AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **92(2)** (2025), Pages 210–236

Autonomous domination

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Abstract

Given a graph, its vertices can be thought of as locations that are subject to a sequence of attacks. Guards who defend the graph respond to attacks by moving along graph edges (or remaining stationary). A set of vertices of the graph is said to be autonomous dominating if mobile guards that start at those locations are able to defend against any infinite series of attacks using local information only, without coordination by a central planner. The autonomous domination number of the graph is the smallest size of an autonomous dominating set. This paper initiates the study of autonomous domination, computing the autonomous domination number for some families of graphs and comparing the parameter to other domination numbers. Examples are presented of "paradoxical" behavior under the addition of edges or guards; these examples show that the autonomous domination number appropriately reflects the difficulty of acting with limited information.

1 Introduction

1.1 Domination and Variations

This paper considers graphs subjected to a series of attacks, with a single unguarded vertex being attacked at each stage, and a guard being required to move along a single edge to the attacked location. The defense of graphs under lengthy (including infinite) attack sequences have been studied in a number of ways. The article [24] surveys the literature. Eternal domination, introduced in [4], involves the defense of a graph under attacks of infinite length with the assistance of a perfectly strategic coordinator. Further studies include [12, 13]. Arbitrarily long (but finite) length series of attacks were considered in [3]. Domination under an infinite series of attacks without any strategic or tactical coordination has been studied under the name foolproof eternal domination [23]. Domination generally is the subject of the monographs [14, 15].

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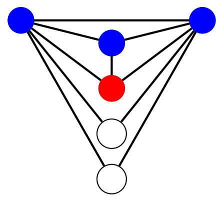
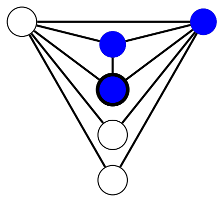
There is a natural intermediate notion between eternal domination and foolproof eternal domination, called autonomous domination, and it is the purpose of the present work to introduce and study this idea. The mobile guards have local, tactical information, but there is no central planner optimizing motion based on global information and strategic thinking. Such a formulation models significant features of real-world decision-making, in which time and information are limited.

1.2 An Example

Here is an example that illustrates the significance of the presence of a master planner. Suppose a graph is defended by three guards, marked in blue in the following figure.



The configuration of guards above is a dominating set. A problem could arise at the red vertex. Each of the three guards is adjacent to the problem vertex. Moreover, if any guard moves to the problem vertex, the new configuration will still be a dominating set. Tactical thinking aiming merely at the maintenance of domination does not reveal that one movement is preferable. Suppose, though, that the guard at the top left moves to the central red vertex. The consequence of that movement is depicted below.



This is still a dominating set, but the guard configuration is no longer secure. If a problem arises at one of the bottom two vertices, it can only be addressed by the guard at the top right, and the other bottom vertex will then be unguarded.

The notions of secure, perfect, and perfect secure domination address, respectively, the ability to respond to a second attack, the uniqueness of the responder, and the uniqueness of the responder to a second attack [6, 32, 36]. Uniqueness implies that no decision needs to be made by a central planner. None of these definitions, however, addresses attack sequences of arbitrary length. The present work begins to fill this gap.

1.3 Distinctions

This subsection briefly distinguishes between autonomous domination and other forms of domination. Autonomous domination deals with attack sequences of arbitrary length, in which attacks occur at a single vertex at each time step. It is thereby distinguished from ordinary domination (which involves defense against a single attack) and secure domination (which involves a sequence of two attacks), and similar to eternal domination and foolproof eternal domination. Autonomous domination does not presume that a strategic planner makes optimal decisions at each step; this distinguishes it from eternal domination. On the other hand, guards move tactically, considering locally available information about graph defense, and so autonomous domination is different from foolproof eternal domination, in which guards move without any consideration of the consequences of such motion.

Only one vertex is attacked at each stage, and only one guard moves, in contrast to m-eternal domination, in which multiple guards move [12]. In autonomous domination, a guard occupying a vertex is considered sufficient for both that vertex and its neighbors; this is in contrast with total domination, in which the graph induced by the guard set has no isolated vertices. Guards move towards, not away from, attacks, in contrast to eternal eviction [21].

In autonomous domination, guards may not accumulate at a single vertex. Instead, at most one guard occupies any location. This distinguishes autonomous domination from Roman domination and similar parameters, in which vertices can hold multiple guards [7, 17].

1.4 Related work and applications

Autonomous domination can be considered a form of self-stabilization [9] and involves a local detection paradigm [2]. Comparison can be made to the notion of distributed graph algorithms. Rather than considering distributed processors that combine to compute a global result, we consider distributed maintainers, whose communication is of a standard simple form at each step. The computation of autonomous domination numbers is done globally, here. Furthermore, supposing the autonomous domination number as given, the initial guard distribution within the graph involves strategic thought and global data. Many distributed algorithms for domination numbers and other parameters have been presented [8, 16, 18, 19, 25, 26, 30, 33, 34].

Autonomous domination can be seen as a toy problem for swarm robotics [10]. The problem (like eternal domination, and unlike foolproof eternal domination) has a hierarchical aspect [27], since the guards must be initially configured (and, in the eternal domination case, subsequently guided). The guards in an autonomous dominating set can be seen as mobile sensors [1, 29, 35], so that the problem of finding an autonomous dominating set is like finding a good mobile sensor deployment. Importantly, the mathematical abstraction of autonomous domination does not consider the cost of sensor movements or of communication at a distance, and these must be accounted for in real applications.

2 Definitions

Informally speaking, an autonomous dominating set is a collection of guards that can cover an arbitrarily long sequence of attacks, while coordinating their movements only by preserving domination. We assume that each guard knows about whether neighboring vertices are also defended by other guards, so that it knows whether its movement away from a location would result in an undefended vertex, and thus be forbidden. At least one guard can cover each attack so that the new configuration is still dominating, and no special choice is made if multiple guards can so move.

We now build up to a formal definition in a series of steps. First comes a definition about dominating sets that is useful when describing how guards move.

Definition 2.1. Given a graph G and dominating sets S and S', we say that S and S' are adjacent if they differ only by a pair of vertices adjacent in G: i.e. there are vertices $v \in S$ and $v' \in S'$ such that (v, v') is an edge of G and $S' = (S \setminus \{v\}) \cup \{v'\}$.

It will be useful at times to use the term 'legal guard move' to refer to the pair (v, v') relating two adjacent dominating sets S and S'. Adjacent dominating sets have the same cardinality.

The following definition is the fundamental one for this paper. It clarifies what precisely is meant when we say, informally, that the guards coordinate their motions using tactical but not strategic information.

Definition 2.2. A non-empty collection $\mathcal{F}$ of subsets $S_i \subset V$ is said to be an autonomously dominating family if the following conditions are satisfied.

1. Each subset S_i is a dominating set.
2. For each subset S_i and each vertex $v \in V \setminus S_i$, there is a subset $S_j \in \mathcal{F}$ which is adjacent to S_i and contains v.
3. For each subset $S \in \mathcal{F}$, every dominating set which is adjacent to S is also in $\mathcal{F}$.

The first condition simply records that the guard configuration at a given time is capable of responding to an attack at any vertex. The second condition is that there is always some guard that can respond to an attack without losing domination. The first and second conditions imply that each member of the family $\mathcal{F}$ is in fact secure dominating. The third is 'closure under all reasonable movements of autonomous guards.' It says that it must be possible for any guard that can move to the attack without losing graph domination to do so. There is no coordinator who, with an eye to the whole, determines which guard moves. What this would mean in a practical situation is that a randomization would pick one guard to move, if multiple guards are free to do so.

