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Estimation of domination number of graphs via safe number

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Abstract

Let G be a graph. A subset D of $V(G)$ is a dominating set of G if every vertex in $V(G) \setminus D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set of G is called the domination number of G. A non-empty subset S of $V(G)$ is a safe set of G if, for a component C of $G - S$ and a component Q of the subgraph of G induced by S, $|V(C)| \leq |V(Q)|$ whenever there is an edge of G between $V(C)$ and $V(Q)$. The minimum cardinality of a safe set of G is called the safe number of G. In this paper, we give some estimations on the domination number of a graph G in terms of the safe number and the maximum degree of G, and discuss their sharpness. We also give their analogies with respect to the "connected version" of our results.

1 Introduction

All graphs in this paper are finite, undirected and simple. Let G be a graph. Let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For

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$x \in V(G)$, let $N_G(x)$, $N_G[x]$ and $\deg_G(x)$ denote the (*open*) *neighborhood*, the *closed neighborhood* and the *degree* of x, respectively; thus $N_G(x) = \{y \in V(G) : xy \in E(G)\}$, $N_G[x] = N_G(x) \cup \{x\}$ and $\deg_G(x) = |N_G(x)|$. Let $\Delta(G)$ denote the *maximum degree* of G; thus $\Delta(G) = \max\{\deg_G(x) : x \in V(G)\}$. For a subset X of $V(G)$, let $N_G(X) = (\bigcup_{x \in X} N_G(x)) \setminus X$, and let $G[X]$ denote the subgraph of G induced by X. For disjoint subsets X and Y of $V(G)$, let $E_G(X, Y)$ denote the set of edges of G joining X and Y. Let $\mathcal{C}(G)$ be the set of components of G. Let K_n, $K_{1,n-1}$, P_n and C_n denote a *complete graph*, a *star*, a *path* and a *cycle* of order n, respectively.

For two subsets X and Y of $V(G)$, X *dominates* Y in G if every vertex in $Y \setminus X$ is adjacent to a vertex in X. A subset D of $V(G)$ is a *dominating set* of G if D dominates $V(G)$ in G. A dominating set D of G is called a *connected dominating set* if $G[D]$ is connected. The minimum cardinality of a dominating set and a connected dominating set of G, denoted by $\gamma(G)$ and $\gamma_c(G)$), respectively, is called the *domination number* and the *connected domination number* of G, respectively. A dominating set and a connected dominating set D of G is a γ*-set* and a γ_c*-set* of G, respectively, if $|D| = \gamma(G)$ and $|D| = \gamma_c(G)$respectively. The domination number and the connected domination number have been widely studied not only for their mathematical value but also for many applications, for example, in communication networks, radio broadcasting, and school bus routing. Recent books on domination concepts reflect the importance of these and related parameters (see [5, 10–12]).

Fujita et al. [9] introduced the notion of safe sets in graphs as follows. A nonempty subset S of $V(G)$ is a *safe set* of G if $|V(C)| \leq |V(Q)|$ for any $C \in \mathcal{C}(G - S)$ and $Q \in \mathcal{C}(G[S])$ with $E_G(V(C), V(Q)) \neq \emptyset$. A safe set S of G is called a *connected safe set* if $G[S]$ is connected. The minimum cardinality of a safe set and a connected safe set of G, respectively, denoted by $s(G)$ and $s_c(G)$, respectively, is called the *safe number* and the *connected safe number* of G, respectively. It models situations like placing emergency refuges in a building, where the capacity of each refuge must be large enough to serve any adjacent area.

The notion on safe sets in graphs was extended to vertex weighted graphs in [3]. In view of some real applications such as network vulnerability, this extension has received considerable attention, especially in the algorithmic aspects of safe sets. Although the scope of the above mentioned paper due to Fujita et al. [9] is not about vertex weighted graphs, they essentially showed that computing the connected safe number in the case (G, w) with a constant weight function w is NP-hard in general. On the other hand, when G is a tree and w is a constant weight function, they gave a linear time algorithm for computing the connected safe number of G. Águeda et al. [1] provided an efficient algorithm for computing the safe number of unweighted graphs with bounded treewidth. Furthermore, Bapat et al. [3] showed that computing the connected weighted safe number for stars, and therefore also for trees, is NP-hard. They also obtained an efficient algorithm computing the safe number for vertex weighted paths. Fujita et al. [8] constructed a linear time algorithm computing the safe number for vertex weighted cycles. Some approximation algorithms on safe sets were also discussed. Indeed, Ehard and Rautenbach [6] showed a polynomial-time approximation scheme (PTAS) for the connected safe number of vertex weighted trees. The parameterized complexity of safe set problems was investigated by Belmonte et al. [4] and a mixed integer linear programing formulation for safe sets was introduced by Hosteins [14]. Very recently, the notion of safe sets was extended to digraphs by Bai et al. [2] and some algorithmic aspects were explored. Thus, various topics on safe sets in graphs have been studied extensively.

Our main aim is to illustrate the mathematical importance of the safe number. Indeed, Fujita and Furuya [7] gave sharp bounds on the integrity of graphs, a useful parameter of a communication network, involving the safe number. In this paper, we seek analogous results by focusing on a relationship between the domination number and the safe number. However, one cannot estimate the domination number by using the safe number alone. To see this, let n be a sufficiently large integer, and T_n be the graph obtained from $K_{1,n}$ by subdividing all of its edges once. Then $\gamma(T_n) = n$ and $s(T_n) = 2$, and hence there is no upper bound of the domination number using the safe number only. Furthermore, $\gamma(K_n) = 1$ and $s(K_n) = \lceil \frac{n}{2} \rceil$, and hence it is impossible to derive a lower bound of the domination number based solely on the safe number.

