}$ for all values of $q \in [0, 1]$. Considering sets and multisets S with $|S| = 2$ and $\{6\} \subset S$ we calculate the following expected value polynomials:

- $\mathcal{E}(G; S, q) = 10q(1-q) + 7(1-q)^2$ when $S = \{2, 6\}, \{3, 6\}, \{6, 9\}$, or $\{6, 10\}$,
- $\mathcal{E}(G; S, q) = 9q(1-q) + 9(1-q)^2$ when $S = \{1, 6\}, \{4, 6\}, \{6, 8\}$, or $\{6, 11\}$,
- $\mathcal{E}(G; S, q) = 14q(1-q) + 7(1-q)^2$ when $S = \{6, 6\}$, and
- $\mathcal{E}(G; S, q) = 13q(1-q) + 9(1-q)^2$ when $S = \{5, 6\}$ or $\{6, 7\}$.

The placement $\{5, 7\}$, corresponding to $\mathcal{E}(G; S, q) = 12q(1-q) + 11(1-q)^2$, is expected to observe more vertices of G for $q \in [0, 2/3)$ than any other 2-set or multiset, including those containing vertex 6. Therefore, even if a set or multiset of vertices S is the best PMU placement using $|S|$ PMUs for all values of q, it is not the case

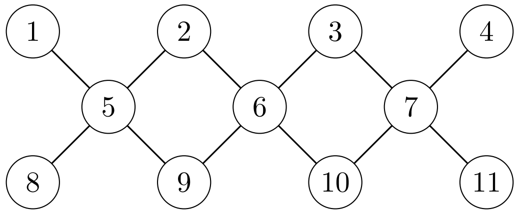

Figure 5: Small graph G on 11 vertices.

that the best PMU placement using $|S|+1$ PMUs is of the form $S \cup \{v\}$ for some $v \in V(G)$.

It is natural to believe that once $|S| \geq \gamma_P(G)$, then $\mathcal{E}(G; S, q)$ will always be bounded above by a power dominating set. Unfortunately, the following describes how this is not always the case.

Theorem 3.14. *For any $q_* \in (0,1) \cap \mathbb{Q}$, there exists a graph G, failed power dominating set F, and power dominating set S with $|F| = |S|$ such that $\mathcal{E}(G; F, q) \leq \mathcal{E}(G; S, q)$ for all $0 \leq q \leq q_*$ and $\mathcal{E}(G; S, q) \leq \mathcal{E}(G; F, q)$ for all $q_* \leq q \leq 1$.*

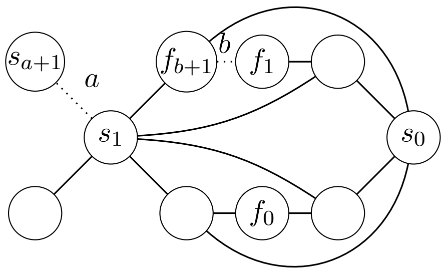

Figure 6: The family of graphs $G_{a,b}$ where the edge $s_1 s_{a+1}$ was subdivided a times and the edge $f_1 f_{b+1}$ was subdivided b times. Note that $\{s_0, s_1\}$ is a power dominating set and $\{f_0, f_1\}$ is a failed power dominating set.

Proof. Consider the graph $G_{a,b}$ as in Fig. 6 where the edge $s_1 s_{a+1}$ is subdivided $a \geq 1$ times and the edge $f_1 f_{b+1}$ is subdivided $b \geq 1$ times. It can be verified that $S = \{s_0, s_1\}$ is a power dominating set that observes all $10 + a + b$ vertices and $F = \{f_0, f_1\}$ is a failed power dominating set that observes only $6 + b$ vertices. We show that a and b can be chosen so that the unique intersection of $\mathcal{E}(G_{a,b}; S, q)$ and $\mathcal{E}(G_{a,b}; F, q)$ is any $q_* \in (0,1) \cap \mathbb{Q}$. Observe that