Remark 2.3. Autonomous domination is about defending a graph in perpetuity. As a result the definition of autonomously dominating family does not privilege one of the many configurations as 'initial.' Instead all possible arrangements of guards are treated equally. This is the reason for making the definition in terms of a family of subsets, rather than a statement about sequences. A definition using sequences could be made as well but would be more complicated. It is simpler to consider a whole family which is closed under the relation of adjacency for dominating sets.

Having defined autonomous domination for a family, the notion for an individual set follows.

Definition 2.4. A set of vertices in a graph is said to be an autonomous dominating set if it belongs to an autonomous dominating family.

It is now possible to define a graph invariant, the autonomous domination number.

Definition 2.5. The autonomous domination number $\gamma_{\mathrm{aut}}(G)$ of a graph G is the smallest size of an autonomous dominating set.

Remark 2.6. The graph given in Section 1.2 has autonomous domination number 4. The discussion of that section showed that fewer guards would not suffice. The eternal domination number of that graph, on the other hand, is 3.

Autonomous domination differs from eternal and eternal foolproof domination in ways made evident by the form of the definition. In Definition 2.2, one obtains eternal domination by dropping the third condition.

Definition 2.7. A set S of vertices V in a graph is said to be a secure dominating set if S is an element of a collection $\mathcal{F}$ of subsets $S_i \subset V$ which satisfies the following conditions.

1. Each subset S_i is a dominating set.
2. For each subset S_i and each vertex $v \in V \setminus S_i$, there is a subset $S_j \in \mathcal{F}$ which is adjacent to S_i and contains v.

In this case, all that is required is the existence of a possible guard (second condition). The master planner will determine which guard moves, and so the third condition is omitted.

On the other hand, the third condition can be strengthened as follows.

> **Strengthened Third Condition:** For each subset S_i, each vertex $v \in V \backslash S_i$, and each vertex $w \in S_i$, if v and w are adjacent, then $(S_i \backslash \{w\}) \cup \{v\}$ is also contained in $\mathcal{F}$.

This gives foolproof eternal domination, in which guards move without considering whether such movement is prudent. The foolproof eternal domination number itself is very simple to compute.

3 Properties and Computational Methods

3.1 Elementary Bounds

The foolproof eternal domination number of a graph is $n - \delta$, where n is the order of the graph and δ is the minimum degree [4, Theorem 3]. It was noted above that foolproof eternal domination arises by strengthening the definition of autonomous domination, so that the following inequality is immediate.

Lemma 3.1. *Let G be a graph with order n and minimum degree δ. Then $\gamma_{\mathrm{aut}}(G) \leq n - \delta$.*

The upper bound can be attained. For example, the complete graphs attain the upper bound.

It is also evident that the autonomous domination number is bounded below by the eternal domination number, since an eternal dominating set is subject to fewer conditions. In eternal domination a 'manager' is able to have an eye to the whole of the network being defended, and this is reflected in the following inequality.

Lemma 3.2. *The autonomous domination number γ_{aut} is bounded below by the eternal domination number γ_{∞}.*

3.2 Some means of computation

If an adversary attacks at a series of independent vertices, each such attack must be addressed by a separate guard. The next lemma records this.

Lemma 3.3. *Let G be a graph and $\mathcal{F}$ an autonomous family of dominating sets. Given any independent set I of vertices of G, there is a set S in the automonous family such that $I \subset S$.*

Proof. Consider a series of attacks at each of the vertices in I. These must be covered by distinct guards, since the set is independent and so a guard will never move from one vertex in I directly to another such vertex. □

When considering concrete examples, the following corollary is useful. It explains how independent dominating sets allow us to explore possible guard configurations.

Corollary 3.4. *Suppose that G is a connected graph, $\mathcal{F}$ an autonomous dominating family, and I an independent dominating set. Suppose further that the subgraph induced by the complement of I is connected. Then any set T of vertices of G which contains I and has the same cardinality as elements of $\mathcal{F}$ is contained in the autonomous family $\mathcal{F}$.*

Proof. By Lemma 3.3 there is some $S \in \mathcal{F}$ which contains I. Let the remaining guards move along paths from their initial locations to the remaining elements of T. These constitute a series of legal guard moves. □

Here are examples that exclude possible weakenings of the hypotheses of Corollary 3.4. Consider a path on five vertices $\{a_1, a_2, a_3, a_4, a_5\}$ and add two leaves ℓ_2 and ℓ_3 adjacent to a_2 and a_3, respectively. The autonomous domination number of this graph is 4; the set $S = \{a_1, a_2, a_3, a_4\}$ is an autonomous dominating set. Consider the set of vertices $\{a_2, a_3, a_4, a_5\}$. This set also contains the dominating set $\{a_2, a_3, a_4\}$, but is not connected to S by a sequence of legal guard moves. The three vertices $\{a_1, a_2, \ell_2\}$ must always have two guards. In order for a guard to move away from these three vertices, it would do so from a_2, thereby abandoning one of the adjacent leaves. This is not a legal guard move. (In this example, both independence and the connectedness of the complement fail.)

Again, with regard to the connectedness hypotheses in Corollary 3.4, consider the disjoint union of two cycles C_5. The autonomous domination number of this graph is 6, by Proposition 4.2. With two guards on one cycle and four on the other, however, we obtain a dominating set of that size which is not autonomous.

The second example involved a graph with two components. It seems reasonable to imagine that the hypothesis about the connectedness of the complement can be dropped. A proof of such a statement would involve some intricate analysis, and the more general result (if it holds) is not needed here.

A useful bound arises in the case that all dominating sets of a given size are secure dominating.

Theorem 3.5. *If every dominating set of size k is secure dominating, then $\gamma_{\text{aut}} \leq k$.*

Proof. Define an autonomous family consisting of all dominating sets of size k. □

The converse does not hold. Consider a K_3 with two leaves from one vertex. The autonomous domination number of this graph is 3, but the set consisting of the three vertices of K_3 is dominating but not secure dominating.

3.3 A partition count

The eternal domination number γ_∞ is bounded above by the clique cover number [4, Theorem 4]. In other words, if the vertices of a graph can be divided among c subsets such that the induced graph on each subset is complete, then $\gamma_\infty \leq c$.

This clique cover number c, though bounding γ_∞, does not provide a bound of γ_{aut}. Observe that a path P_n can be partitioned (by taking adjacent pairs of vertices, with possibly one left over) into $\lceil \frac{n}{2} \rceil$ such subsets, all of whose induced graphs are complete. On the other hand, $\gamma_{\text{aut}}(P_n)$, computed in Proposition 4.1, exceeds this bound by an amount growing arbitrarily large as n increases.

There is, however, the following technical proposition involving a partition of the set of vertices, which determines the autonomous domination number at least in some cases. It will be applied in Section 5 to compute autonomous domination numbers of Cartesian products of complete graphs.

Before giving the specific statement, here is an informal summary. Suppose that a graph consists of a relatively small number of relatively large cliques whose interconnections are sparse. Then one obtains an autonomous dominating set by assigning one guard to each clique. Because there are few connections between cliques, the guards will not leave their initial cliques.

Proposition 3.6. *Suppose that the set of vertices of a graph G is partitioned into k cliques S_i each satisfying $|S_i| > k$. Suppose as well that for each vertex $v \in S_i$ and for all $j \neq i$, there is at most one vertex in S_j adjacent to v. Then $\gamma_{\text{aut}}(G) = k$.*

Proof. Let T be a set of vertices of G with $|T| < k$. Then there is an i so that $S_i \bigcap T = \emptyset$; without loss of generality let it be S_1. By the connectivity hypothesis, each vertex in T is adjacent to at most one vertex in S_1. Since $|S_1| > k$, we see that some vertex in S_1 is not adjacent to T, so that T is not a dominating set. We have shown that $\gamma_{\text{aut}}(G) \geq \gamma(G) \geq k$.