On the other hand, using both the safe number and the maximum degree, we can construct upper and lower bounds of the domination number. Let G be a connected graph with $\Delta(G) = 2$; thus, G is either a path or a cycle. It is known that $\gamma(G) = \left\lceil \frac{|V(G)|}{3} \right\rceil$. Furthermore, if G is a path, then $s(G) = \left\lceil \frac{|V(G)|}{3} \right\rceil$, while if G is a cycle, then $s(G) = \left\lceil \frac{|V(G)|}{2} \right\rceil$ (see [9]). Thus $\gamma(G) = \left\lceil \frac{|V(G)|}{3} \right\rceil \leq s(G)$. The inequality is best possible because $\gamma(P_n) = s(P_n)$. Inspired by the brief discussion, we proceeded to consider the case where $\Delta(G) \geq 3$ and obtained the following results.

Theorem 1.1 *Let G be a connected graph with $s(G) = s \geq 2$ and $\Delta(G) = \Delta \geq 3$. Then $\gamma(G) \leq \lfloor \frac{s}{2} \rfloor (s\Delta - 2s + 3)$.*

Theorem 1.2 *Let G be a connected graph with $s(G) = s \geq 1$ and $\Delta(G) = \Delta \geq 2$. Then $\gamma(G) \geq \frac{2s-1}{\Delta+1}$.*

We remark that Theorems 1.1 and 1.2 provide bounds on the safe number in terms of $\gamma(G)$ and $\Delta(G)$. These bounds are sharp in a sense, just as in Theorems 1.1 and 1.2.

For a positive integer n, we can easily verify that $\gamma(K_{2n+1}) = 1 = \frac{2(n+1)-1}{2n+1} = \frac{2s(K_{2n+1})-1}{\Delta(K_{2n+1})+1}$ and $\gamma(C_{6n+3}) = 2n + 1 = \frac{2(3n+2)-1}{2+1} = \frac{2s(C_{6n+3})-1}{\Delta(C_{6n+3})+1}$. Hence Theorem 1.2 is best possible.

We also give analogies of Theorems 1.1 and 1.2 for the connected domination number and the connected safe number. Moreover, we characterize the graphs for which equality holds in an upper bound on γ_c.

To state the result, we define the class $\mathcal{G}(s_c, \Delta)$ of graphs as follows (see Figure 1). Let $s_c \geq 1$ and $\Delta \geq 2$ be integers. Let T be a tree of order s_c with $\Delta(T) \leq \Delta$. For $x \in V(T)$ and i with $1 \leq i \leq \Delta - \deg_T(x)$, let $R_{x,i}$ be a path of order s_c, and let $y_{x,i}$ be an endvertex of $R_{x,i}$. Let G be the graph obtained from T and $R_{x,i}$ by adding the edges $xy_{x,i}$ ($x \in V(T)$, $1 \leq i \leq \Delta - \deg_T(x)$). Let $\mathcal{G}(s_c, \Delta)$ be the family of such graphs G. Note that every element of $\mathcal{G}(s_c, \Delta)$ is a tree, and in particular, $\mathcal{G}(1, \Delta) = \{K_{1,\Delta}\}$ and $\mathcal{G}(s_c, 2) = \{P_{3s_c}\}$. Then the following theorems hold.

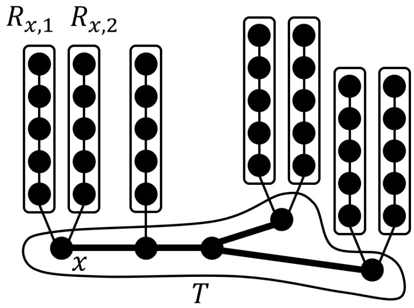

Figure 1: A graph in $\mathcal{G}(5, 3)$

Theorem 1.3 *Let G be a connected graph with $s_c(G) = s_c \geq 1$ and $\Delta(G) = \Delta \geq 2$. Then $\gamma_c(G) \leq s_c(s_c\Delta - 2s_c - \Delta + 5) - 2$, and the equality holds only if G is isomorphic to a graph in $\mathcal{G}(s_c, \Delta)$.*

Theorem 1.4 *Let G be a connected graph with $s_c(G) = s_c \geq 2$ and $\Delta(G) = \Delta \geq 2$. Then $\gamma_c(G) \geq \frac{2s_c-3}{\Delta-1}$.*

For a positive integer n, note that $\gamma_c(K_{2n+1}) = 1 = \frac{2(n+1)-3}{2n-1} = \frac{2s_c(K_{2n+1})-3}{\Delta(K_{2n+1})-1}$ and $\gamma_c(C_{2n+1}) = 2n - 1 = \frac{2(n+1)-3}{2-1} = \frac{2s_c(C_{2n+1})-3}{\Delta(C_{2n+1})-1}$. Hence Theorem 1.4 is best possible.

In Section 2, we introduce some known results and prepare useful lemmas. In Section 3, we prove Theorems 1.1 and 1.2, and in Subsection 3.1, we discuss the sharpness of Theorem 1.1. In Section 4, we prove Theorems 1.3 and 1.4.

