AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **92(2)** (2025), Pages 194–209

k-coalitions in graphs

Abbas Jafari   Saeid Alikhani*

Department of Mathematical Sciences
Yazd University, 89195-741, Yazd
Iran
abbasjafaryb91@gmail.com   alikhani@yazd.ac.ir

Davood Bakhshesh

Department of Computer Science
University of Bojnord, Bojnord
Iran
d.bakhshesh@ub.ac.ir

Abstract

In this paper, we propose and investigate the concept of k-coalitions in graphs, where $k \geq 1$ is an integer. A k-coalition refers to a pair of disjoint vertex sets that jointly constitute a k-dominating set of the graph, meaning that every vertex not in the set has at least k neighbors in the set. We define a k-coalition partition of a graph to be a vertex partition in which each set is either a k-dominating set with exactly k members or forms a k-coalition with another set in the partition. The maximum number of sets in a k-coalition partition is called the k-coalition number of the graph and is represented by $C_k(G)$. We present fundamental findings regarding the properties of k-coalitions and their connections with other graph parameters. Also we obtain the exact values of 2-coalition numbers of some specific graphs and we also study graphs with large 2-coalition number.

1 Introduction

Consider a simple and undirected graph G with vertex set $V = V(G)$. Two vertices are said to be neighbors if they are adjacent. For an integer $k \geq 1$, a k-dominating set of G is a set S of vertices such that each vertex in $V \setminus S$ is adjacent to at least k vertices in S. The smallest possible size of a k-dominating set of G is referred to as

* Corresponding author.

ISSN: 2202-3518

the k-domination number of G and is denoted by $\gamma_k(G)$. The interested reader may refer to [11, 12] for a comprehensive overview of dominating sets in graphs.

A coalition in a graph G is a pair of nonempty sets S_1 and S_2 that are not dominating sets of G, but their union $S_1 \cup S_2$ is a dominating set of G. Such a pair forms a coalition and the members of such a pair are coalition partners. A vertex partition $\mathfrak{X} = \{S_1, \ldots, S_k\}$ of the vertex set $V(G)$ is called a coalition partition of G if every set $S_i \in \mathfrak{X}$ is either a dominating set of G with cardinality $|S_i| = 1$, or not a dominating set but forms a coalition with some $S_j \in \mathfrak{X}$. The coalition number of a graph is the maximum number of sets in a coalition partition.

The concept of a coalition in graphs was introduced by Haynes, Hedetniemi, Hedetniemi, McRae, and Mohan in [10]. Their fundamental studies have set the stage for much of the subsequent research on coalition numbers and coalition graphs. Notably, they explored upper bounds on coalition numbers, providing essential insights and bounds that help understanding the maximum coalition number possible in various graph classes [7]. Additionally, they developed the concept of coalition graphs, which are derived from the original graph by focusing on the coalition structure among the vertices, allowing for deeper analysis of the interactions and relationships within graph coalitions [8]. Extending their previous work, they introduced self-coalition graphs, a specific type of coalition graph where the coalitions possess a self-referential property, adding another layer of complexity and applicability to the study of coalition graphs [9].

Recent studies have continued to build upon these foundational concepts, expanding the scope and depth of coalition studies in graphs. Significant contributions in this area include the exploration of total coalitions, independent coalitions, connected coalitions, and specific investigations into coalition numbers in tree structures and singleton coalition graph chains. Alikhani et al. (2024) delved into total coalitions, provided a detailed analysis of how coalitions can encompass all vertices in a graph and the implications of such total structures. This study offers new metrics and bounds for total coalition numbers, expanding the understanding of coalition dynamics in comprehensive vertex sets [1]. Henning and Jogan [13] characterized graphs with smallest possible total coalition number. Moreover they characterized graphs G with $\delta(G) = 1$ satisfying $C_t(G) = k$ for all $k \geq 3$.

Alikhani et al. also explored the independence properties within coalitions, defining and characterizing independent coalition graphs, leading to new theoretical insights and practical applications in graph theory [2]. Moreover, Alikhani et al. investigated connected coalitions, where coalitions form connected subgraphs, providing critical results on the connectivity aspects of coalitions, which are vital for applications requiring robust and resilient coalition structures [3]. Mojdeh and Masoumi [15], introduced the concept of edge coalition and characterized graphs G with small and large size number of edge coalition. Also existence and characterization of independent coalition and double coalitions are investigated in [14, 16].

In addition, Bakhshesh, Henning, and Pradhan (2023) focused on tree structures, a fundamental graph class, for determining coalition numbers. Their findings offer specific insights and bounds applicable to trees, enriching the overall understanding of coalition numbers in hierarchical and acyclic graph structures [4] and in an upcoming publication, Bakhshesh, Henning, and Pradhan explored chains formed by singleton coalitions. This study provides a novel perspective on coalition structures by examining the sequential and chain-like properties of singleton coalitions, contributing to the broader theory of coalition graphs [5].

Building on these established concepts, the exploration of k-coalitions in graphs represents a promising extension. A *k-coalition* consists of a pair of disjoint vertex sets that together form a k-dominating set of the graph, meaning that each vertex not in the set has at least k neighbors within the set. We define a k-coalition partition of a graph as a vertex partition where each set is either a k-dominating set with exactly k members or forms a k-coalition with another set in the partition. The maximum number of sets in a k-coalition partition is referred to as the *k-coalition number* of the graph and denoted by $C_k(G)$. This generalization has the potential to tackle more complex real-world problems, where entities often participate in multiple overlapping groups.