Let T be a set with k vertices whose intersection with some S_i is empty. Then T is not dominating, by the same counting argument as above. The graph G does, however, have dominating sets of cardinality k, namely those containing one vertex of each clique S_i. These are secure dominating sets (move guards within their cliques) and so by Theorem 3.5, we conclude that $\gamma_{\text{aut}}(G) = k$. □

4 Elementary Examples

It is not difficult to check that $\gamma_{\text{aut}}(P_2) = 1$ and $\gamma_{\text{aut}}(P_3) = 2$. For longer paths, the following result gives the autonomous domination number.

Proposition 4.1. *The autonomous domination number of the path P_n is $n - 2$ for $n \geq 4$.*

Proof. In the case $n = 4$, there is no dominating set of size $n - 3 = 1$. Two guards suffice, since they will not cluster at one end.

Now consider $n \geq 5$. Let $\{a_1, a_2, \ldots, a_n\}$ be the vertices of P_n, with edges (a_i, a_{i+1}). The set $I = \{a_2, a_4, \ldots, a_{2\lfloor \frac{n}{2} \rfloor}\}$ is independent and dominating, so by Lemma 3.3 it is a subset of an autonomous dominating set. Let S be a dominating set with $|S| \leq n - 3$ which contains I.

We now show that the guards can congregate on one side of the path, leaving the other side vulnerable. Let j be an odd integer such that $a_j \in S$ and $a_{j+2} \notin S$ and $j + 2 \leq n$. Then the legal guard moves (a_{j+1}, a_{j+2}) followed by (a_j, a_{j+1}) show that the guards outside I can accumulate among the vertices with high indices.

A series of moves of this kind means that the odd vertices of low index are vacant. More specifically: by the cardinality of S we arrive at a dominating set which contains $\{a_2, a_4\}$ and which does not contain a_1, a_3, or a_5. Then (a_4, a_5) is a legal guard move. The resulting set of vertices is no longer secure dominating, since only the guard at a_2 defends a_3, but (a_2, a_3) is not a legal guard move since it would leave a_1 without defense.

Having established the lower bound, now consider any dominating set S of size $n-2$. Either the omitted vertices are adjacent, or they are not. If they are adjacent, they must be interior to the path (by the fact that S is dominating), in which case S is secure dominating. If they are not adjacent it is immediate that S is secure dominating. By Theorem 3.5 and the lower bound already established, we see that $\gamma_{\text{aut}}(P_n) = n - 2$. □

The foolproof eternal domination number of P_n is $n-1$, and the eternal domination number is $\lceil \frac{n}{2} \rceil$, given in [4]. Consequently, the autonomous domination number is intermediate, but closer to the foolproof for large n.

A similar relation is present in the case of cycles. For small cycles, it is straightforward to check that $\gamma_{\text{aut}}(C_3) = 1$, $\gamma_{\text{aut}}(C_4) = 2$, and $\gamma_{\text{aut}}(C_5) = 3$. The general result is the following.

Proposition 4.2. *The autonomous domination number of the cycle C_n is $n-3$ for $n \geq 6$.*

Proof. Let n be at least 8, and let $\{a_1, \ldots, a_n\}$ be the vertices of C_n, with edges (a_i, a_{i+1}) and (a_1, a_n). Let I be the set of all odd-indexed vertices with index strictly less than n. Let S be any dominating set of size $|S| \leq n-4$ which contains I. Reasoning as in the proof of Proposition 4.1, S is connected through a series of legal guard moves to a dominating set which contains the vertices of I as well as the vertex a_n and all vertices with sufficiently large even index. In particular, by cardinality, the vertices $\{a_2, a_4, a_6, a_8\}$ are unguarded. The series of legal guard moves (a_3, a_2) and (a_7, a_8) yields a dominating set which is not secure dominating, since the guard at a_5 must defend both a_4 and a_6 but cannot do so.

In the case of $n = 6$, there is no secure dominating set of size 2. In the case of $n = 7$, the independent set $\{a_1, a_4, a_6\}$ is dominating and so would necessarily be contained in an autonomous family whose elements have size 3. But (a_6, a_7) is a legal guard move, yielding $\{a_1, a_4, a_7\}$ which is not secure dominating, since only a_4 defends a_3 and a_5.

Every dominating set of size $n-3$ is secure dominating (there is no room to end up with an 'isolated guard'), so by Theorem 3.5 we conclude that $\gamma_{\text{aut}}(C_n) = n-3$ for $n \geq 6$. □

The comparable results are that the eternal domination number of C_n is $\lceil \frac{n}{2} \rceil$ and the foolproof eternal domination number is $n-2$, given in [4].

Proposition 4.3. *The autonomous domination number of the complete bipartite graph $K_{m,n}$ is the maximum of m and n.*

Proof. Let n be greater than or equal to m. Since $K_{m,n}$ has an independent dominating set of n vertices, the autonomous domination number must be at least n. Since each set of vertices of cardinality n is secure dominating, we find that n autonomous guards suffice. □

5 Autonomous Domination Numbers for Certain Families of Graphs

Let $p \leq q$ be natural numbers. The eternal domination number of the Cartesian product $K_p \square K_q$ is p, while the foolproof eternal domination number is $pq-(p+q)+2$, so that the two differ widely in general [4, Proposition 2]. For this family of graphs the autonomous domination number happens to coincide with the eternal domination number.

Proposition 5.1. *If $p \leq q$, then $\gamma_{\text{aut}}(K_p \square K_q) = p$.*

Proof. First consider the case that $p < q$. Partition the graph $K_p \square K_q$ into the p sets of vertices of the copies of K_q, each of which has size q. Then apply Proposition 3.6 to infer that $\gamma_{\text{aut}}(K_p \square K_q) = p$.

It remains to treat the case $p = q$. By [4, Proposition 2] the eternal domination number is p, so the elementary bound of Lemma 3.2 implies that γ_{aut} is at least p. We now show that p guards suffice.

Enumerate the vertices of each K_p with non-negative integers $\{0, \ldots, p-1\}$. The vertices of the product $K_p \square K_p$ are then ordered pairs (i, j) of such integers.

Distribute p guards to p vertices with distinct first coordinates; the second coordinate of each location is arbitrary. This is a dominating set by the completeness of the factor K_p.

Consider a legal guard move whose result is that there is an integer i such that there is no guard at (i, j) for all j. In other words, the collection of vertices $(i, \bullet)$ in $K_p \square K_p$ has been vacated. For this to be the case, there must have been guards at (j_k, k) for all k (the first coordinates need not be distinct) so that every complete subgraph of the form $(i, \bullet)$ remains dominated.

Therefore, at every time, legal guard movements will be such that either every guard has a distinct first coordinate, or every guard has a distinct second coordinate, and all such subsets of vertices are dominating sets.

Since the eternal domination number p is a lower bound for the autonomous domination number, we conclude that $\gamma_{\text{aut}}(K_p \square K_p) = p$. □

Ladder graphs are also an instance in which eternal and foolproof eternal domination numbers diverge widely. In this case the autonomous domination number differs from the foolproof domination number by one. The eternal domination number of $P_2 \square P_n$ is n, and the foolproof eternal domination number is $2n-2$, shown in [4, Theorem 8].

Proposition 5.2. *For $n \geq 3$, the ladder graph $P_2 \square P_n$ has autonomous domination number $\gamma_{\text{aut}}(P_2 \square P_n) = 2n-3$.*

Proof. Let the vertices of $P_2 \square P_n$ be $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ with edges (a_i, a_{i+1}), (b_i, b_{i+1}), and (a_i, b_i). Let S be a dominating set of size $n \leq |S| \leq 2n-4$ which contains the independent dominating set $\{a_1, b_2, a_3, b_4, \ldots\}$. By simple legal guard moves, it is no loss of generality to suppose that at least two of the a vertices and two of the b vertices are omitted from S. (Consider, for example, (b_{2k}, b_{2k+1}) and (a_{2k}, b_{2k}), which moves a guard from the a level to the b level.)