2 Preliminaries

In this section, we list some results which will be used in what follows. We start with general bounds for the (connected) safe number.

Theorem 2.1 (Fujita et al. [9]) *For a connected graph G, $s(G) \leq s_c(G) \leq \lceil \frac{|V(G)|}{2} \rceil$.*

Theorem 2.2 (Fujita et al. [9]) *For a tree T, $s_c(T) \leq \lceil \frac{|V(T)|}{3} \rceil$.*

Next we introduce general upper bounds of the (connected) domination number and related results. The following lemma is well-known.

Lemma 2.3 (Ore [15]) *Let H be a graph having no isolated vertices. Then $\gamma(H) \leq \frac{|V(H)|}{2}$.*

Lemma 2.4 *Let H be a graph of odd order, and let H' be a subgraph of H having no isolated vertices. Then $\gamma(H') \leq \frac{|V(H)|-1}{2}$.*

Proof. By Lemma 2.3, $\gamma(H') \leq \frac{|V(H')|}{2} \leq \frac{|V(H)|}{2}$. Since $|V(H)|$ is odd, this implies that $\gamma(H') \leq \frac{|V(H)|-1}{2}$. □

Lemma 2.5 (Hedetniemi and Laskar [13]) *Let G be a connected graph. Then $\gamma_c(G) \leq |V(G)| - \Delta(G)$.*

Lemma 2.6 (Sampathkumar and Walikar [16]) *Let T be a tree of order at least 3, and suppose that T has exactly l leaves. Then $\gamma_c(G) = |V(G)| - l$.*

The following lemma plays a key role in some of our arguments.

Lemma 2.7 *Let G be a connected graph, and let X be a subset of $V(G)$ such that $G[X]$ is connected. Then*

$$\sum_{x \in X} |N_G(x) \setminus X| \leq \Delta(G)|X| - 2|X| + 2,$$

and the equality holds only if $G[X]$ is a tree and $\deg_G(x) = \Delta(G)$ for every $x \in X$.

Proof. Since $G[X]$ is connected, we can take a spanning tree T of $G[X]$. Then

$$\sum_{x \in X} |N_G(x) \cap X| = 2|E(G[X])| \geq 2|E(T)| = 2(|X| - 1). \tag{2.1}$$

Since $\Delta(G) \geq \deg_G(x) = |N_G(x) \cap X| + |N_G(x) \setminus X|$ for all $x \in X$, it follows from (2.1) that

$$\begin{aligned} \Delta(G)|X| &\geq \sum_{x \in X} \deg_G(x) \\ &= \sum_{x \in X} |N_G(x) \cap X| + \sum_{x \in X} |N_G(x) \setminus X| \\ &\geq 2(|X| - 1) + \sum_{x \in X} |N_G(x) \setminus X|. \end{aligned} \tag{2.2}$$

This leads to $\sum_{x \in X} |N_G(x) \setminus X| \leq \Delta(G)|X| - 2|X| + 2$. Considering (2.1), the equality in (2.2) holds only if $G[X] = T$ (i.e., $G[X]$ is a tree) and $\deg_G(x) = \Delta(G)$ for every $x \in X$. □

3 Domination number versus safe number

In this section, we prove Theorems 1.1 and 1.2 and discuss their sharpness.

Proof of Theorem 1.1. Let S be a safe set of G with $|S| = s(G)$. Let S_1 be the set of isolated vertices of $G[S]$, and let D_0 be a γ-set of $G[S \setminus S_1]$. For each $C \in \mathcal{C}(G - S)$, let D_C be a γ-set of C. Let

$$D = D_0 \cup \left(\bigcup_{C \in \mathcal{C}(G-S)} D_C \right).$$

Claim 1 *The set D is a dominating set of G.*

Proof of Claim 1. It is clear that D is a dominating set of $G - S_1$. Let $x \in S_1$. It suffices to show that D dominates $\{x\}$ in G. Since G is connected and $|V(G)| > \Delta(G) \geq 3$, there exists a vertex $y \in N_G(x)$. By the definition of S_1, $y \notin S$. Since S is a safe set of G and $G[\{x\}] \in \mathcal{C}(G[S])$, this implies that $G[\{y\}] \in \mathcal{C}(G - S)$, and hence $y \in D$. Hence D dominates $\{x\}$ in G. (■)

For each $C \in \mathcal{C}(G - S)$, since G is connected and $S \neq \emptyset$, there exists a vertex $x_C \in S$ such that $N_G(x_C) \cap V(C) \neq \emptyset$. Fix a vertex $x \in S$. Let

- $\mathcal{C}_x = \{C \in \mathcal{C}(G - S) : x_C = x\}$,
- $\mathcal{C}_x^{(1)} = \{C \in \mathcal{C}_x : |V(C)| = 1\}$,
- $\mathcal{C}_x^{(2)} = \{C \in \mathcal{C}_x : |V(C)| \equiv 0 \pmod 2\}$, and
- $\mathcal{C}_x^{(3)} = \mathcal{C}_x \setminus (\mathcal{C}_x^{(1)} \cup \mathcal{C}_x^{(2)})$ $(= \{C \in \mathcal{C}_x : |V(C)| \geq 3,\ |V(C)| \equiv 1 \pmod 2\})$.