$$\mathcal{E}(G_{a,b}; S, q) = (10 + a + b)(1-q)^2 + (12 + a)(1-q)q$$

and

$$\mathcal{E}(G_{a,b}; F, q) = (6 + b)(1-q)^2 + (6 + b)(1-q)q.$$

Let $g(q) = \mathcal{E}(G_{a,b}; S, q) - \mathcal{E}(G_{a,b}; F, q)$ and notice g can be simplified to

$$g(q) = (q-1)\big((b-2)q - (a+4)\big).$$

In this form, it can be seen that the zeros of g are 1 and $\frac{a+4}{b-2}$. Then given any $q_* \in (0,1) \cap \mathbb{Q}$, choose positive integers a and b such that $q_* = \frac{a+4}{b-2}$.

Since S is a power dominating set and F is a failed power dominating set, $\mathcal{E}(G_{a,b}; F, 0) < \mathcal{E}(G_{a,b}; S, 0)$. Then for all $0 \leq q \leq \frac{a+4}{b-2}$, it follows that $\mathcal{E}(G_{a,b}; F, q) \leq \mathcal{E}(G_{a,b}; S, q)$. Since $g(q)$ is a quadratic whose roots are $\frac{a+4}{b-2}$ and 1, it necessarily follows that $\mathcal{E}(G_{a,b}; S, q) \leq \mathcal{E}(G_{a,b}; F, q)$ for $\frac{a+4}{b-2} \leq q \leq 1$. □

4 Probability of observing the entire graph

Given a graph G, an initial placement of PMUs S, and a probability q of PMU failure, we now investigate the probability that the entire vertex set will be observed at the termination of the fragile power domination process. Corollary 3.13 shows that when restricting to sets as PMU placements, the best probability of observing the entire graph G occurs when $S = V(G)$. We examine this in Section 4.1. We then consider how the probability of observing the entire graph is related to the failed power domination number and the k-PMU-defect-robust power domination number in Section 4.2. We conclude with examples and calculate the exact probability for stars in Section 4.3.

4.1 Using the entire the vertex set

If a PMU is placed on every vertex, the probability that this placement observes the entire graph is connected to the total number of power dominating sets. In [6], Brimkov, et. al. introduced the *power domination polynomial* as a tool to count the number of power dominating sets of a graph.

Definition 4.1. Given a graph G on n vertices, let $p(G;i)$ be the number of power dominating sets of size i. Define the *power domination polynomial* as

$$\mathcal{P}(G;x) = \sum_{i=1}^{n} p(G;i)x^i.$$

Using power domination polynomial notation, the probability of observing the entire graph with a PMU placed on every vertex is as follows.

Observation 4.2. *Let G be a graph on n vertices and q be a probability of PMU failure. Then,*

$$\text{Prob}\Big[|\operatorname{Obs}(G; V(G), q)| = n\Big] = \sum_{i=1}^{n} p(G;i)q^{n-i}(1-q)^i.$$

If $i > \overline{\gamma}_P(G)$ then all subsets of $V(G)$ of with i vertices are power dominating sets. This gives a lower bound for the probability that $V(G)$ observes the entire graph.

Proposition 4.3. *Let G be a graph on n vertices with $\overline{\gamma}_P(G) = f$ and let q be a probability of PMU failure. Then*

$$\text{Prob}\Big[|\operatorname{Obs}(G; V(G), q)| = n\Big] \geq \sum_{i=f+1}^{n} \binom{n}{i} q^{n-i}(1-q)^i.$$

For a connected graph on at least 3 vertices, $p(G;i) = \binom{n}{i}$ for $n-2 \leq i \leq n$ [6, Corollary 4]. Moreover, any such graph has $\overline{\gamma}_P(G) \leq n-3$. This gives the following lower bound.