Suppose that S contains a_{2k} and does not contain a_{2k+2}. The pairs (a_{2k+1}, a_{2k+2}) and (a_{2k}, a_{2k+1}) are legal guard moves whether or not b_{2k+1} is occupied. Thus excess guards in the a-subgraph can be moved to vertices of higher index. The same can be done with guards among the b subgraph vertices. After many such moves, we obtain a dominating set S' connected to S through a series of legal guard moves does not contain $\{a_2, a_4, b_1, b_3\}$. Then (b_2, b_3) and (a_3, a_4) are legal guard moves, but the resulting dominating set is not secure dominating since the guard at a_1 is isolated and cannot defend both b_1 and a_2.

To show that $2n - 3$ autonomous guards suffice, consider any dominating set S of size $2n - 3$. All vertices but those on the ends of the ladder have degree 3. By cardinality, every vertex of degree 3 must have a guard in its closed neighborhood, and so vertices not at the end of the ladder are always defended. Since S is presumed dominating, it includes either at least one of a_1 and b_1, or both a_2 and b_2. In either case the set is secure dominating. The same reasoning applies for the other end of the ladder. Since all dominating sets of size $2n - 3$ are secure dominating, Theorem 3.5 gives the conclusion. $\square$

6 Counterintuitive examples

The following examples show that autonomous domination reflects noteworthy features of tactical action under limited information. The addition of an edge, which offers new ways for guards to defend locations, can also lead them away from the most strategically suitable positions. Additional guards, which in theory make a defensive arrangement stronger, can allow for poor positioning due to temporary security arising from the abundance of defenders.

These examples reveal the utility of the concept of autonomous domination. The parameter does not decrease monotonically with edge addition, and it is not a superhereditary property. These facts suggest that autonomous domination is a good abstraction that can account for difficulties that arise when real, limited agents act under constraints.

6.1 Edge addition

The autonomous domination number can decrease when an edge is added. A simple example is completing a path on three vertices to the complete graph on these three vertices, in which case the autonomous domination number decreases from 2 to 1.

Adding an edge can increase the autonomous domination number, however. A new edge means that guards can be drawn away from one portion of the graph and into another, and this could leave the source portion vulnerable.

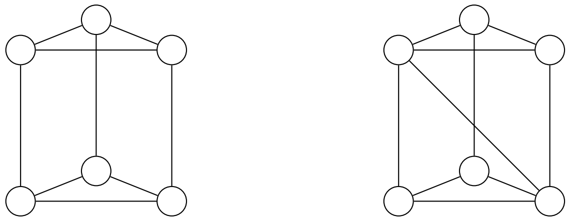


Figure 1: Edge addition can increase the autonomous domination number. In the graph on the right, the upper left and upper central vertices form a dominating set that is not secure dominating.

The graph on the left in Figure 1 has autonomous domination number 2, as it is $K_2 \square K_3$. The additional diagonal edge added to obtain the graph on the right causes the autonomous domination number to increase.

Lemma 6.1. *The autonomous domination number of the graph on the right in Figure 1 is* 3.

Proof. The autonomous domination number must be at least 2, since the graph has an independent set of size 2. Consider the independent dominating set consisting of the upper left and lower central vertices. This is adjacent to the dominating set consisting of the upper left and upper central vertices. But this is not secure dominating since an attack on the upper right vertex cannot be safely covered.

Every 3-subset of the graph is dominating, so more than three guards are not required. □

Another simple example is adding an edge to connect the disjoint union of C_5 and K_n. The graph $C_5 \bigcup K_n$ has autonomous domination number 4. Add a single edge to connect the two components. Guards can now be 'siphoned off' of the C_5 subgraph in a way that leaves it exposed to attack. More concretely, the autonomous domination number of the new (connected) graph is $n + 3$. With fewer than $n + 3$ guards, it is possible to move all but two guards to the K_n. The two guards that remain on the cycle do not defend it securely.

Example 6.2. The addition of a single edge can increase the autonomous domination number by an arbitrary amount even for connected graphs. Construct a graph consisting of two cliques $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_{n+1}\}$. Add edges (a_i, b_i) for each $i \leq n$. Additionally, let b_1 and b_2 be adjacent to the vertices all a vertices except a_n and a_{n-1}. For i satisfying $2 < i < m \ll n$, let b_i be adjacent to all but the last i vertices among the a. This graph has autonomous domination number 2. If the edge (b_1, a_{n-1}) is added, however, the autonomous domination number increases dramatically. For any $1 \leq k \leq m$, $k + 1$ guards will fail; place one guard at b_k and the others at the uppermost (k total) a vertices. Those guards can vacate the a level, moving to the corresponding b vertices, and the configuration remains dominating. An attack at b_{n+1} leads to failure.

Reasoning like in Example 6.2 will be useful in Section 7. Edges are added to connect cliques in such a way that the eternal domination (and ordinary domination) numbers remain fixed, while the autonomous domination number increases because of certain unstable configurations involving the congregation of guards within a single large clique.

6.2 Failure through excess of guards

In can be that n guards suffice for autonomous domination but $n+1$ guards do not. Thus autonomous domination is not a super-hereditary property.

Definition 6.3. Define a graph with the vertices

$$\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, b_5\}$$

and the following edges.

- Edges such that the set $\{a_i\}_{i=1}^4$ induces a complete subgraph.
- Edges such that the set $\{b_i\}_{i=1}^5$ induces a complete subgraph.
- (a_i, b_i) for $1 \le i \le 4$.
- (b_1, a_2).
- There are no other edges.

The graph is depicted in Figure 2. The eternal domination number of this graph is 2. This is also the autonomous domination number. Yet there is no autonomous set with three guards. The reason is that an arbitrary dominating set of cardinality 3 is connected through a series of legal guard moves with $\{b_1, a_3, a_4\}$, which is connected by a pair of legal guard moves with $\{b_1, b_3, b_4\}$. This last set is not secure dominating, since it is not adjacent to a dominating set containing b_5. Informally, with three guards it is possible that they all move 'upstairs,' and this is a bad decision.

Because autonomous domination is not super-hereditary, showing that n guards fail does not suffice to demonstrate that n is a lower bound on the autonomous domination number. Instead, every possibility between the eternal domination number and n must be considered.

One can generalize the reasoning of Definition 6.3. By stacking two large complete graphs, and making some of the vertices of the upper graph highly connected to vertices of the lower one, it is possible to produce a graph where two guards remain in their "assigned" cliques, but three or more can migrate upwards (and so fail), unless there are so many guards that the upper complete graph becomes essentially full before the lower one empties. In other words, if n guards succeed but $n+1$ fail, it could be that many more guards than $n+2$ are required to succeed again.

6.3 Mitigation

Without a central planner or particular structural details about a specific graph or family of graphs, it is difficult to conceive of a way to completely eliminate the

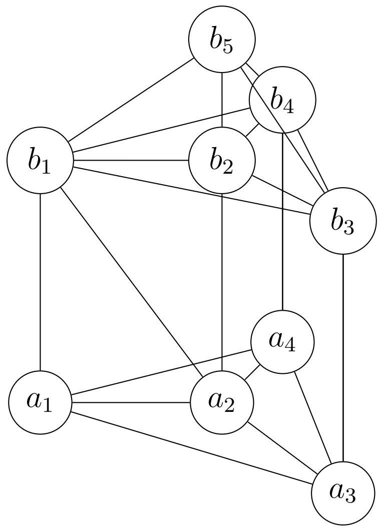


Figure 2: Autonomous domination is not super-hereditary. Two guards dominate the graph autonomously, but three guards can congregate at the upper level, a security failure.

paradoxical behaviors described here. Generally speaking, it is a poor use of resources to have many guards concentrate themselves in a highly connected region. In an implementation, one can imagine adding some sort of concentration measure to the guards' tactical motion planning. In other words, the guards move with a preference for remaining relatively dispersed. Such considerations are interesting in practice, but too far afield of the theoretical parameter now under discussion to warrant further treatment here.