In the next section, after introducing k-coalition partition and k-coalition number, we prove that any graph G has a k-coalition partition, and we obtain bounds on the k-coalition number. Moreover, we study the k-coalition number of certain graphs such as complete graphs, trees, paths, cycles, and corona of paths and cycles with $\overline{K_l}$ in Section 3. In Section 4, we study graphs with large 2-coalition numbers and in Section 5 we characterize trees T of order n with $C_2(T) = n$ and $C_2(T) = n - 1$. Finally, we conclude the paper in Section 6.

2 Existence and some bounds

In this section, we prove that any graph G has a k-coalition partition, and also we present some bounds on the k-coalition number.

Definition 1 Two sets $U_1 \subseteq V$ and $U_2 \subseteq V$ form a k-coalition (are k-coalition partners) if neither is a k-dominating set, but their union is a k-dominating set. We define a k-coalition partition $\Theta = \{U_1, \ldots, U_r\}$ of a graph G as a vertex partition in which each set of Θ is either a k-dominating set with exactly k members or forms a k-coalition with another set in the partition. We denote the k-coalition number of a graph G which is the maximum number of sets in a k-coalition partition of G, by $C_k(G)$.

A domatic partition is a partition of the vertex set into dominating sets, in other words, a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ such that every set V_i is a dominating set of G. Cockayne and Hedetniemi [6] introduced the domatic number of a graph $d(G)$ as the maximum order k of a domatic partition. For more details on the domatic number, refer to [17, 18, 19]. Now we propose the notion of k-domatic number of G.

Definition 2 A k-domatic partition is a partition of the vertex set into k-dominating sets, in other words, a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of $V(G)$ such that every V_i is a k-dominating set of G. The k-domatic number of a graph $d_k(G)$ is the maximum order k of a k-domatic partition.

Theorem 2.1 *For any integer $k \geq 1$ there is a k-coalition partition for a given graph G.*

Proof. We consider two cases:

Case 1. If $d_k(G) \geq 2$, then consider a k-domatic partition $\Phi = \{X_1, \ldots, X_s\}$ of $V(G)$ and let $1 \leq i < s$. Without loss of generality, assume that X_i is a minimal k-dominating set of G. If it is not, then there exists a minimal k-dominating set $X'_i \subseteq X_i$. In this case, we replace X_i with X'_i and add all members of $X_i \backslash X'_i$ to X_s. To construct a k-coalition partition Θ of G, we split each minimal k-dominating set X_i with $i < s$ into two non-empty sets $X_{i,1}$ and $X_{i,2}$ and add them to Θ. If $|X_i| = 1$, then $k = 1$ and we simply add X_i to Θ without splitting it. Note that neither $X_{i,1}$ nor $X_{i,2}$ is a k-dominating set, but their union is a k-dominating set. Next, we consider the set X_s. If X_s is a minimal k-dominating set, we split it into two non-empty sets $X_{s,1}$ and $X_{s,2}$ and add them to Θ to complete the construction. If X_s is not a minimal k-dominating set, there exists a set $X'_s \subseteq X_s$ that is minimal and k-dominating. We split X'_s into two non-empty sets $X'_{s,1}$ and $X'_{s,2}$ and add them to Θ. Let $X''_s = X_s \backslash X'_s$. It is important to observe that X''_s cannot be a k-dominating set, as this would imply that $d_k(G) > s$, which contradicts the fact that Φ is a k-domatic partition of G. If X''_s form a k-coalition with any set in Θ, we add it to Θ and finish the construction. Otherwise, we remove $X'_{s,2}$ from Θ and add $X'_{s,2} \cup X''_s$ to Θ.

Case 2. If $d_k(G) = 1$, it is sufficient to consider X_1 as X_s in Case 1. ∎

The following theorem gives a lower bound on $C_k(G)$ for connected graphs of order n by means of the k-domatic number.

Theorem 2.2 *If G is a connected graph and $k \geq 2$, then $C_k(G) \geq 2d_k(G)$.*

Proof. Let G have a k-domatic partition $\mathcal{C} = \{C_1, C_2, \ldots, C_s\}$ with $d_k(G) = s$. Without loss of generality, we assume that the sets $\{C_1, C_2, \ldots, C_{s-1}\}$ are minimal k-dominating sets. If any set C_i is not minimal, we can find a subset $C'_i \subseteq C_i$ that is a minimal k-dominating set and add the remaining vertices to the set C_s. Since $k \geq 2$, it follows that C_i is not a singleton set. Furthermore, if we partition a minimal k-dominating set, containing more than one element, into two non-empty subsets, then we obtain two non-k-dominating sets that together form a k-coalition. Consequently, we divide each non-singleton set C_i into two sets $C_{i,1}$ and $C_{i,2}$ that form a k-coalition. This results in a new partition $\mathcal{C}'$ consisting of non-k-dominating sets, each of which pairs with another non-k-dominating set in $\mathcal{C}'$ to form a coalition. Next, we consider the k-dominating set C_s. If C_s is a minimal k-dominating set, we divide it into two non-k-dominating sets, add these sets to $\mathcal{C}'$, and obtain a k-coalition partition of cardinality at least $2s$. Since $s = d_k(G)$, it follows that $C_k(G) \geq 2d_k(G)$.