Corollary 4.4. *Let G be a connected graph on $n \geq 3$ vertices and let q be a probability of PMU failure. Then*

$$\text{Prob}\Big[|\operatorname{Obs}(G; V(G), q)| = n\Big] \geq \binom{n}{n-2} q^2(1-q)^{n-2} + nq(1-q)^{n-1} + (1-q)^n.$$

4.2 Relating to other power domination parameters

In this section we produce estimates for observing the entire graph using the failed power domination number and the k-PMU-defect-robust power domination number. We now introduce a specialized case of a bound on the tails of random variables.

Theorem 4.5 (Hoeffding's Inequality, [9, Theorem 1])**.** *Let $X_1, \ldots, X_m$ be independent Bernoulli random variables, and let $X = \sum_{i=1}^m X_i$. Then for any $t > 0$,* $\text{Prob}\big[X \leq \mathbb{E}[X] - t\big] \leq e^{-2t^2/m}$.

It now suffices to observe that if S is an initial PMU placement and $q = 1-p \in (0,1)$ is a probability of PMU failure, then $|S^*|$ is distributed as a Binomial$(|S|, p)$ random variable. In particular, for each $v \in S$, we can define a Bernoulli random variable S_v such that $S_v = 1$ with probability p, $S_v = 0$ otherwise, and $|S^*| = \sum_{v \in S} S_v$. This leads to the following.

Proposition 4.6. *Let G be a graph on n vertices, let S be a set of vertices, and let $q \in (0,1)$ be a probability of PMU failure. If $|S| > \overline{\gamma}_P(G)/(1-q)$, then*

$$\text{Prob}\big[|\operatorname{Obs}(G; S, q)| = n\big] \geq 1 - \exp\Big(-2|S|\Big(1 - q - \frac{\overline{\gamma}_P(G)}{|S|}\Big)^2\Big).$$

Proof. Observe

$$\text{Prob}\big[|\operatorname{Obs}(G; S, q)| = n\big] \geq \text{Prob}\big[|S^*| \geq \overline{\gamma}_P(G) + 1\big] = 1 - \text{Prob}\big[|S^*| \leq \overline{\gamma}_P(G)\big].$$

The result now follows from Hoeffding's Inequality by taking $t = (1-q)|S| - \overline{\gamma}_P(G)$. □

This bound can be slightly improved asymptotically by, for instance, the bound derived in [7, Theorem 1].

Proposition 4.7. *Let G be a graph on n vertices, let S be a set of vertices, and let $q \in (0,1)$ be a probability of PMU failure. Define $H(a,p) = a\ln\frac{a}{p} + (1-a)\ln\frac{1-a}{1-p}$ to be the relative entropy between the Bernoulli(a) and Bernoulli(p) distributions. If $|S| > \overline{\gamma}_P(G)/(1-q)$, then*

$$\text{Prob}\left[|\operatorname{Obs}(G;S,q)| = n\right] \geq 1 - \exp\Big(-|S|\,H\Big(\frac{\overline{\gamma}_P(G)}{|S|}, 1-q\Big)\Big).$$

For an estimate of $\text{Prob}\left[|S^*| \leq \overline{\gamma}_P(G)\right]$ when $|S| \leq \overline{\gamma}_P(G)/(1-q)$, we turn to estimating the binomial distribution by the normal distribution.

Theorem 4.8 (De Moivre-Laplace Theorem)**.** *Let $q = 1-p \in [0,1]$, and let X be a Binomial(m,p) random variable. Let $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2}\,dt$ denote the standard normal cumulative distribution function. For fixed z_1 and z_2,*

$$\lim_{n\to\infty} \text{Prob}\left[z_1 \leq \frac{X - mp}{\sqrt{mpq}} \leq z_2\right] = \Phi(z_2) - \Phi(z_1).$$