7 Realizability of Parameter Values

It is straightforward to observe that the eternal domination number is bounded below by the domination number, and similarly that the autonomous domination number is bounded below by the eternal domination number (Lemma 3.2). A question following from these bounds is whether every possible collection of three natural numbers satisfying the inequalities constitute the domination number, eternal domination number, and autonomous domination number of some graph. This is so (excluding a trivial exception), as will be shown below. The proof will require reference to a number of families of graphs constructed for the purpose. These will be presented first, and their various parameters computed. Theorem 7.14 then collects these results. Note for comparison that realizability for triples given by the independence number, eternal domination number, and clique covering number has been considered previously [22].

Some general remarks can be made about the families of graphs to come. In a

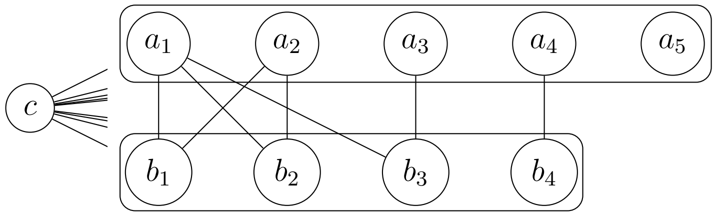


Figure 3: Depicts the graph $\mathcal{A}_2$. The domination number is 1 and the eternal domination number is 2, but autonomous guards can congregate at the upper level, so that γ_{aut} is larger.

sense, the final family, $\mathcal{F}_{\ell,m,n}$ of Definition 7.11, is generic. More precisely, given natural numbers $1 \ll a \ll b \ll c$, a member of this family has these numbers as domination, eternal domination, and autonomous domination number, respectively. The difficulty comes when considering cases where two of the parameters a, b, and c are equal or almost equal. That leads us to introduce the additional families of graphs here.

Recall that γ denotes the domination number, γ_∞ denotes the eternal domination number, and γ_{aut} denotes the autonomous domination number.

Definition 7.1. Given a natural number n define a graph $\mathcal{A}_n$ with vertices

$$\{a_1, a_2, \dots, a_{2n+1}, b_1, b_2, \dots, b_{2n}, c\}$$

and the following edges.

- The vertices a_i induce a complete subgraph.
- The vertices b_i induce a complete subgraph.
- (a_i, b_i) for all $1 \leq i \leq 2n$
- (a_i, b_j) for all $1 \leq i \leq n$ and $1 \leq j \leq 2n - i$ (observe that some of these were included in the previous item)
- (c, a_i) for all $1 \leq i \leq 2n + 1$
- (c, b_j) for all $1 \leq j \leq 2n$

The graph $\mathcal{A}_2$ is sketched in Figure 3. Some simplification has been made there to avoid clutter. The boxes denote induced complete subgraphs, and the vertex c, adjacent to all other vertices, is shown with half edges.

Proposition 7.2. *The parameters of $\mathcal{A}_n$ are these: $\gamma(\mathcal{A}_n) = 1$, $\gamma_\infty(\mathcal{A}_n) = 2$, and $\gamma_{\text{aut}}(\mathcal{A}_n) = n + 2$.*

Proof. The vertex c is adjacent to every other vertex, so $\gamma(\mathcal{A}_n) = 1$. The vertices a_{2n+1} and b_1 are not adjacent, so $\gamma_\infty(\mathcal{A}_n) \geq 2$. That two guards suffice follows from observing that $\{c, a_1, a_2, \dots, a_{2n+1}\}$ and $\{b_1, b_2, \dots, b_{2n}\}$ are cliques.

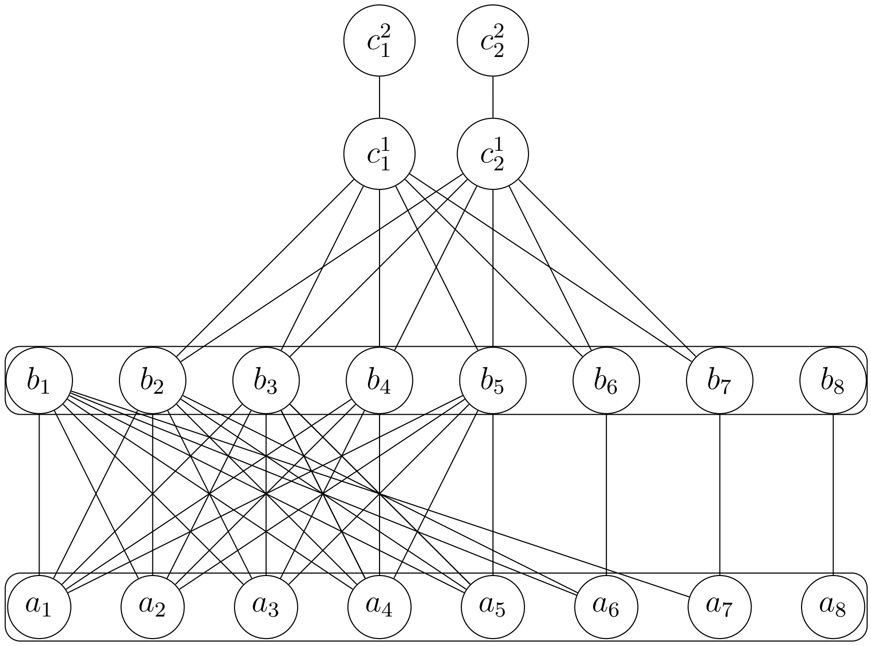


Figure 4: Depicts the graph $\mathcal{B}_{2,3}$. The domination and eternal domination numbers are equal, while the autonomous domination number is greater.

Consider any natural number g satisfying $2 \leq g \leq n$. There is no autonomous dominating set of size g. Any dominating set is adjacent to one which includes c. Since c is adjacent to every vertex, we see that any two sets of the same size, both of which contain c, are connected through a series of legal guard moves. An arbitrary dominating set of size g is thus connected to $\{a_{2n-g+1}, a_{2n-g+2} \ldots, a_{2n}, c\}$. This set is adjacent to $\{a_{g-1}, a_{2n-g+1}, a_{2n-g+2} \ldots, a_{2n}\}$, which is a dominating set. It is not secure dominating, however, since an attack on a_{2n+1} leads to failure.

It remains to show that $\mathcal{A}_n$ has an autonomous dominating set of size $n+2$. Any dominating set of $n+2$ vertices among the a_i is secure dominating. No set consisting only of b vertices is dominating, since none is adjacent to a_{2n+1}. Thus the only remaining case to check is that a dominating set including c is secure dominating. If c is occupied, along with at least one vertex from a or b, the set is secure dominating, since the guard at c can move to the other level to defend an attack there. □

Definition 7.3. Given a non-negative integer m and a natural number n, define the graph $\mathcal{B}_{m,n}$ with vertices

$$\{a_1, a_2, \ldots a_{2n+m}, b_1, b_2, \ldots, b_{2n+m}, c_1^1, c_1^2, c_2^1, c_2^2, \ldots, c_m^1, c_m^2\}$$

and the following edges.

- The vertices a_i induce a complete subgraph.
- The vertices b_i induce a complete subgraph.