If C_s is not a minimal k-dominating set, we aim to find a subset $C'_s \subseteq C_s$ that is minimal. We then partition C'_s into two non-k-dominating sets that together form a k-coalition. Let C''_s be the complement of C'_s in C_s, and append $C'_{s,1}$ and $C'_{s,2}$ to $\mathcal{C}'$. If C''_s can merge with any non-k-dominating set to form a k-coalition, we can obtain a k-coalition partition of cardinality at least $2s+1$ by adding C''_s to $\mathcal{C}'$. Thus, $C_k(G) \geq 2d_k(G)+1$. However, if C''_s cannot form a k-coalition with any set in $\mathcal{C}'$, we remove $C'_{s,2}$ from $\mathcal{C}'$ and add the set $C'_{s,2} \cup C''_s$ to $\mathcal{C}'$. This results in a k-coalition partition of cardinality at least $2s$. Therefore, $C_k(G) \geq 2d_k(G)$.

Based on the above arguments, we conclude that $C_k(G) \geq 2d_k(G)$, completing the proof. ■

Corollary 2.3 *For even k, $C_k(G) \geq d_{k/2}(G)$.*

Proof. Consider a graph G possessing a $k/2$-domatic partition, denoted as $\mathcal{C} = \{C_1, C_2, \ldots, C_s\}$. By definition, any two sets in $\mathcal{C}$ are considered as k-coalition partners if neither qualifies as a k-dominating set. Define the subset $\mathcal{C}'$ of $\mathcal{C}$ by:

$$\mathcal{C}' = \{C_i \in \mathcal{C} \mid C_i \text{ is a } k\text{-dominating set}\}.$$

Applying the same process to $\mathcal{C}'$ as applied for $\mathcal{C}$ in the proof of Theorem 2.2, we arrive at the desired conclusion. ■

Lemma 2.4 *For any graph G and any k-coalition partition $\mathcal{C}$ of G, any set of $\mathcal{C}$ forms a k-coalition with at most $\Delta(G) - k + 2$ sets of $\mathcal{C}$.*

Proof. Consider a vertex v in graph G and a set $S \in \mathcal{C}$ such that $v \in S$. If S is a k-dominating set, then by definition, S forms a k-coalition with no other set in $\mathcal{C}$, thereby confirming the result. Now, suppose S is not a k-dominating set. Then, there exists a vertex $x \notin S$ that is not k-dominated by S. For any set $A \in \mathcal{C}$ that does not include x and forms a k-coalition with S, in order for $A \cup S$ to k-dominate vertex x, the set $A \cup S$ must include at least k vertices from $N(x)$ (the neighborhood of x). Let $x \in A$. To maximize the number of sets that form a k-coalition with S, the set S must contain at most $k-1$ neighbors of x, leaving the remaining neighbors of x to be covered by all coalition partners of S except A. Therefore, in the worst case, S forms a k-coalition with at most $1 + |N(x)| - (k-1) \leq \Delta(G) - k + 2$ sets. This completes the proof. ■

Theorem 2.5 *For any graph G with the maximum degree $\Delta(G)$ and any integer $k > \delta(G)$, we have $C_k(G) \leq \Delta(G) - k + 3$.*

Proof. Let x be a vertex of G of degree $\deg(x) = \delta(G)$. Let $\mathcal{C}$ be a c_k-partition of G of the cardinality $C_k(G)$. Let $X \in \mathcal{C}$ such that $x \in X$. If $N(x) \subseteq X$, then any set of $\mathcal{C} \setminus \{X\}$ must form a k-coalition only with X. Hence, by Lemma 2.4, $C_k(G) \leq 1 + \Delta(G) - k + 2 = \Delta(G) - k + 3$. Now, assume that $N(x) \nsubseteq X$. Let $A \neq X$ and $B \neq X$ be two sets of $\mathcal{C}$. If A and B forms a k-coalition, then

$A \cup B$ is a k-dominating set. Since $x \notin A \cup B$, x must have at least k neighbors in $A \cup B$, which is a contradiction because x has $\delta(G) < k$ neighbors. Hence, every set of $\mathcal{C}$ must only form a k-coalition with X. Hence, by Lemma 2.4, we have $C_k(G) \leq \Delta(G) - k + 2 + 1 = \Delta(G) - k + 3$. ∎

Lemma 2.6 *If $\Delta(G) < k$ then $C_k(G) = 2$.*

Proof. Suppose A and B form a k-coalition. Since $\deg(v) < k$, it follows that $A \cup B$ contains all vertices and no proper subset of the vertices can be a k-dominating set. Consequently, $C_k(G) = 2$. ∎

Theorem 2.7 *For any graph G with $\Delta(G) \geq \delta(G)+1$, we have $C_{\delta(G)}(G) \leq 2\Delta(G) - 2\delta(G) + 4$.*

Proof. Let $k = \delta(G)$ and let x be a vertex of G of degree $\delta(G)$. Let $\mathcal{C}$ be a k-coalition partition of G of the cardinality $C_k(G)$. Let $X \in \mathcal{C}$ such that $x \in X$.