In the context of our problem p and q are fixed, so let $z_1 = -|S|p/\sqrt{|S|pq}$ and $z_2 = \big(\overline{\gamma}_P(G) - |S|p\big)/\sqrt{|S|pq}$. Then since $0 \leq |S^*|$,

$$\text{Prob}\left[|S^*| \leq \overline{\gamma}_P(G)\right] = \text{Prob}\left[z_1 \leq \frac{|S^*| - |S|p}{\sqrt{|S|pq}} \leq z_2\right] \approx \Phi(z_2) - \Phi(z_1)$$

up to some error term 2ε. Note that we have an error contribution of up to ε from each of the two approximations $\Phi(z_1)$ and $\Phi(z_2)$. In fact, the Berry-Esseen theorem [8] actually gives a bound on ε. Namely, $\varepsilon \leq C(p^2+q^2)/\sqrt{|S|pq}$ where $C < 0.4748$ is some positive constant. The calculation of this term is standard once observing $(|S^*| - |S|p)/\sqrt{|S|pq} = (|S|pq)^{-1/2}\sum_{v\in S}(S_v - p)$ where $S_v = 1$ with probability $p = 1-q$.

Putting this all together,

$$\text{Prob}\left[|\operatorname{Obs}(G;S,q)| = n\right] \geq 1 - \Phi\Big(\frac{\overline{\gamma}_P(G) - |S|p}{\sqrt{|S|pq}}\Big) + \Phi\Big(\frac{-|S|p}{\sqrt{|S|pq}}\Big) - \frac{.9496(p^2+q^2)}{\sqrt{|S|pq}}.$$

Notice that $\Phi\Big(\frac{-|S|p}{\sqrt{|S|pq}}\Big) \to 0$ and $\frac{.9496(p^2+q^2)}{\sqrt{|S|pq}} \to 0$ as $|S| \to \infty$. Moreover, $\Phi\Big(\frac{\overline{\gamma}_P(G)-|S|p}{\sqrt{|S|pq}}\Big) \to 0$ exactly when $p > \overline{\gamma}_P(G)/|S|$, i.e., when $|S| > \overline{\gamma}_P(G)/(1-q)$. Indeed, if $(1-q)|S|$, the expected number of PMUs which do not fail, is larger than the failed power domination number, then we expect the entire graph to be observed.

A higher probability of observing the entire graph is related to PMU-defect-robust power domination. A k-rPDS is a good choice for a PMU placement because any k PMUs can fail and not compromise the observability of the graph.

Theorem 4.9. *Given a graph G on n vertices, a k-PMU-defect-robust power dominating set or multiset S, and a probability of PMU failure q, the probability that S observes the entire graph G is*

$$\text{Prob}\Big[|\operatorname{Obs}(G;S,q)| = n\Big] \geq 1 - \sum_{i=0}^{|S|-k-1} \binom{|S|}{i} q^{|S|-i}(1-q)^i.$$

Equality holds when $|S| - k = \gamma_P(G)$.

Proof. Let S be a k-rPDS, then all subsets or submultisets $W \subseteq S$ with $|W| \geq |S|-k$ are power dominating sets. The probability that S observes all of G is at least the probability that $|W| \geq |S|-k$, or one minus the probability that $|W| < |S|-k$. The inequality then follows.

Note that if $|S| - k = \gamma_P(G)$, then $W \subseteq S$ is a power dominating set if and only if $|W| \geq |S| - k$ and so equality holds. □

4.3 Examples calculating the probability of entire network observance

Consider the complete bipartite graph $K_{3,3}$ and probability of PMU failure $q = 0.1$. Proposition 4.7 yields at least a 64% chance of complete network observance, that is, $\text{Prob}\Big[|\operatorname{Obs}(K_{3,3};S,0.1)| = 6\Big] \geq 0.64$. Note that any set of 2 distinct vertices of $K_{3,3}$ forms a 0-rPDS power dominating set, so Theorem 4.9 yields at least an 81% chance of complete network observance, that is, $\text{Prob}\Big[|\operatorname{Obs}(K_{3,3};S,0.1)| = 6\Big] \geq 0.81$. Similarly, for any 3-set S', Proposition 4.7 yields at least a 93.925% chance of complete network observance. Since a 3-set is necessarily a 1-rPDS of $K_{3,3}$, Theorem 4.9 yields at least a 97.2% chance of complete network observance.