- (a_i, b_i) for all $1 \leq i \leq 2n$.
- (c_i^1, c_i^2) for all $1 \leq i \leq m$.
- (b_i, a_j) for each $0 \leq i \leq n$ and $0 \leq j \leq 2n + m - i$.
- (b_i, a_j) for each $n + 1 \leq i \leq n + m$ and $0 \leq j \leq n + m - 1$.
- (b_i, c_j^1) for all $2 \leq i \leq 2n + m - 1$ and all $1 \leq j \leq m$.
- There are no other edges.

A sketch of such a graph is in Figure 4. There the parameter values are $m = 2$ and $n = 3$.

Proposition 7.4. *The domination parameters of $\mathcal{B}_{m,n}$ are:* $\gamma(\mathcal{B}_{m,n}) = \gamma_\infty(\mathcal{B}_{m,n}) = m + 2$, *and* $\gamma_{\mathrm{aut}}(\mathcal{B}_{m,n}) = m + n + 2$.

Proof. The set $I = \{a_{2n+m}, b_1, c_1^2, c_2^2, \ldots, c_m^2\}$ is an independent dominating set of size $m + 2$. There is no smaller dominating set, since at least one guard is needed for each of the m edges (c_i^1, c_i^2). None of these defends b_1, and no b vertex is adjacent to all a vertices, nor vice versa.

The eternal domination number is also $m + 2$, since one guard can be allocated to each of the m edges (c_i^1, c_i^2), one can be allocated to the vertices a_i, and finally one to the vertices b_j.

We now show that the autonomous domination number is not less than $m+n+2$. Consider $i \leq n$. We show that $m + i + 1$ guards will fail. Let m guards be arranged among the edges (c_i^1, c_i^2). The vertex b_i is adjacent to all but the last i a vertices. By stationing a guard at b_i, and another i guards at the last i a vertices, the guards can vacate the a level while remaining a dominating set. This configuration is not secure, since an attack at b_{2n+m-i} cannot be defended.

Now we must show that $m + n + 2$ guards suffice for autonomous domination. Consider any dominating set S of this size. Then S necessarily contains at least m vertices from among c_i^j.

There are a few cases to consider.

Case 1: S contains a and b vertices. This case is straightforward. Since $\{a_i\}$ and $\{b_i\}$ are each cliques, S is secure dominating.

All of the nuance is in the following two cases. We must confirm that when guards congregate at a single level, an attack within that level will not throw off the coverage of the other level.

Case 2: S contains no a vertices. Since b_{2n+m} is the only b vertex adjacent to a_{2n+m}, we see that $b_{2n+m} \in S$. If there is an i so that both of the vertices c_i^1 and c_i^2 are in S, then S is secure dominating, since problems at the b level can be addressed by the c_i^1 guard, and problems at the a level can be addressed by the guards already at the b level, on the hypothesis that S is dominating.

Therefore, suppose that there is no i such that both of the vertices c_i^1 and c_i^2 are in S. Then S contains $n + 2$ of the b vertices. With that many guards, there is necessarily redundancy in coverage of a vertices, so that at least one b vertex is free

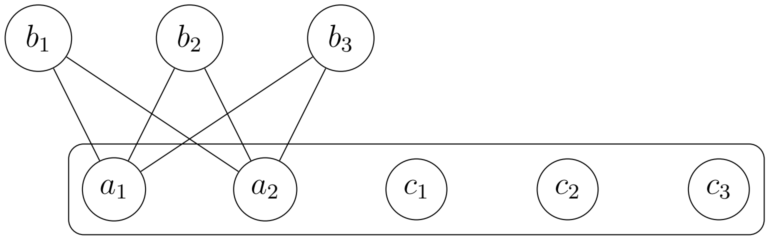


Figure 5: Depicts the graph $\mathcal{C}_{3,3}$. The domination number is 1, while the eternal and autonomous domination numbers are greater and distinct.

to move within the b level. This claim can be made more formal in the following manner. Let j be the least natural number so that b_{2n+m-j} is unoccupied. If $j = n+2$, then S is secure dominating, since b guards moving to lower indices cover more a vertices. If $j < n+2$, then b_k is necessarily occupied for some $k \leq j$. This means that the dominating set S contains j vertices with index higher than $2n + m - j$, and the vertex b_k. But these $j+1$ vertices are already a dominating set, so that the remaining vertex or vertices in S is/are free to cover any attacks at the remaining b vertices.

Case 3: S contains no b vertices. Since no a vertex is adjacent to both b_1 or b_{2n+m}, and neither of those b vertices is defended by any c vertex, we see that S must contain a_{2n+m} along with at least one other a vertex. If more than m guards are among the c vertices, the set is secure dominating, since a c_i^1 vertex can guard the interior b vertices, and every a vertex other than a_{2n+m} is adjacent to b_1.

Suppose, then, that S contains exactly m of the c vertices. This means that it contains $n+2$ a vertices. To be dominating, S must contain the n vertices $\{a_{n+m+1}, a_{n+m+2}, \ldots, a_{2n+m}\}$. Therefore it also contains two vertices from $\{a_1, a_2, \ldots, a_{n+m}\}$. At least one of these vertices, however, defends the vertices $\{b_1, b_2, \ldots, b_{n+m}\}$. ':w! Therefore, the second guard can move freely at the a level to defend any attack, while maintaining domination.

In each of the three cases, then, we find that S, a dominating set of size $m+n+2$, is secure dominating. □

Definition 7.5. Given a natural numbers $m \geq 2$ and $n \geq 2$, define the graph $\mathcal{C}_{m,n}$ whose vertices are

$$\{a_1, a_2, b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_n\}$$

having the following edges.

- The set of vertices $\{a_1, a_2, c_1, c_2, \ldots, c_n\}$ induces a complete subgraph.
- Each b_i is adjacent to a_1 and a_2.
- There are no other edges.

The graph $\mathcal{C}_{3,3}$ is depicted in Figure 5.

Proposition 7.6. *The domination parameters of $\mathcal{C}_{m,n}$ are these:* $\gamma(\mathcal{C}_{m,n}) = 1$, $\gamma_\infty(\mathcal{C}_{m,n}) = m+1$, *and* $\gamma_{\text{aut}}(\mathcal{C}_{m,n}) = m+n$.

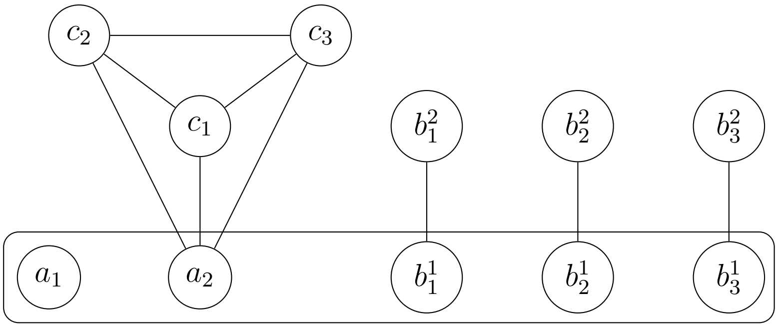


Figure 6: Depicts the graph $\mathcal{D}_{3,3}$. The domination and eternal domination numbers differ by one, while the autonomous domination number is greater.

Proof. Since a_1 is adjacent to every vertex the domination number is 1.

The set $\{b_1, b_2, \ldots .b_m, c_1\}$ is an independent set, so that the eternal domination number is at least $m+1$. In fact $m+1$ suffice. Assign stationary guards to each b_i, and a single additional guard for the remaining clique.

Let any number k from $m+1$ to $m+n-1$ be given, and consider any dominating set of size k. Such a dominating set is adjacent to one containing the vertex a_1. Then, via a series of legal guard moves, we arrive at the dominating set $\{a_1, c_1, c_2, \ldots, c_{k-m+1}, b_1, b_2, \ldots, b_{m-2}\}$. This is not a secure dominating set, since an attack at b_{m-1} can only be covered by the guard at a_1, and then no guard is adjacent to b_m.