- If $N(x) \cap X \neq \emptyset$, then, any set of $\mathcal{C} \setminus X$ must form a k-coalition only with X. Hence, by Lemma 2.4, $C_k(G) \leq \Delta(G) - k + 3$. Since $\Delta(G) \geq \delta(G) = k$, we have $C_{\delta(G)}(G) \leq 2\Delta(G) - 2\delta(G) + 4$.
- If $N(x) \cap X = \emptyset$, then we consider the following cases.
  - There exist at least three sets A, B and C in k-coalition partition $\mathcal{C} \setminus \{X\}$ which have intersect with $N(x)$, i.e., $A \cap N(x) \neq \emptyset$, $B \cap N(x) \neq \emptyset$ and $C \cap N(x) \neq \emptyset$. In this case for every two partners S_i and S_j which form k-coalition, we have $S_i = X$, or $S_j = X$. Therefore by Lemma 2.4, $C_k(G) \leq \Delta(G) - k + 3$.
  - There exist exactly two sets A and B with $A \cap N(x) \neq \emptyset$, $B \cap N(x) \neq \emptyset$. Then $N(x) \subseteq A \cup B$. Hence, there is no set $\mathcal{C} \setminus \{X, A, B\}$ forming k-coalition with A or B. Hence, by Lemma 2.4 the set X is in at most $\Delta(G) - k + 2$ k-coalition and therefore $C_k(G) \leq 1 + \Delta(G) - k + 2 + 2 = \Delta(G) - k + 5$. Since $\Delta(G) \geq k + 1$, we have $C_k(G) \leq 2(\Delta(G) - k + 2)$.
  - There exists exactly one set $A \in \mathcal{C}$ with $A \cap N(x) \neq \emptyset$. So $N(x) \subseteq A$. Then, if A and X form a k-coalition, then by Lemma 2.4, each of the sets A and X form k-coalitions with at most $\Delta(G) - k + 2$ sets. Since we assumed that A and X form a k-coalition, then $C(G) \leq 2(\Delta(G) - k + 1) + 2 = 2(\Delta(G) - k + 2)$. Now, assume that A and X do not form a k-coalition. Then, $A \cup X$ is not a k-dominating set. Hence, there exists a vertex w which is not k-dominated by A and X. Consider $|(X \cup A) \cap N(w)| = t \leq k - 1$. Therefore, every member of $\mathcal{C} \setminus \{X, A\}$ must either contain the vertex w or include at least $k - t$ vertices of $N(w)$. Consequently, we have

  $$C_k(G) \leq \max_{0 \leq t \leq k-1} \left(\frac{\Delta(G) - t}{k - t}\right) + 1 + 2.$$

  This value can reach its maximum when $t = k - 1$ and $C_k(G) \leq \Delta(G) - k + 4$. ∎

In the following, we show that the k-coalition number of any k-regular graph is 3 or 4.

Theorem 2.8 *If G is a k-regular graph, then $3 \leq C_k(G) \leq 4$.*

Proof. Suppose that two vertices v_1 and v_2 in G are adjacent. Then

$$\{V \setminus \{v_1, v_2\}, \{v_1\}, \{v_2\}\}$$

is a k-coalition partition of G and so $C_k(G) \geq 3$. Now let $\mathcal{C}$ be a k-coalition partition of G with the cardinality $C_k(G)$. If there exists X in $\mathcal{C}$ which contains two adjacent vertices, then for some vertex x in X, $N(x) \cap X \neq \emptyset$. By Lemma 2.4, $C_k(G) \leq \Delta(G) - k + 3 = 3$. Therefore in this case $C_k(G) = 3$.

If no member of $\mathcal{C}$ contains two adjacent vertices, then we consider the following two cases:

Case 1. If there exists a vertex x such that $N(x)$ has intersection with just one set or more than two sets in $\mathcal{C}$, then by the proof of Theorem 2.7,

$$C_k(G) \leq \max\{\Delta(G) - k + 4,\ 2(\Delta(G) - k + 2)\} = 4.$$

Case 2. Otherwise, let X be a member of $\mathcal{C}$, and consider $x \in X$ as a vertex. Again, we consider two cases:

- If S_1 or S_2 forms a k-coalition with X, then by Lemma 2.4

$$C_k(G) \leq 1 + \Delta(G) - k + 2 + 1 = \Delta(G) - k + 4 = 4.$$

- If neither S_1 nor S_2 forms a k-coalition with X and X is a k-dominating set, we are done. Now let X form a k-coalition with S_0. Since any vertex $v_0 \in N(x)$ (which is not in $S_0 \cup X$) is dominated by $S_0 \cup X$, and the neighborhood of any vertex cannot be in the only one set in a partition, so there is a vertex $v_0 \in N(X)$ which is adjacent to a vertex in S_0. So X just forms a k-coalition with S_0 and so $\mathcal{C} = \{X, S_0, S_1, S_2\}$. Therefore we have the result.

■

3 k-coalition number of specific graphs

First let us recall the definition of the corona of two graphs F and H. By taking a single instance of graph F and $|V(F)|$ copies of the graph H, and linking the i-th vertex of F to each vertex in the i-th instance of H, we obtain a graph denoted as $F \circ H$. This graph is regarded as the corona product of F and H.

In this section, we study the k-coalition number of certain graphs, such as complete graphs, trees, paths P_n, cycles C_n, $P_n \circ \overline{K_l}$ and i $C_n \circ \overline{K_l}$. We start with complete graphs.

Since $\pi = \{\{v_1, v_2, \ldots, v_{k-1}\}, \{v_k\}, \{v_{k+1}\}, \ldots, \{v_n\}\}$ is a k-coalition partition of K_n, we have the following observation:

Observation 3.1 *For every* $2 \leq k \leq n-1$, $C_k(K_n) = n - k + 2$.

The following theorem characterizes the k-coalition number of the complete bipartite graph.