We now determine the probability of observing the entire star S_n given *any* PMU placement.

Proposition 4.10. *Let S_n denote the star with universal vertex v_0 and let q be a probability of PMU failure. Then for any set $S \subseteq V(S_n)$,* $\text{Prob}\Big[|\operatorname{Obs}(S_n;S,q)| = n\Big]$ *is equal to*

$$\begin{cases} (1-q)^{|S|}, & \text{if } v_0 \notin S \text{ and } |S| = n-2 \\ (1-q)^{|S|} + |S|q(1-q)^{|S|-1}, & \text{if } v_0 \notin S \text{ and } |S| = n-1 \\ 1-q, & \text{if } v_0 \in S \text{ and } |S| \leq n-2 \\ 1-q+q(1-q)^{|S|-1}, & \text{if } v_0 \in S \text{ and } |S| = n-1 \\ 1-q+q(1-q)^{|S|-1} + (|S|-1)q^2(1-q)^{|S|-2}, & \text{if } v_0 \in S \text{ and } |S| = n \\ 0, & \text{otherwise.} \end{cases}$$

Proof. First suppose $v_0 \notin S$. If S does not have at least $n-2$ leaves, then S_n can never be fully observed. If $|S| = n-2$, then we need every vertex to fully observe S_n, which occurs with probability $(1-q)^{|S|}$. If $|S| = n-1$, then to fully observe S_n

either no PMUs fail or one PMU fails. These are disjoint events which occur with probabilities $(1-q)^{|S|}$ and $|S|q(1-q)^{|S|-1}$ respectively.

Now suppose $v_0 \in S$. If $|S| \leq n-2$ then the only way that S_n is fully observed is if $v_0 \in S^*$, which occurs with probability $1-q$. If $|S| = n-1$, the probability that S_n is fully observed is equal to

$$\text{Prob}\left[v_0 \in S^* \vee \{v_0 \notin S^* \wedge |S^*| = n-2\}\right] = (1-q) + q(1-q)^{|S|-1}.$$

Finally, if $|S| = n$ then the probability that S_n is fully observed is equal to

$$\begin{aligned}&\text{Prob}\left[v_0 \in S^* \vee \{v_0 \notin S^* \wedge |S^*| \geq n-2\}\right]\\ &\quad = (1-q) + q\,\text{Prob}\left[\big(|S^*| = n-2 \vee |S^*| = n-1\big) : v_0 \notin S^*\right]\\ &\quad = (1-q) + q\left(\left(|S|-1\right)q(1-q)^{|S|-2} + (1-q)^{|S|-1}\right).\end{aligned}$$

Therefore the result holds. □

We now apply this theorem by way of example. Consider the star S_{20} and probability of PMU failure $q = 0.1$ with initial PMU placement $S = V(S_{20})$. Then Proposition 4.10 yields an approximate 94.20% chance of complete network observance, that is, $\text{Prob}\Big[|\,\text{Obs}\,(S_{20}; V(S_{20}), q)\,| = 20\Big] \approx 0.9420$. Hence, it is not possible to utilize *sets* of vertices in order to obtain any higher probability of observing the entire graph. However, if we instead place two PMUs on the universal vertex v_0 to create a 1-rPDS, Theorem 4.9 yields a 99% chance of complete network observance. It is only with *multisets* that one can achieve over 95% chance of complete network observance, and with substantially fewer PMUs.

5 The expected value polynomial for graph families

We determine the expected value polynomial for a PMU placement for the following graph families: a generalization of barbell graphs, stars, and complete multipartite graphs.

5.1 Generalized barbells

Given a graph G, let $\overline{G}$ denote the *complement of* G where $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{xy : xy \notin E(G)\}$.

Lemma 5.1. *Let* $G \in \{C_n, W_n, K_n, \overline{C_{n+2}} : n \geq 3\}$. *Construct* H *from* G *by attaching a leaf to an arbitrary vertex of* G. *Then for any* $v \in V(G)$, $\{v\}$ *is a power dominating set for* H.