Note that

$$|\mathcal{C}_x^{(1)}| + |\mathcal{C}_x^{(2)} \cup \mathcal{C}_x^{(3)}| = |\mathcal{C}_x| \leq |N_G(x) \setminus S|. \tag{3.1}$$

Since $s \geq 2$, $\sum_{C \in \mathcal{C}_x^{(1)}} |D_C| = |\mathcal{C}_x^{(1)}| \leq \frac{s|\mathcal{C}_x^{(1)}|}{2}$. For $C \in \mathcal{C}_x^{(2)} \cup \mathcal{C}_x^{(3)}$, since S is a safe set of G, it follows from Lemma 2.3 that $|D_C| \leq \frac{|V(C)|}{2} \leq \frac{s}{2}$. This implies that $\sum_{C \in \mathcal{C}_x^{(2)} \cup \mathcal{C}_x^{(3)}} |D_C| \leq \frac{s|\mathcal{C}_x^{(2)} \cup \mathcal{C}_x^{(3)}|}{2}$. Since $G[S \setminus S_1]$ has no isolated vertex, it follows from Lemma 2.3 that $|D_0| = \gamma(G[S \setminus S_1]) \leq \frac{|S| - |S_1|}{2} \leq \frac{s}{2}$. Consequently, by Lemma 2.7, Claim 1 and (3.1),

$$\begin{aligned}
\gamma(G) &\leq |D_0| + \sum_{x \in S} \left(\sum_{C \in \mathcal{C}_x^{(1)} \cup \mathcal{C}_x^{(2)} \cup \mathcal{C}_x^{(3)}} |D_C| \right) \\
&\leq \frac{s}{2} + \sum_{x \in S} \left(\frac{s|\mathcal{C}_x^{(1)}|}{2} + \frac{s|\mathcal{C}_x^{(2)} \cup \mathcal{C}_x^{(3)}|}{2} \right) \\
&\leq \frac{s}{2} \left(1 + \sum_{x \in S} |N_G(x) \setminus S| \right) \\
&\leq \frac{s(1 + (\Delta|S| - 2|S| + 2))}{2} \\
&= \frac{s(s\Delta - 2s + 3)}{2},
\end{aligned}$$

which proves the theorem for the case where s is even.

Thus we may assume that s is odd, and it suffices to show that

$$\gamma(G) \leq \frac{(s-1)(s\Delta - 2s + 3)}{2}.$$

Fix $x \in S$. Since $s \geq 3$, $\sum_{C \in \mathcal{C}_x^{(1)}} |D_C| = |\mathcal{C}_x^{(1)}| \leq \frac{(s-1)|\mathcal{C}_x^{(1)}|}{2}$. For $C \in \mathcal{C}_x^{(2)}$, since S is a safe set of G and $|V(C)|$ is even, $|V(C)| \leq s-1$, and so $|D_C| \leq \frac{|V(C)|}{2} \leq \frac{s-1}{2}$. For $C \in \mathcal{C}_x^{(3)}$, since S is a safe set of G, it follows from Lemma 2.3 that $|D_C| \leq \frac{|V(C)|-1}{2} \leq \frac{s-1}{2}$. This implies that $\sum_{C \in \mathcal{C}_x^{(2)} \cup \mathcal{C}_x^{(3)}} |D_C| \leq \frac{(s-1)|\mathcal{C}_x^{(2)} \cup \mathcal{C}_x^{(3)}|}{2}$. Since $|S|$ $(= s)$ is odd and $G[S \setminus S_1]$ has no isolated vertex, it follows from Lemma 2.4 that $|D_0| = \gamma(G[S \setminus S_1]) \leq \frac{|S|-1}{2} = \frac{s-1}{2}$. Consequently, by Lemma 2.7, Claim 1 and (3.1),

$$\begin{aligned}
\gamma(G) &\leq |D_0| + \sum_{x \in S} \left(\sum_{C \in \mathcal{C}_x^{(1)} \cup \mathcal{C}_x^{(2)} \cup \mathcal{C}_x^{(3)}} |D_C| \right) \\
&\leq \frac{s-1}{2} + \sum_{x \in S} \left(\frac{(s-1)|\mathcal{C}_x^{(1)}|}{2} + \frac{(s-1)|\mathcal{C}_x^{(2)} \cup \mathcal{C}_x^{(3)}|}{2} \right) \\
&\leq \frac{s-1}{2} \left(1 + \sum_{x \in S} |N_G(x) \setminus S| \right) \\
&\leq \frac{(s-1)(1 + (\Delta|S| - 2|S| + 2))}{2} \\
&= \frac{(s-1)(s\Delta - 2s + 3)}{2}.
\end{aligned}$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Let D be a γ-set of G. Then

$$|V(G)| = \left| \bigcup_{x \in D} N_G[x] \right| \leq \sum_{x \in D} (d_G(x) + 1) \leq \gamma(G)(\Delta + 1).$$

This together with Theorem 2.1 implies that

$$s \leq \left\lceil \frac{|V(G)|}{2} \right\rceil \leq \frac{|V(G)| + 1}{2} \leq \frac{\gamma(G)(\Delta + 1) + 1}{2},$$

as desired. □

3.1 Examples

In this subsection, we first construct graphs which attain the equality of Theorem 1.1 for the case where $s \geq 3$ is odd. Let $\Delta \geq 3$ be an integer, and let $n = \frac{s-1}{2}$. Let A_s be the graph such that

$$\begin{aligned}
V(A_s) &= \{u_i : 1 \leq i \leq n\} \cup \{v_i : 0 \leq i \leq n\} \quad \text{and} \\
E(A_s) &= \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_1 v_0\} \cup \{u_i v_i : 1 \leq i \leq n\}
\end{aligned}$$

(see Figure 2).