Theorem 3.2 *Let* $K_{s,t}$ *be the complete bipartite graph with partitions* $X = \{v_1, v_2, \ldots, v_s\}$ *and* $Y = \{v'_1, v'_2, \ldots, v'_t\}$*, where* $s \leq t$*. The coalition number* $C_k(K_{s,t})$ *is characterized as follows:*

(i) *If* $k < s$*, then* $\max\{s + t - 4k + 4, t - k + 2\} \leq C_k(K_{s,t}) \leq s + t - 2k + 1$.
(ii) *If* $1 < k = s$*, then* $C_k(K_{s,t}) = 4$.
(iii) *If* $s < k \leq t$*, then* $C_k(K_{s,t}) = 2$.
(iv) *If* $t < k$*, then* $C_k(K_{s,t}) = 2$.

Proof. Let $X = \{v_1, v_2, \ldots, v_s\}$ and $Y = \{v'_1, v'_2, \ldots, v'_t\}$ represent the two partitions of $K_{s,t}$. Consider the following cases:

(i) Using the following k-coalition partitions the lower bound follows.

$$\Big\{\{v_1, v_2, \ldots, v_{k-1}, v'_1, v'_2, \ldots, v'_k\}, \{v_k, v_{k+1}, \ldots, v_{2k-1}, v'_{k+1}, v'_{k+2}, \ldots, v'_{2k-1}\}, \{v_{2k}\}, \ldots, \{v_s\}, \{v'_{2k}\}, \ldots, \{v'_t\}\Big\},$$

and

$$\Big\{\{v_1, v_2, \ldots, v_s, v'_1, v'_2, \ldots, v'_{k-1}\}, \{v'_k\}, \ldots, \{v_{k+1}\}, , \ldots, \{v'_t\}\Big\}.$$

Every S_i and S_j forming a k-coalition must satisfy $|S_j \cup S_i| \geq 2k$ (ensuring k vertices in each part) and the following partition gives the upper bound.

$$\Big\{\{u_1, u_2, \ldots, u_{2k-1}\}, \{u_{2k}\}, \ldots, \{u_{s+t}\}\Big\}, \quad u_i \in X \cup Y.$$

(ii) Let $\Theta = \{S_1, S_2, \ldots\}$ be a k-coalition partition and S_1, S_2 form a k-coalition. Since $\deg_X(v) = k$ for all $v \in Y$, it follows that $Y \subseteq (S_1 \cup S_2)$ or $X \subseteq (S_1 \cup S_2)$. So the Θ can at most have four member. According the following partition we have $|\Theta| = 4$.

$$\{\{v_1\}, \{v_2, \ldots, v_s\} \{v'_1\}, \{v'_2 \ldots, v'_t\}\}.$$

Note that if there exists a k-dominating set S in Θ, then $\Theta = \{X, Y\}$.

(iii) Let $\Theta = \{S_1, S_2, \ldots\}$ be a k-coalition partition. Suppose S_1 and S_2 form a k-coalition. Since $\deg_X(v) < k$ for all $v \in Y$, it follows that $Y \subseteq (S_1 \cup S_2)$. If both $Y \cap S_1 \neq \emptyset$ and $Y \cap S_2 \neq \emptyset$, then $|\Theta| = 2$. Without loss of generality, assume $Y \subseteq S_1$. Furthermore, since $\forall v \in X, \deg_Y(v) \geq k$, S_1 is a k-dominating set, which contradicts Definition 2. Note that a k-dominating set must contain all vertices of Y. Consequently, $C_k(K_{s,t}) = 2$.

(iv) This follows from Lemma 2.6.

■

Using Theorem 2.7, we have the following result.

Corollary 3.3 *For every tree G, $C_2(G) \leq \Delta(G) + 1$.*

Corollary 3.4 *For every $k \in \mathbb{N}$, there exists a tree T with maximum degree k and $C_2(T) = k + 1$.*

Proof. Let T be a tree obtained from a star $K_{1,k}$ with attaching two pendant vertices to each leave of $K_{1,k}$ (see Figure 1). If L is the set of leaves of T, then the 2-coalition number of this tree is $k + 1$, as demonstrated by the following partition:

$$\{\{u\}, \{v_2\}, \{v_3\}, \ldots, \{v_k\}, L \cup \{v_1\}\}.$$

■

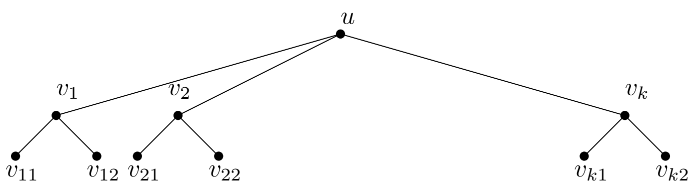

Figure 1: A tree T with maximum degree k and $C_2(T) = k + 1$.

Now by using Theorem 2.5 we prove the following result.