Proof. Throughout what follows, let x be the leaf attached to some vertex of G.

If $G = K_n$ then any vertex $v \in V(G)$ dominates $V(G)$ and observes x in either the domination step or at most one zero forcing step.

If $G = C_n$, then any $v \in V(G)$ is a power dominating set for C_n, and the power domination process is unaffected by attaching x. Then x becomes observed after at most one additional zero forcing step.

When $G = W_n$, the leaf x can be adjacent to either the cycle or the universal vertex. If x is adjacent to the cycle, then for any $v \in V(G)$ the center of the wheel is observed in domination step, after which the $G = C_n$ case is recovered. If x is adjacent to the universal vertex, power domination proceeds as on W_n before observing x in at most one additional step.

Finally, let $G = \overline{C_{n+2}}$, label the vertices of G cyclically $v_0, \ldots, v_{n-1}$ with indices modulo n, and by symmetry assume x is adjacent to v_0 in H. Let $v_i \in V(G)$ be arbitrary. If $i = 0$ then we recover the case of $\overline{\gamma}_P\left(\overline{C_{n+2}}\right)$ from [5]. Otherwise, notice that v_i dominates everything except v_{i-1} and v_{i+1}. Then there are three unobserved vertices remaining. At least one of v_{i-2} or v_{i+2} is not adjacent to x, without loss of generality say v_{i+2} is not adjacent to x. Then v_{i+2} observes v_{i-1} as its only unobserved neighbor. Finally, one of v_{i-1} or v_{i-2} is not a neighbor of x and hence observes v_{i+1}. Since all of $V(G)$ has been observed, x becomes observed. □

We now introduce the generalized barbell graph.

Definition 5.2. Let G_1, G_2 be graphs and pick $x_1 \in V(G_1)$, $x_2 \in V(G_2)$. Add the edge x_1x_2 and subdivide it $m \geq 0$ times. The resultant graph is a *generalized barbell graph*, denoted by $B(G_1, x_1, G_2, x_2, m)$.

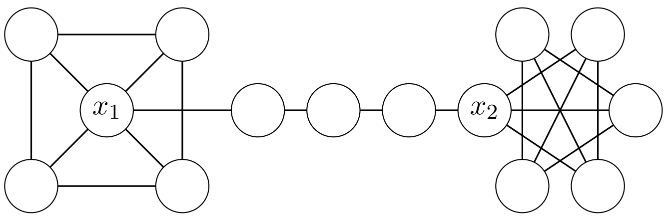

Figure 7: The generalized barbell graph $B(W_5, x_1, \overline{C_6}, x_2, 3)$.

The m vertices of a generalized barbell graph $B(G_1, x_1, G_2, x_2, m)$ between x_1 and x_2 are referred to as the *central path* of the graph. If $m = 0$, then G_1 and G_2 are connected by an edge, and the central path is empty. Note that $B(K_n, x_1, K_n, x_2, 0)$ for any x_1, x_2 is the usual barbell graph. Fig. 7 demonstrates the generalized barbell graph $G(W_5, x_1, \overline{C_6}, x_2, 3)$. We now determine the expected value polynomial for certain generalized barbell graphs.

Proposition 5.3. *Let $G_1, G_2 \in \{K_1, K_n, W_n, C_n, \overline{C_{n+2}} : n \geq 3\}$ with $|V(G_1)| = \ell$ and $|V(G_2)| = n$, and construct the generalized barbell graph $G = B(G_1, x_1, G_2, x_2, m)$ for any $x_i \in V(G_i)$. Let $S_{r,s,t}$ be a subset or submultiset of $V(G)$ containing r vertices from G_1, s vertices from the central path, and t vertices from G_2. Note if $m = 0$ then $s = 0$ necessarily. Then for a given probability of PMU failure q, $\mathcal{E}(G; S_{r,s,t}, q)$ is given by*

$$(\ell+m+n)(1-q^r)(1-q^t)+(\ell+m+1)(1-q^r)q^t+(m+n+1)q^r(1-q^t)+(m+2)q^{r+t}(1-q^s).$$