A set with $m+n$ vertices either contains all the peripheral vertices b_i and c_j, or it contains one of the central vertices a_k. In either case such a set is secure dominating.

□

Definition 7.7. Given natural numbers m and n, define the graph $\mathcal{D}_{m,n}$ with vertices

$$\{a_1, a_2, b_1^1, b_1^2, b_2^1, b_2^2, \ldots, b_m^1, b_m^2, c_1, c_2, c_3, \ldots, c_n\}$$

and the following edges.

- The set of vertices $\{a_1, a_2, b_1^1, b_2^1, \ldots, b_m^1\}$ induces a complete subgraph.
- (b_i^1, b_i^2) for each i, $1 \leq i \leq m$.
- The set of vertices $\{a_2, c_1, c_2, \ldots, c_n\}$ induces a complete subgraph.
- There are no other edges.

The graph $\mathcal{D}_{3,3}$ is depicted in Figure 6. The rectangle encloses vertices inducing a complete subgraph.

Proposition 7.8. *The domination parameters of $\mathcal{D}_{m,n}$ are these: $\gamma(\mathcal{D}_{m,n}) = m+1$, $\gamma_\infty(\mathcal{D}_{m,n}) = m+2$, and $\gamma_{\text{aut}}(\mathcal{D}_{m,n}) = m+n+1$.*

Proof. The vertices $\{b_1^1, b_2^1, \ldots, b_m^1, a_2\}$ form a dominating set. No smaller dominating set exists. At least m guards are required for the m leaves b_i^2, and another is required for vertices c_i.

The set $\{a_1, b_1^2, b_2^2, \ldots, b_m^2, c_1\}$ is an independent set of size $m+2$, giving a lower bound for the eternal domination number. In fact $m+2$ guards suffice, since $\{a_1, a_2\}$, $\{\{b_i^1, b_i^2\}_{i=1}^m\}$, and $\{c_i\}_{i=1}^n$ is a partition of the vertices into $m+2$ subsets all inducing complete subgraphs.

Let k be a number of guards $m+2 \leq k \leq m+n$. Consider a dominating set of size k containing the independent set of vertices $\{a_1, b_1^2, b_2^2, \ldots, b_m^2, c_1\}$. This set has size $m+2$, so that there are $k-m-2$ remaining guards. Then via a series of legal guard moves we obtain the set

$$\{a_1, b_1^2, b_2^2, \ldots, b_m^2, c_1, c_2, \ldots, c_{k-m-1}\},$$

which is adjacent to

$$\{a_1, b_1^1, b_2^2, \ldots, b_m^2, c_1, c_2, \ldots, c_{k-m-1}\},$$

which is adjacent to

$$\{a_2, b_1^1, b_2^2, \ldots, b_m^2, c_1, c_2, \ldots, c_{k-m-1}\},$$

which is adjacent to

$$\{b_1^1, b_2^2, \ldots, b_m^2, c_1, c_2, \ldots, c_{k-m}\},$$

which is not secure dominating, since an attack at b_1^2 is only covered by the guard at b_1^1, leaving a_1 unprotected.

A dominating set of $m+n+1$ vertices is secure dominating, by counting the leaves and the size of the clique $\{a_2, c_1, c_2, \ldots, c_n\}$. □

Definition 7.9. Let m and n be natural numbers. Define the graph $\mathcal{E}_{m,n}$ with vertices

$$\{a_1, a_2, a_3, \ldots, a_{m+3}, b_1, b_2, c_1, c_2, \ldots, c_n\}$$

and the following edges.

- The vertices a_i induce a complete subgraph.
- (a_i, b_j), for each pair $(i, j) \in \{1, 2\} \times \{1, 2\}$.
- (a_{m+3}, c_i) for each $i \in \{1, 2, \ldots, n\}$.

The graph $\mathcal{E}_{3,3}$ is depicted in Figure 7. The vertices a_i, forming a clique, are enclosed by the rectangle and their edges are omitted.

Proposition 7.10. *The domination parameters of $\mathcal{E}_{m,n}$ are these: $\gamma(\mathcal{E}_{m,n}) = 2$, $\gamma_\infty(\mathcal{E}_{m,n}) = n+3$, and $\gamma_{\mathrm{aut}}(\mathcal{E}_{m,n}) = m+n+3$.*

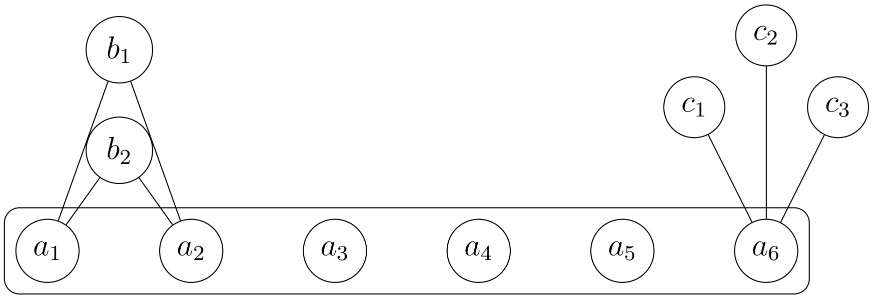


Figure 7: Depicts the graph $\mathcal{E}_{3,3}$. The domination number is 2, while the eternal and autonomous domination numbers are greater and distinct.

Proof. There is no single vertex adjacent to all vertices, but each vertex is adjacent to either a_1 or a_{m+3}. This establishes the domination number.

The set $\{b_1, b_2, a_3, c_1, c_2, \ldots, c_n\}$ is an independent set of size $n+3$, establishing a lower bound for the eternal domination number. That $n+3$ guards suffice is seen by noting that stationary guards can be assigned to each b_i and c_j, and one final guard is needed for the complete graph induced by $\{a_i\}$.

Let k be a natural number $n+3 \leq k \leq m+n+2$ and consider a dominating set of size k which contains the independent set $\{b_1, b_2, a_3, c_1, c_2, \ldots, c_n\}$. Through a series of legal guard moves we obtain a dominating set which contains b_1 and b_2 but neither a_1 nor a_2 nor a_3. This set is adjacent to one containing a_1 but not b_1, and the latter is adjacent to one containing a_2 but not b_2. That set is adjacent to the dominating set containing a_3 but not a_2. This set is not secure dominating. An attack at one of the vertices b_i can only be addressed by the guard at a_1, at which point the other b vertex is left without defense.

That $m+n+3$ guards suffice follows from counting. With $m+n+3$ guards at least two of $\{a_1, a_2, b_1, b_2\}$ will be occupied, hence neither b_1 nor b_2 will be unguarded. $\square$

This final family of graphs is, in a sense, the generic one for this problem.

Definition 7.11. Given natural numbers ℓ, m, and n, define the graph $\mathcal{F}_{\ell,m,n}$ with vertices

$$\{a_1, a_2, \ldots, a_\ell, b_1, b_2, \ldots, b_m, c_1, c_2, c_3, \ldots, c_{\ell+n}\}$$

and the following edges.

- (a_i, c_i) for each $i \in \{1, 2, \ldots, \ell\}$.
- (b_i, c_j) for each $i \in \{1, 2, \ldots, m\}$ and each $j \in \{1, 2, \ldots, \ell\}$.
- The vertices c_i induce a complete subgraph.
- There are no other edges.

The graph $\mathcal{F}_{4,3,3}$ is depicted in Figure 8. The vertices c_i, inducing a complete subgraph, are enclosed in the rectangle and their edges are omitted.