Theorem 3.5 *For any path P_n*

$$C_2(P_n) = \begin{cases} 1 & n = 1 \\ 2 & n = 2, 3 \\ 3 & n \geq 4. \end{cases}$$

Proof. It is easy to verify that form $n \leq 3$, $C_2(P_1) = 1$, $C_2(P_2) = 2$, and $C_2(P_3) = 2$. Now consider a path P_n with $n \geq 4$. By Theorem 2.5, for any path P_n we have $C_2(P_n) \leq \Delta(P_n) - 2 + 3 = 3$. Now, we have a 2-coalition partition of cardinality 3 for P_n as follows.

$$\{\{v_1, v_n\}, \{v_{2i} | 1 < 2i < n\}, \{v_{2i+1} | 1 < 2i + 1 < n\}\}.$$

■

To obtain the 2-coalition number of cycles we need the following easy lemma:

Lemma 3.6 *If $S \subseteq V(C_n)$ is a 2-dominating set of C_n, then $|S| \geq \dfrac{n}{2}$.*

Proof. For every $v \in V \backslash S$, $\deg_S(v) = 2$ and $\sum_{v \in V \backslash S} \deg_S(v) = 2(n - |S|)$. So by definition,

$$2(n - |S|) \leq \sum_{v \in S} \deg(v) = 2|S|,$$

and so $\frac{n}{2} \leq |S|$. ■

Theorem 3.7 *(i) If $\Theta = \{S_0, S_1, \ldots, S_t\}$ is a 2-coalition partition for C_n with $|\Theta| \geq 4$ and every $S_i \in \Theta$ does not contain consecutive vertices of C_n, then n is even and $C_2(C_n) = 4$.*

(ii) The 2-coalition number of odd cycles is 3.

Proof.

(i) We consider two cases:

Case 1: If there is a vertex $v_0 \in S_0$ which is adjacent to different members S_1 and S_2, then every $S_i \in \Theta \backslash \{S_0, S_1, S_2\}$ form a 2-coalition with S_0. So

$$\frac{n}{2} \leq |S_i \cup S_0| = |S_i| + |S_0|.$$

Since $\deg_{V \backslash \{S_1, S_2\}}(v_0) = 0$, so S_1 and S_2 form a 2-coalition. Therefore

$$\frac{n}{2} + \frac{n}{2} \leq |S_i| + |S_0| + |S_1| + |S_2| \leq n = |V|,$$

and so

$$|S_i| + |S_0| = \frac{n}{2}, \quad |S_1| + |S_2| = \frac{n}{2}.$$

Thus n is even and $|\Theta| = 4$.

Case 2: We do not have any vertex adjacent to a different member. So the vertices must be alternatively in the same member S_0. Therefore n is even and S_0 is 2-dominating set. Hence $|S_0| = 2$ and $n = 4$.

(ii) Suppose that $\Theta = \{S_1, S_2, \ldots, S_t\}$ is a 2-coalition of C_n, where $n > 4$ is odd. On the other hand, for every C_n with $n \geq 4$, there exists a 2-coalition partition with three elements, explicitly given by:

$$\big\{\{v_1\}, \{v_{2i} \mid 2 \leq 2i \leq n\}, \{v_{2i+1} \mid 2 \leq 2i + 1 \leq n\}\big\}.$$

Thus it follows that $|\Theta| \geq 3$. Now, assume $|\Theta| \geq 4$. By Theorem 3.7, this implies that there exists some $S_i \in \Theta$ containing two consecutive vertices. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, and without loss of generality suppose that v_1 and v_2 are both elements of S_i. Since $\deg_{V \backslash S_i}(v_1) \leq 1$, every $S_j \in \Theta \setminus \{S_i\}$ must form a 2-coalition with S_i. It is important to note that S_i cannot be a 2-dominating set, as S_i is an element of the 2-coalition partition. Therefore, $V \setminus S_i$ must contain consecutive vertices. Let $v_r, v_{r+1} \in V \setminus S_i$ be two consecutive vertices, and let S_k and $S_{k'}$ be the members of Θ containing v_r and v_{r+1}, respectively. Now we observe that S_i cannot form a k-coalition with any member of $\Theta \setminus \{S_k, S_{k'}\}$, implying that $|\Theta| \leq 3$. This contradicts the assumption that $|\Theta| \geq 4$. Therefore, the initial assumption leads to a contradiction, and the result follows. ∎

Corollary 3.8 *The 2-coalition number of C_n is:*

$$C_2(C_n) = \begin{cases} 4 & n \text{ is even} \\ 3 & n \text{ is odd.} \end{cases}$$

Using Theorem 2.7, we obtain the following result.

Corollary 3.9 *For any cycle C_n and any path P_n, we have $C_2(C_n \circ K_1) = 4$ and $C_2(P_n \circ K_1) = 4$.*

Proof. Let V' contain vertices of degree one and $V'' = V \setminus V'$. By Theorem 2.7, we have $C_2(C_n \circ K_1) \leq 4$ and $C_2(P_n \circ K_1) \leq 4$. Now, we present a 2-coalition partition with 4 elements for P_n and C_n, as follows:

$$\big\{V \setminus \{v_{n-2}, v_{n-1}, v_n\}, \{v_{n-2}\}, \{v_{n-1}\}, \{v_n\}\big\}$$

such that $v_{n-2}, v_{n-1}, v_n \in V''$. ∎

Theorem 3.10 *The k-coalition number of $C_n \circ \overline{K_l}$ is:*

$$C_k(C_n \circ \overline{K_l}) = \begin{cases} 2 & l \leq k-3 \\ 3 & l = k-2 \\ 4 & l = k-1 \\ 2 & l \geq k \end{cases}$$

Proof. Let $\Theta = \{S_1, S_2, \ldots, S_t\}$ be a k-coalition partition and S_i and S_j form a k-coalition. Suppose that $V' = \{v |\deg(v) = 1\}$ and $V'' = \{v |\deg(v) = l+2\}$. We consider the following cases:

Case 1: $l \leq k-3$. Since the degree of any vertices is less than k, so $S_i \cup S_j$ must contain whole of the vertices. Therefore $\Theta = \{S_i, S_j\}$.