Proof. Partition $S_{r,s,t} = R \sqcup S \sqcup T$ as disjoint vertex subsets of size r, s, and t corresponding to G_1, the central path, and G_2 respectively. Denote by R^* the random set which contain a vertex $v \in R$ with probability $1 - q$, and similarly write S^* and T^*. Notice that if $|R^*| \geq 1$ then by Lemma 5.1, G_1, the central path, and x_2 will be observed. Similarly, if $|T^*| \geq 1$ then G_2, the central path, and x_1 will be observed.

On the other hand, if $|R^*| = |T^*| = 0$ but $|S^*| > 0$, then the entire central path, x_1, and x_2 will be observed. Hence, we can expand $\mathcal{E}\left(G; S_{r,s,t}, q\right)$ using the definition of expected value to obtain

$$(\ell+m+n)\,\mathrm{Prob}\left[\exists v \in R^* \wedge \exists v \in T^*\right] + (\ell+m+1)\,\mathrm{Prob}\left[\exists v \in R^* \wedge \nexists v \in T^*\right] +$$
$$(m+n+1)\,\mathrm{Prob}\left[\nexists v \in R^* \wedge \exists v \in T^*\right] + (m+2)\,\mathrm{Prob}\left[\nexists v \in R^* \wedge \nexists v \in T^* \wedge \exists v \in S^*\right].$$

The result follows from some basic probability calculations. □

5.2 Stars

We now calculate the expected value polynomial for stars. First, we note that for any graph G, $S \subseteq V(G)$, and probability of PMU failure q, we can write

$$\mathcal{E}\left(G; S, q\right) = |V(G)| - \sum_{v \in V(G)} \mathrm{Prob}\left[v \text{ is not observed}\right].$$

Theorem 5.4. *Let S_n denote the star on n vertices with center vertex c. If S is a set of vertices containing ℓ leaves, define the following:*

$$C = \begin{cases} q & c \in S \\ 1 & \textit{otherwise} \end{cases} \quad X = \begin{cases} 1-(1-q)^{\ell-1} & \ell = n-1 \\ 1 & \textit{otherwise} \end{cases} \quad Y = \begin{cases} 1-(1-q)^{\ell} & \ell = n-2 \\ 1 & \textit{otherwise.} \end{cases}$$

If q is a probability of PMU failure, then

$$\mathcal{E}\left(S_n; S, q\right) = n - \left(q^{|S|} + \ell q C X + (n-\ell-1) C Y\right).$$

Proof. We consider Prob[v is not observed] for each vertex in turn. The center vertex is only unobserved if every PMU fails, giving $q^{|S|}$. For a leaf v with a sensor, v is not observed only if all of the following occur:

1. its own sensor fails,
2. any PMU on the center fails, and
3. v is not observed via a zero forcing step.

Observe that 1. occurs with probability q and 2. occurs with probability 1 if there is no such sensor or q if there is such a sensor. For 3., note that in this case, all leaves must have started with a PMU and all must have failed, which occurs with probability $1 - (1 - q)^{\ell-1}$. We obtain

$$\mathrm{Prob}\left[v \text{ is not observed}\right] = qCX.$$

For a leaf v without a sensor, v is not observed if either one of the following occur:

1. any PMU on the center fails, or
2. v is not observed via a zero forcing step.

Note that 1. occurs with probability C. For 2., either there is no zero forcing step possible or all remaining leaves have a PMU, i.e. $\ell = n-2$ and we do not have the case that all leaf PMUs succeed. That is, this occurs with probability 1 or $1-(1-q)^{\ell}$. We obtain

$$\text{Prob}\,[v \text{ is not observed}] = CY. \qquad \square$$

5.3 Complete multipartite graphs

We now calculate the expected value polynomial for complete multipartite graphs with parts of size at least 2, using a similar method as in the previous section.