Proposition 7.12. *The domination parameters of $\mathcal{F}_{\ell,m,n}$ are these: $\gamma(\mathcal{F}_{\ell,m,n}) = \ell$, $\gamma_\infty(\mathcal{F}_{\ell,m,n}) = \ell + m + 1$, and $\gamma_{\text{aut}}(\mathcal{F}_{\ell,m,n}) = \ell + m + n$.*

Proof. One vertex of each pair $\{a_i, c_i\}$ must be occupied in order to obtain a dominating set, so that the domination number is at least ℓ. Provided that at least one guard is stationed at one of the c_i, the set is dominating, since each vertex c_i is adjacent to every b and c vertex.

The set $\{a_1, a_2, \ldots, a_\ell, b_1, b_2, \ldots, b_m, c_{\ell+1}\}$ is an independent set of size $\ell + m + 1$, which gives a lower bound for the eternal domination number. That many guards suffice, since stationary guards can be assigned to each a_i and b_j and one guard can defend the remaining vertices c_i.

Let k be a natural number $\ell + m + 1 \leq k \leq \ell + m + n - 1$. Consider a dominating set of size k which contains the independent set $\{a_1, a_2, \ldots, a_\ell, b_1, b_2, \ldots, b_m, c_{\ell+1}\}$. This is adjacent to one with c_1 replacing a_1. Then all the guards on the b_i can move to c_i without losing domination. In the end, since the ℓ leaves are occupied, as well as interior vertices $\{c_{\ell+1}, \ldots, c_{\ell+n}\}$, then because $k \leq \ell + m + n - 1$ we see that at most $m - 1$ of the vertices $c_1, \ldots, c_\ell$ are occupied. There is then no defense against the series of attacks $b_1, b_2, \ldots, b_m$.

We finally show that $\ell+m+n$ guards suffice for autonomous domination. Consider any dominating set of that size. Since the set is dominating, at least one member of each pair $\{a_i, c_i\}$ is contained in the set. Moreover, either no c_i is in the set, in which case all of the vertices b_i are, or at least one of the c_i is in the set. In the former case secure domination is straightforward: every vertex a_i is occupied, and every vertex b_i is occupied, and there are $n > 0$ guards remaining necessarily contained in the set $\{c_i\}$ which induces a complete subgraph. In the latter case, let j be the number of vertices c_i contained in the dominating set $1 \leq j \leq \ell$. Then $\ell - j$ guards occupy leaves, since the set is dominating. At most n guards are contained in the subset of vertices $\{c_{\ell+1}, \ldots, c_{\ell+n}\}$, which means at least $\ell+m+n-(j+(\ell-j)+n) = m$ guards are contained in the vertices b_i. But there are only m such vertices. Thus the only empty vertices are either among pairs $\{a_i, c_i\}$ (which contain a guard by hypothesis) or are among the vertices $\{c_{\ell+1}, \ldots, c_{\ell+n}\}$ (which induce a complete subgraph and contain a guard). Thus the set is secure dominating. □

Proposition 7.13. *Suppose that $\gamma_\infty(G) = 1$. Then $\gamma_{\text{aut}}(G) = 1$.*

Proof. If $\gamma_\infty(G) = 1$ then there is no independent set of vertices of size 2. Therefore G is necessarily a complete graph. □

The previous proposition is the reason for excluding the case $a = b = 1$ and $c \neq 1$ in the following theorem. Otherwise there is no limit on how the various domination numbers can be related, beyond their ordering.

Theorem 7.14. *Let a, b, and c be natural numbers such that $a \leq b \leq c$ and either $c = 1$ or $b > 1$. Then there is a graph G such that $\gamma(G) = a$, $\gamma_\infty(G) = b$, and $\gamma_{\text{aut}}(G) = c$.*

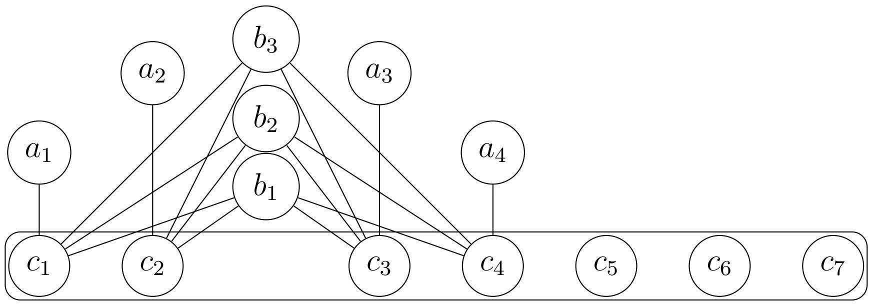


Figure 8: Depicts the graph $\mathcal{F}_{4,3,3}$. The domination, eternal domination, and autonomous domination numbers are distinct.

Proof. It is necessary only to put together the examples collected previously. We organize the cases in a sort of lexicographic order, thinking of the sizes of a, b, and c. There are some extraordinary situations for small parameter values, leading to a number of cases.

Case $c = 1$

K_n is such a graph.

Case $a = 1$ *and* $b = 2$

If $c > 2$, then $\mathcal{A}_{c-2}$ is such a graph. If $c = 2$, then P_3 is such a graph.

Case $a = 1$ *and* $b > 2$

If $b = c$, then $K_{1,b}$ is such a graph. If $b < c$, the graph $\mathcal{C}_{b-1,1-b+c}$ is such a graph.

Case $a = 2$ *and* $b = 2$

If $c = 2$ then P_4 is such a graph. If $c > 2$ then $\mathcal{B}_{0,c-2}$ is such a graph.

Case $a = 2$ *and* $b = 3$

If $c = 3$, then $K_{2,3}$ is such a graph. If $c > 3$, then $\mathcal{D}_{1,c-2}$ is such a graph.

Case $a = 2$ *and* $b > 3$

If $b = c$, then $K_{2,b}$ is such a graph. If $b < c$, then $\mathcal{E}_{c-b,b-3}$ is such a graph.

Case $a \geq 3$ *and* $b = a$

If $c = b$, then the path on c vertices with a leaf attached to each path vertex is such a graph. If $c > b$, the graph $\mathcal{B}_{b-2,c-b}$ is such a graph.

Case $a \geq 3$ *and* $b = a + 1$

The graph $\mathcal{D}_{a-1,c-a}$ is such a graph.

Case $a \geq 3$ *and* $b - a \geq 2$

The graph $\mathcal{F}_{a,b-a-1,c-b+1}$ is such a graph. □

8 Conclusion

This paper has introduced a new invariant for graphs, the autonomous domination number. This new invariant is computable, relates in clear ways to previously studied domination invariants, and models interesting features of decision-making under partial information.

8.1 Further Questions

The following are avenues for future research in autonomous domination.

- Compute the autonomous domination numbers of well-known graph classes.
- Compute the autonomous domination numbers of random graphs.
- Find an algorithm to compute the autonomous domination number of a tree.
- Theorem 7.14 gives a family of graphs whose domination and autonomous domination numbers are equal. Determine whether these are all such (connected) graphs.
- Characterize graphs whose eternal domination and autonomous domination numbers are equal.
- Characterize graphs whose autonomous domination and foolproof eternal domination numbers are equal (cf. Lemma 3.1).
- Determine whether there is a graph with two autonomous dominating sets of minimum size that are not connected by a series of legal guard moves.
- If some graphs have minimum-size autonomous dominating sets that are not connected by legal guard moves (as in the preceding item), consider weighting the vertices of the graph, and finding minimum-weight autonomous families.
- Determine whether the hypothesis regarding the connectedness of the complement in Corollary 3.4 can be removed.
- Investigate the computational complexity of finding the autonomous domination number of a graph (cf. [5, 11, 20, 28, 31]).
- Investigate distributed algorithms for finding the autonomous domination number of a graph.

Acknowledgements

I thank Professor Kiran Bhutani for asking the question answered by Theorem 7.14. I thank anonymous referees for numerous helpful remarks, and for proposing a number of open questions.

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(Received 5 Feb 2025; revised 30 May 2025)