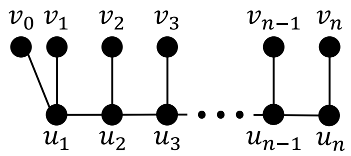

Figure 2: Graph A_s

Lemma 3.1 *The following statements hold:*

(i) *We have* $\gamma(A_s) = \gamma(A_s - v_0) = n$.

(ii) *If a dominating set* D *of* A_s *contains* v_0, *then* $|D| \geq n + 1$.

Proof. Let D be a dominating set of A_s. Since D dominates $\{v_i : 1 \leq i \leq n\}$ in A_s, $D \cap \{u_i, v_i\} \neq \emptyset$ for every i $(1 \leq i \leq n)$, and so

$$\left|\bigcup_{1\leq i\leq n} (D \cap \{u_i, v_i\})\right| = \sum_{1\leq i\leq n} |D \cap \{u_i, v_i\}| \geq n. \tag{3.2}$$

In particular, we have $\gamma(A_s) \geq |D| \geq n$. Arguing similarly, we also have $\gamma(A_s - v_0) \geq n$. On the other hand, since $\{u_i : 1 \leq i \leq n\}$ is a dominating set of A_s and $A_s - v_0$, $\gamma(A_s) \leq n$ and $\gamma(A_s - v_0) \leq n$, which proves (i). Furthermore, if $v_0 \in D$, then by (3.2), $|D| = |(D \cap \{v_0\}) \cup (\bigcup_{1\leq i\leq n}(D \cap \{u_i, v_i\}))| \geq 1 + n$, which proves (ii). □

Let H_0 be a copy of A_s, and let $\mathcal{X} = \{(x, i) : x \in V(H_0),\ 1 \leq i \leq \Delta - \deg_{H_0}(x)\}$. Take $|\mathcal{X}|$ vertex-disjoint copies $H_{x,i}$ $((x, i) \in \mathcal{X})$ of A_s. For each $(x, i) \in \mathcal{X}$, let $v_{x,i}$ be the vertex $H_{x,i}$ corresponding to v_0. Let $G_{s,\Delta}$ be the graph obtained from H_0 and $H_{x,i}$ by adding edges $xv_{x,i}$ $((x, i) \in \mathcal{X})$. Then $G_{s,\Delta}$ is connected and $\Delta(G_{s,\Delta}) = \Delta$. Hence the following proposition shows the sharpness of Theorem 1.1 for the case where s is odd.

Proposition 3.2 *We have* $s(G_{s,\Delta}) = s$ *and* $\gamma(G_{s,\Delta}) = \frac{(s-1)(s\Delta-2s+3)}{2}$.

Proof. Since $s = 2n + 1$, we have

$$|\mathcal{X}| = \sum_{x\in V(H_0)} (\Delta - \deg_{H_0}(x)) = \Delta|V(H_0)| - 2|E(H_0)| = s\Delta - 2(s - 1) \geq s + 2. \tag{3.3}$$

We first prove that $s(G_{s,\Delta}) = s$. Since $V(H_0)$ is a safe set of $G_{s,\Delta}$, we have $s(G_{s,\Delta}) \leq |V(H_0)| = s$. Let S be a safe set of $G_{s,\Delta}$. It suffices to show that $|S| \geq s$. If $S \cap V(H_{x,i}) \neq \emptyset$ for all $(x, i) \in \mathcal{X}$, then by (3.3), $|S| \geq \sum_{(x,i)\in\mathcal{X}} |S \cap V(H_{x,i})| \geq s+2$. Thus we may assume that $S \cap V(H_{x,i}) = \emptyset$ for some $(x, i) \in \mathcal{X}$. Then there exists $C \in \mathcal{C}(G_{s,\Delta} - S)$ with $V(H_{x,i}) \subseteq V(C)$. Since $G_{s,\Delta}$ is connected and S is a safe set of $G_{s,\Delta}$, this implies that $s = |V(H_{x,i})| \leq |V(C)| \leq |S|$, as desired.

Next we prove that $\gamma(G_{s,\Delta}) = \frac{(s-1)(s\Delta-2s+3)}{2}$. In view Theorem 1.1 and the first statement of the proposition, it suffices to show that $\gamma(G_{s,\Delta}) \geq \frac{(s-1)(s\Delta-2s+3)}{2}$. Take