Case 2: $l = k-2$. Since $\deg(v) = 1 < k$ for $v \in V'$, it follows that $V' \subset S_i \cup S_j$. If V' intersects with both S_i and S_j, it is clear that $|\Theta| = 2$. Now let $V' \subset S_i$. Since S_i is not a k-dominating set, there are at least 2 consecutive vertices v_1 and v_2 in $V \setminus S_i$. Every $S_t \in \Theta \setminus \{S_i\}$ must form a k-coalition with S_i and contains at least one of v_1, v_2. Therefore $|\Theta| \leq 3$. We can have a k-coalition with three elements as follows:

$$\big\{V \setminus \{v_{n-1}, v_n\}, \{v_{n-1}\}, \{v_n\}\big\} \quad \text{such that } v_{n-1}, v_n \in V''.$$

Case 3: $l = k-1$. This is similar to the proof of Corollary 3.9.

Case 4: $l \geq k$. In this case the vertices can be partitioned into two disjoint sets V' and V'' such that $V'\{v| \deg(v) = 1\}$ and $V'' = \{v| \deg(v) = l+2\}$. Let S_i and S_j be a k-coalition, so $V' \subset S_i \cup S_j$. If $V' \subset S_i$ then S_i is a k-dominating set, since $V \setminus S_i \subset V''$ and $\deg_{V'}(v) = l+2 > k$ for $v \in V''$, and it contracts the definition of k-coalition. Therefore sets $V' \cap S_i$ and $V' \cap S_j$ are not empty and it implies $\Theta = \{S_i, S_j\}$. ■

Similar to the proof of Theorem 3.10 we have the following result:

Theorem 3.11 *The k-coalition number of $P_n \circ \overline{K_l}$ is:*

$$C_k(P_n \circ \overline{K_l}) = \begin{cases} 2 & l \leq k-3 \\ 2 & l = k-2 \quad n \leq 3 \\ 3 & l = k-2 \quad n \geq 4 \\ 4 & l = k-1 \\ 2 & l \geq k \end{cases}$$

4 Graphs with large 2-coalition number

Characterization of graphs of order n whose coalition number is n or $n-1$ is an interesting subject, see [3, 4]. In this section, we study graphs with large 2-coalition number.

Theorem 4.1 *If $C_2(G) = n$, then $\deg(v) \geq n-2$ for every vertex v.*

Proof. Since $C_2(G) = n$, for every $v_1 \in V(G)$, there exists $v_2 \in V(G)$ which forms a 2-coalition. Therefore v_1 and v_2 must be adjacent to all vertices of $V(G) \setminus \{v_1, v_2\}$. ■

Lemma 4.2 *If $C_2(G) = n$ and $\deg(v) = n-2$, then there is only one 2-coalition partner for v.*

Proof. Since $C_2(G) = n$, there is a vertex v' which forms a 2-coalition with v. Therefore all vertices of $V(G) \setminus \{v, v'\}$ must be adjacent to v and since $\deg(v) = n-2$, v' is not adjacent to v. ■

Lemma 4.3 *For any even integer n, there is an $(n-2)$-regular graph H with $C_2(H) = n$.*

Proof. Suppose $H = K_n \setminus M$, where M is a perfect matching in K_n. Obviously the graph H is $(n-2)$-regular with $C_2(H) = n$. ■

Corollary 4.4 *If G is an $(n-2)$-regular graph with $C_2(G) = n$, then n is even and G is isomorphic to graph H in the proof of Lemma 4.3.*

We close this section with the following remark.

Remark 4.5 *If $C_2(G) = n$ and n is odd, then the number of full vertices of G is odd.*

5 Trees with large 2-coalition number

Characterization of trees of order n whose coalition number is n or $n-1$ is an interesting subject. In this section, we study trees with large 2-coalition number. First we obtain another upper bound for the 2-coalition number of trees.

Theorem 5.1 *For any tree T of order n, $C_2(T) \leq \frac{n}{2} + 1$.*

Proof. Let π be a k-coalition partition of T and $L \in \pi$ is the set which consists of leaves of T (if leaves are contained in two parts, then $c_2(T) = 2$). Any $X \neq L$ in π forms a 2-coalition with L. So by Lemma 2.4, there are at most $\Delta(T) - 2 + 3 = \Delta(T) + 1$ sets in π. But it is easy to see that for any tree T with k leaves, $\Delta(T) \leq k$, and so

$$C_2(T) \leq k + 1.$$

On the other hand, since there are k leaves in L, we have at most $n - k$ vertices which are not in L. If any set $X \neq L$ in π is a singleton (worst case), then

$$C_2(T) \leq 1 + n - k.$$

From these two upper bounds, we have:

$$2C_2(T) \leq n + 2,$$

which implies the result. ■

Corollary 5.2 *Let T be a tree of order n.*

(i) If $C_2(T) = n$, then $T = P_2$.

(ii) If $C_2(T) = n - 1$, then $T = P_4$.

Proof.

(i) Suppose that $C_2(T) = n$. By Theorem 5.1, we have $n \leq \frac{n}{2} + 1$, so $n \leq 2$. Therefore $T = P_2$.