Theorem 5.5. *Let $G = K_{r_1,\ldots,r_k}$ with $k \geq 2$ and $r_i \geq 2$ for all i, let $S \subseteq V(G)$ be a set such that S contains ℓ_i vertices from the ith partition of G, and let q be a probability of PMU failure. Denote $\ell = |S|$. Then*

$$\mathcal{E}\,(G; S, q) = |V(G)| - \sum_{j=1}^{k} \left(\sum_{v \in S \cap A_j} q^{\ell - \ell_j + 1} X_{v,j} + \sum_{v \in \overline{S} \cap A_j} q^{\ell - \ell_j} Y_{v,j} \right)$$

where for all $v \in S$, $v \in A_j$ for some j we define

$$X_{v,j} = \begin{cases} 1-(1-q)^{\ell_j - 1} & \ell_j = r_j \\ 1 & \text{otherwise} \end{cases}$$

and for all $v \notin S$, $v \in A_j$ for some j we define

$$Y_{v,j} = \begin{cases} 1-(1-q)^{\ell_j} & \ell_j = r_j - 1 \\ 1 & \text{otherwise} \end{cases}$$

Proof. We consider Prob[v is observed] $= 1 -$ Prob[v is not observed] for each $v \in V(G)$.

If $v \in S \cap A_j$, then v is not observed if all of the following occur:

1. the PMU at v fails,
2. v is not observed in the domination step, that is, every PMU in A_i for $i \neq j$ fails, and
3. v is not observed via a zero forcing step.

Observe that 1. occurs with probability q and 2. occurs with probability $q^{\ell-\ell_j}$. For 3., the only way that v can be observed in a zero forcing step is if all other vertices in A_j have a succeeding PMU, i.e., $\ell_j = r_j$, so in this case Prob[v is not observed] $=$

$1-(1-q)^{\ell_j-1}$. If that is not possible, then we do not have $\ell_j = r_j$ and v cannot ever be observed via a zero forcing step, which gives Prob[v is not observed] = 1. As each of these sensor failures occur independently, we obtain

$$\text{Prob}[v \text{ is not observed}] = q^{\ell-\ell_j+1} X_{v,j}.$$

If $v \in \overline{S} \cap A_j$, then v is not observed if v is not observed in the domination step, that is, every PMU in A_i for $i \neq j$ fails, and v is not observed via a zero forcing step. In a similar way to the previous case, we obtain

$$\text{Prob}[v \text{ is not observed}] = q^{\ell-\ell_j} Y_{v,j}.$$ □

6 Future work

In this paper, we introduced the concept of power domination with random sensor failure and tools to study the fragile power domination process. Many of our results depend on particular PMU placements or specific graphs. We demonstrated connections to power domination variations such as failed power domination, fault-tolerant power domination, and PMU-defect-robust power domination. Are there more general structural conditions or other graph parameters that could be used to study fragile power domination?

In [2], it is shown one can always find a power dominating set with vertices all having degree three or higher. Fig. 1 suggests that, when choosing PMU placements, high degree vertices may be a better choice. Particularly, the set $\{4, 4\}$ containing two vertices of degree 4 observes more vertices than the set $\{2, 5\}$ containing a leaf and a degree 4 vertex for some values of q. Is this a behavior that can be characterized? Is there a relationship between the degrees of vertices in a PMU placement and the expected value polynomial? An easy result to see is that if a single PMU is placed at the vertex of lowest degree, we obtain $\mathcal{E}(G; S, q) \geq (\delta(G)+1)(1-q)$ as only the closed neighborhood of said vertex is observed.

In Theorem 3.14, we demonstrated a family of graphs for which the intersection of the expected value polynomials for failed power dominating sets and power dominating sets can occur at any rational probability. What can be said about the graph structures that realize this phenomenon?

We utilized the power domination polynomial from [6] to study the probability of observing the entire graph in Section 4. There are also results on graph products presented in [6] that might be extendable to fragile power domination.

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