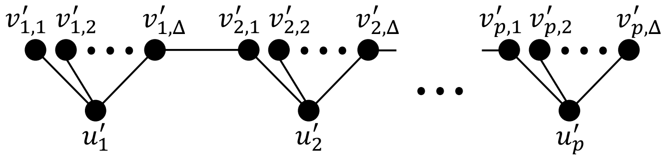

Figure 3: Graph $B_{\Delta,p}$

a γ-set D of $G_{s,\Delta}$ so that $|D \setminus V(H_0)|$ is as small as possible. Suppose that $v_{x,i} \in D$ for some $(x,i) \in \mathcal{X}$. Then $D \cap V(H_{x,i})$ is a dominating set of $H_{x,i}$. Together with Lemma 3.1(ii), this implies that $|D \cap V(H_{x,i})| \geq n+1$. Let $D_{x,i}$ be a γ-set of $H_{x,i}$. By Lemma 3.1(i), $|D_{x,i}| = n$. Let $D' = (D \setminus V(H_{x,i})) \cup D_{x,i} \cup \{x\}$. Then D' is a dominating set of $G_{s,\Delta}$ and $|D'| \leq (|D| - |D \cap V(H_{x,i})|) + |D_{x,i}| + 1 \leq \gamma(G_{s,\Delta}) - (n+1) + n + 1 = \gamma(G_{s,\Delta})$, and hence D' is a γ-set of $G_{s,\Delta}$. However, $|D \setminus V(H_0)| - |D' \setminus V(H_0)| = |D \cap V(H_{x,i})| - |D' \cap V(H_{x,i})| \geq (n+1) - n > 0$, which contradicts the minimality of $|D \setminus V(H_0)|$. Thus $v_{x,i} \notin D$ for all $(x,i) \in \mathcal{X}$. Then $D \cap V(H_0)$ is a dominating set of H_0. Furthermore, $D \cap V(H_{x,i})$ is a dominating set of $H_{x,i} - v_{x,i}$ for each $(x,i) \in \mathcal{X}$. Hence by Lemma 3.1(i) and (3.3),

$$\begin{aligned}
\gamma(G_{s,\Delta}) &= |D| \\
&= |D \cap V(H_0)| + \sum_{(x,i)\in\mathcal{X}} |D \cap V(H_{x,i})| \\
&\geq n + (s\Delta - 2s + 2)n \\
&= \frac{(s-1)(s\Delta - 2s + 3)}{2},
\end{aligned}$$

as desired. □

On the other hand, the inequality $\gamma(G) \leq \frac{(s-1)(s\Delta-2s+3)}{2}$ does not hold for even s. Let $\Delta \geq 3$ be an integer, and $p \geq 2$ be an integer such that p is even for the case Δ is even. Let $s = p(\Delta+1)$. Note that s is even regardless of the parity of p. Let $B_{\Delta,p}$ be the graph such that

$$V(B_{\Delta,p}) = \{u'_i : 1 \leq i \leq p\} \cup \{v'_{i,j} : 1 \leq i \leq p,\ 1 \leq j \leq \Delta\} \text{ and}$$
$$E(B_{\Delta,p}) = \{u'_i v'_{i,j} : 1 \leq i \leq p,\ 1 \leq j \leq \Delta\} \cup \{v'_{i,\Delta} v'_{i+1,1} : 1 \leq i \leq p-1\}$$

(see Figure 3). Since the distance between any pair of vertices in $\{u'_i : 1 \leq i \leq p\}$ is at least 3 in $B_{\Delta,p}$, we obtain the following lemma, for which the proof is omitted.

Lemma 3.3 *If a subset D of $V(B_{\Delta,p})$ dominates $\{u'_i : 1 \leq i \leq p\}$ in $B_{\Delta,p}$, then $|D| \geq p$.*

Let $\mathcal{I} = \{(i,j,k) : 1 \leq i \leq p,\ 1 \leq j \leq \Delta,\ 1 \leq k \leq \Delta - \deg_{B_{\Delta,p}}(v'_{i,j})\}$. Take $|\mathcal{I}|$ vertex-disjoint copies $H_{i,j,k}$ ($(i,j,k) \in \mathcal{I}$) of $A_{s+1} - v_0$, where A_{s+1} is the graph defined in the first paragraph of the subsection. For each $(i,j,k) \in \mathcal{I}$, let $u_{i,j,k}$ be the vertex $H_{i,j,k}$ corresponding to u_1. Let $G'_{\Delta,p}$ be the graph obtained from $B_{\Delta,p}$ and

$H_{i,j,k}$ $((i,j,k) \in \mathcal{I})$ by adding edges $v'_{i,j}u_{i,j,k}$ $((i,j,k) \in \mathcal{I})$. Then $G'_{\Delta,p}$ is connected and $\Delta(G'_{\Delta,p}) = \Delta$. Furthermore, arguing similarly as in the proof of the first statement of Proposition 3.2, we have $s(G'_{\Delta,p}) = p(\Delta+1)$ $(= s)$. Since

$$\frac{s(s\Delta - 2s + 2)}{2} + p - \frac{(s-1)(s\Delta - 2s + 3)}{2} = \frac{2p + s(\Delta - 3) + 3}{2} > 0,$$

the following proposition shows that the inequality $\gamma(G) \leq \frac{(s-1)(s\Delta-2s+3)}{2}$ does not hold for the case where $G = G'_{\Delta,p}$ and $s = p(\Delta+1)$, and the bound on Theorem 1.1 is asymptotically sharp for even s.

Proposition 3.4 *We have* $\gamma(G'_{\Delta,p}) \geq \frac{s(s\Delta-2s+2)}{2} + p$.