(ii) If $C_2(T) = n - 1$, then by Theorem 5.1, we have $n - 1 \leq \frac{n}{2} + 1$, so $n \leq 4$. Therefore $T = P_4$. ■

Theorem 5.3 *If there is a vertex x in the tree T such that the distance between x and all leaves of T is at least 2, then $C_2(T) \geq 3$.*

Proof. Suppose that $S = \{v \in V(T) | d(v, x) \geq 2\}$ and $S_1 = \{w \in V(T) | d(v, w) = 1\}$. Then S, S_1 and $\{x\}$ is a c_2-partition. Therefore we have the result. ■

Now, we show that by using Theorem 5.3 we have another proof for 2-coalition number of paths.

Corollary 5.4 *For any $n \geq 5$, $C_2(P_n) = 3$.*

Proof. By Theorem 2.7, $C_2(P_n) \leq \Delta(P_n) - l + 3 = 3$. On the other hand by Theorem 5.3, for $n \geq 5$, $C_2(P_n) \geq 3$. Therefore the result is obtained. ■

6 Conclusion

This paper introduces the concept of k-coalition in graphs and investigates some properties related to k-coalition number. We proved that any graph G has a k-coalition partition, and also we presented some bounds on the k-coalition number. Utilizing these bounds, we have determined the precise values of k-coalition number of some specific graphs. We studied the graphs G with large $C_k(G)$.

Here we state some unresolved problems and potential research directions related to the k-coalition number of graphs.

(i) What is the exact values of k-coalition number of specific graphs, such as paths, cycles, trees and unicyclic graphs for $k \geq 3$.

(ii) Study Nordhaus and Gaddum lower and upper bounds on the sum and the product of the k-coalition number of a graph and its complement.

(iii) What is the k-coalition number of graph operations, such as corona, Cartesian product, join, lexicographic product, and so on?

(iv) Associated with every k-coalition partition π of a graph G, there is a graph called the k-coalition graph of G with respect to π, denoted $kCG(G, \pi)$, the vertices of which correspond one-to-one with the sets $V_1, V_2, \ldots, V_k$ of π and two vertices are adjacent in $kCG(G, \pi)$ if and only if their corresponding sets in π form a k-coalition. Study of k-coalition graph is an interesting subject.

Acknowledgements

The authors would like to express their gratitude to the referees for their careful reading and very helpful comments.

References

[1] S. Alikhani, D. Bakhshesh and H. R. Golmohammadi, Total coalitions in graphs, *Quaest. Math.* 47(11) (2024), 2283–2294.

[2] S. Alikhani, D. Bakhshesh, H. R. Golmohammadi and S. Klavžar, On independent coalition in graphs and independent coalition graphs, *Discuss. Math. Graph Theory* 45 (2025), 533–544.

[3] S. Alikhani, D. Bakhshesh, H. R. Golmohammadi and E. V. Konstantinova, Connected coalitions in graphs, *Discuss. Math. Graph Theory* 44 (2024), 1551–1566.

[4] D. Bakhshesh, M. A. Henning and D. Pradhan, On the coalition number of trees, *Bull. Malays. Math. Sci. Soc.* 46 (2023), 95.

[5] D. Bakhshesh, M. A. Henning and D. Pradhan, Singleton coalition graph chains, *Comput. Appl. Math.* 43(2) (2024), 85.

[6] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks* 7 (1977), 247–261.

[7] T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae and R. Mohan, Upper bounds on the coalition number, *Australas. J. Combin.* 80 (2021), 442–453.

[8] T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae and R. Mohan, Coalition graphs, *Commun. Comb. Optim.* 8(2) (2023), 423–430.

[9] T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae and R. Mohan, Self-coalition graphs, *Opuscula Math.* 43(2) (2023), 173–183.

[10] T. W. Haynes, J. T. Hedetniemi, S. T. Hedetniemi, A. A. McRae and R. Mohan, Introduction to coalitions in graphs, *AKCE Int. J. Graphs Combin.* 17(2) (2020), 653–659.

[11] T. W. Haynes, S. T. Hedetniemi and M. A. Henning (Eds.), *Topics in Domination in Graphs*, Dev. Math. vol 64. Springer, Cham. 2020.

[12] T. W. Haynes, S. T. Hedetniemi and M. A. Henning (Eds.), *Structures of Domination in Graphs*, Dev. Math. vol 66. Springer, Cham., 2021.

[13] M. A. Henning and S. N. Jogan, A characterization of graphs with given total coalition numbers, *Discrete Appl. Math.* 358 (2024), 395–403.

[14] M. A. Henning and D. A. Mojdeh, Double Coalitions in Graphs, *Bull. Malays. Math. Sci. Soc.* 48, 51 (2025); `https://doi.org/10.1007/s40840-025-01831-7`.

[15] D. A. Mojdeh and I. Masoumi, Edge coalitions in graphs, Available at: `https://arxiv.org/abs/2302.10926`.

[16] M. R. Samadzadeh and D. A. Mojdeh, Independent coalition in graphs: existence and characterization, *Ars Math. Contemp.* 24 (2024), #P3.09.

[17] B. Zelinka, Domination in the generalized Petersen graphs, *Czech. Math. J.* 52 (127) (2002), 11–16.

[18] B. Zelinka, Domatic number and degrees of vertices of a graph, *Math. Slovaca* 33 (1983), 145–147.

[19] B. Zelinka, On domatic numbers of graphs, i *Math. Slovaca* 31 (1981), 91–95. ‘

(Received 12 Oct 2024; revised 15 Apr 2025, 26 May 2025)