Proof. Since $\sum_{1\leq j\leq\Delta} \deg_{B_{\Delta,p}}(v'_{1,j}) = \sum_{1\leq j\leq\Delta} \deg_{B_{\Delta,p}}(v'_{p,j}) = \Delta + 1$ and for each i with $2 \leq i \leq p-1$, $\sum_{1\leq j\leq\Delta} \deg_{B_{\Delta,p}}(v'_{i,j}) = \Delta + 2$, we have

$$|\mathcal{I}| = \sum_{\substack{1\leq i\leq p \\ 1\leq j\leq\Delta}} (\Delta - \deg_{B_{\Delta,p}}(v'_{i,j})) = p\Delta^2 - (p(\Delta+2) - 2). \tag{3.4}$$

Let D be a γ-set of $G'_{\Delta,p}$. Fix $(i,j,k) \in \mathcal{I}$. Note that $H_{i,j,k}$ has exactly $\frac{|V(B_{\Delta,p})|}{2}$ $(= \frac{s}{2})$ leaves, and they are also leaves of $G'_{\Delta,p}$. Since the distance between two leaves of $H_{i,j,k}$ in $G'_{\Delta,p}$ is at least 3, we have $|D \cap V(H_{i,j,k})| \geq \frac{s}{2}$. Furthermore, since $N_{G'_{\Delta,p}}(u'_i) = N_{B_{\Delta,p}}(u'_i)$ for every i with $1 \leq i \leq p$, $D \cap V(B_{\Delta,p})$ dominates $\{u'_i : 1 \leq i \leq p\}$ in $G_{\Delta,p}$. This together with Lemma 3.3 leads to $|D \cap V(B_{\Delta,p})| \geq p$. Consequently, it follows from (3.4) that

$$\begin{aligned}
\gamma(G'_{\Delta,p}) &= |D| \\
&= |D \cap V(B_{\Delta,p})| + \sum_{(i,j,k)\in\mathcal{I}} |D \cap V(H_{i,j,k})| \\
&\geq p + \frac{s}{2}(p\Delta^2 - (p(\Delta+2) - 2)) \\
&= \frac{s(s\Delta - 2s + 2)}{2} + p,
\end{aligned}$$

as desired. □

We here propose the following open problem.

Problem 1 *For an even integer* $s \geq 2$ *and an integer* $\Delta \geq 3$*, determine the minimum value* $f(s,\Delta)$ *such that* $\gamma(G) \leq f(s,\Delta)$ *for every connected graph* G *with* $s(G) = s$ *and* $\Delta(G) = \Delta$.

4 Connected domination number versus connected safe number

We start with some properties of graphs in $\mathcal{G}(s_c, \Delta)$.

Lemma 4.1 *Let $s_c \geq 1$ and $\Delta \geq 2$ be integers, and let $G \in \mathcal{G}(s_c, \Delta)$. Then the following hold.*

(i) *We have $\Delta(G) = \Delta$ and $s_c(G) = s_c$.*

(ii) *We have $\gamma_c(G) = s_c(s_c\Delta - 2s_c - \Delta + 5) - 2$.*

Proof. Let T, $R_{x,i}$ and $y_{x,i}$ be as in the definition of G, and let $\mathcal{X} = \{(x, i) : x \in V(T),\ 1 \leq i \leq \Delta - \deg_T(x)\}$. If $s_c = 1$, then G is a star of Δ leaves, and hence $\Delta(G) = \Delta$ and $s_c(G) = \gamma_c(G) = 1$. Thus we may assume that $|V(T)| = s_c \geq 2$.

For each $(x, i) \in \mathcal{X}$, since $|V(R_{x,i})| = s_c \geq 2$, there exists an endvertex $z_{x,i}$ of $R_{x,i}$ other than $y_{x,i}$. Then $\{z_{x,i} : (x, i) \in \mathcal{X}\}$ is the set of leaves of G.

(i) By the definition of G, it is clear that $\Delta(G) = \Delta$.

Since $V(T)$ is a connected safe set of G, we have $s_c(G) \leq |V(T)| = s_c$. Let S be a connected safe set of G with $|S| = s_c(G)$. It suffices to show that $|S| \geq s_c$. If $\Delta = 2$, then $G \simeq P_{3s_c}$, and so $s_c(G) = s_c$. Thus we may assume that $\Delta \geq 3$. Suppose that a leaf x of T does not belong to S. Since S is a connected safe set of G and $\Delta - \deg_G(x) \geq 3 - 1 = 2$, $S \cap V(R_{x,i}) = \emptyset$ for some $i \in \{1, 2\}$, say $S \cap V(R_{x,1}) = \emptyset$. Then there exists a component C of $G - S$ with $V(R_{x,1}) \subseteq V(C)$, and hence $|S| \geq |V(C)| \geq |V(R_{x,1})| = s_c$. Thus we may assume that all leaves of T belong to S. Since G is a tree, this leads to $V(T) \subseteq S$, and so $|S| \geq |V(T)| = s_c$, as desired.

(ii) Since T is a tree of order s_c,

$$|\mathcal{X}| = \sum_{x \in V(T)} (\Delta - \deg_T(x)) = s_c\Delta - 2|E(T)| = s_c\Delta - 2(s_c - 1). \tag{4.1}$$

Since $V(G)$ is the disjoint union of $V(T)$ and $V(R_{x,i})$ ($(x, i) \in \mathcal{X}$), it follows from (4.1) that

$$|V(G)| = |V(T)| + \sum_{(x,i) \in \mathcal{X}} |V(R_{x,i})| = s_c + s_c|\mathcal{X}| = s_c(s_c\Delta - 2s_c + 3).$$

Since G is a tree, this together with Lemma 2.6 and (4.1) implies that

$$\begin{aligned} \gamma_c(G) &= |V(G)| - |\mathcal{X}| \\ &= s_c(s_c\Delta - 2s_c + 3) - (s_c\Delta - 2s_c + 2) \\ &= s_c(s_c\Delta - 2s_c - \Delta + 5) - 2, \end{aligned}$$

as desired. $\square$

Proof of Theorem 1.3. Recall that $\mathcal{G}(1, \Delta) = \{K_{1,\Delta}\}$. If $s_c = 1$, then $G \simeq K_{1,\Delta}$, and so $\gamma_c(G) = 1$, as desired. Thus we may assume that $s_c \geq 2$.

Let S be a connected safe set of G with $|S| = s_c$. Since $G[S]$ is connected, it follows from Lemma 2.7 that

$$|\mathcal{C}(G - S)| \leq \sum_{x \in S} |N_G(x) \setminus S| \leq s_c\Delta - 2s_c + 2. \tag{4.2}$$

Since S is a connected safe set of G,

$$|V(C)| \leq |S| = s_c \text{ for every } C \in \mathcal{C}(G - S). \tag{4.3}$$

For each $C \in \mathcal{C}(G - S)$, let u_C be a vertex of C such that $N_G(u_C) \cap S \neq \emptyset$. For each $C \in \mathcal{C}(G - S)$, we define a subset D_C of $V(C)$ as follows: If $|V(C)| = 1$, let $D_C = \emptyset$; if $|V(C)| \geq 2$, take a connected dominating set D_C of C so that

(D1) $u_C \in D_C$, and

(D2) subject to (D1), $|D_C|$ is as small as possible.

Note that $D_C = \{u_C\}$ if $C \in \mathcal{C}(G - S)$ satisfies $|V(C)| = 2$. For each $C \in \mathcal{C}(G - S)$, if $|V(C)| \leq 2$, then $|D_C| \leq 1 \leq s_c - 1$; if $|V(C)| \geq 3$, then by Lemma 2.5 and (4.3), we have $|D_C| \leq \gamma_c(C) + 1 \leq (|V(C)| - \Delta(C)) + 1 \leq (s_c - 2) + 1$. In either case, we obtain

$$|D_C| \leq s_c - 1 \text{ for every } C \in \mathcal{C}(G - S). \tag{4.4}$$

Claim 2 *Let $C \in \mathcal{C}(G - S)$. If $|D_C| = s_c - 1$, then C is a path of order s_c and u_C is an endvertex of C.*

Proof of Claim 2. Suppose that $|D_C| = s_c - 1$. If $|V(C)| \leq 2$, then $s_c - 1 = |D_C| \leq 1 \leq s_c - 1$ and this forces $|V(C)| = s_c = 2$, and so C is a path of order s_c $(= 2)$. Thus we may assume that $|V(C)| \geq 3$. Then $s_c - 1 = |D_C| \leq \gamma_c(C) + 1 \leq (|V(C)| - \Delta(C)) + 1 \leq (s_c - 2) + 1$. This forces $|V(C)| = s_C$ and $\Delta(C) = 2$. If C is either a cycle, or a path and u_C is not an endvertex of C, then there exists a connected dominating set $\tilde{D}_C$ of C such that $u_C \in \tilde{D}_C$ and $|\tilde{D}_C| = |V(C)| - 2 = s_c - 2 < |D_C|$, which contradicts (D2). Since $\Delta(C) = 2$, it follows that C is a path and u_C is an endvertex of C. (■)

We can easily check that $D := S \cup (\bigcup_{C \in \mathcal{C}(G-S)} D_C)$ is a connected dominating set of G. By (4.2) and (4.4),

$$\begin{aligned}
\gamma_c(G) &\leq |D| \\
&= |S| + \sum_{C \in \mathcal{C}(G-S)} |D_C| \\
&\leq s_c + |\mathcal{C}(G - S)|(s_c - 1) \\
&\leq s_c + (s_c\Delta - 2s_c + 2)(s_c - 1) \\
&= s_c(s_c\Delta - 2s_c - \Delta + 5) - 2.
\end{aligned} \tag{4.5}$$

We will use Lemma 4.1 to complete the proof of Theorem 1.3. Suppose that the equality (4.5) holds. We prove that G is isomorphic to a graph in $\mathcal{G}(s_c, \Delta)$.

By (4.2), $|\mathcal{C}(G - S)| = \sum_{x \in S} |N_G(x) \setminus S| = s_c\Delta - 2s_c + 2$. In particular, for each $C \in \mathcal{C}(G - S)$, $E_G(V(C) \setminus \{u_C\}, S) = \emptyset$ and $|N_G(u_C) \cap S| = 1$. Furthermore, it follows from Lemma 2.7 that $G[S]$ is a tree and $\deg_G(x) = \Delta$ for every $x \in S$. This implies that $|\{C \in \mathcal{C}(G - S) : N_G(x) \cap V(C) \neq \emptyset\}| = \Delta - \deg_G(x)$ for every $x \in S$. By (4.4), $|D_C| = s_c - 1$ for every $C \in \mathcal{C}(G - S)$. This together with Claim 2 implies that C is a path of order s_c and u_C is an endvertex of C. Consequently, G is isomorphic to a graph in $\mathcal{G}(s_c, \Delta)$.

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. Let D be a γ_c-set of G. Then by Lemma 2.7,

$$|V(G)| \leq |D| + \sum_{x \in D} |N_G(x) \setminus D| \leq |D| + (\Delta|D| - 2|D| + 2) = \gamma_c(G)(\Delta - 1) + 2.$$

This together with Theorem 2.1 implies that

$$s_c \leq \left\lceil \frac{|V(G)|}{2} \right\rceil \leq \frac{|V(G)| + 1}{2} \leq \frac{\gamma_c(G)(\Delta - 1) + 3}{2},$$

as desired. □

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