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On the star-critical Ramsey number of forests versus near-complete graphs

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Abstract

For given graphs G_1, G_2 and G, by $G \to (G_1, G_2)$ we mean that if the edges of G are arbitrarily colored by red and blue, then there is either a red monochromatic copy of G_1 or a blue monochromatic copy of G_2 in G. The Ramsey number $R(G_1, G_2)$ is defined as the smallest positive integer n such that $K_n \to (G_1, G_2)$ and the star-critical Ramsey number $R_*(G_1, G_2)$ is defined as $\min\{\delta(G) : G \subseteq K_{R(G_1,G_2)},\ G \to (G_1, G_2)\}$. The size Ramsey number $\hat{R}(G_1, G_2)$ is defined as the minimum number of edges of a graph G such that $G \to (G_1, G_2)$. In this paper, the star-critical Ramsey number of a forest versus a near complete graph is computed exactly and a sharp bound is given for their size Ramsey numbers.

1 Introduction

In this paper, we are only concerned with undirected simple finite graphs and we follow [3] for terminology and notations that are not defined here. For a given graph G, we denote its *vertex set*, *edge set*, *maximum degree* and *minimum degree* of G by $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$, respectively. For a vertex $v \in V(G)$, we use $\deg(v)$ and $N(v)$ to denote the degree and the set of neighbors of v in G, respectively. Also, for given disjoint subsets A and B of $V(G)$, by $E[A, B]$ we mean the set of edges of the bipartite subgraph of G with partite sets A and B. In this paper, for a given graph G, we use $l(G)$ to denote the number of vertices of the largest component of G and we use $k_i(G)$ to denote the number of components of G with exactly i vertices. Moreover, the *variety* of a graph G, denoted by $q(G)$, is defined as $q(G) = |\{i : k_i(G) \neq 0\}|$, that is the number of components of G with different sizes and we set $C(G) = \{i : k_i(G) \neq 0\}$. A *clique* in a graph is a set of mutually

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adjacent vertices and the maximum size of a clique in a graph G is called the *clique number* of G. As usual, the star graph on $n+1$ vertices is denoted by $K_{1,n}$ and the complete graph on n vertices is denoted by K_n. We use $K_k(n_1,\ldots,n_k)$ to denote the complete k-partite graph in which the i-th part, $1 \leq i \leq k$, has n_i vertices.

For a given red/blue coloring of the edges of a graph G, we use G^r and G^b to denote the spanning subgraphs of G induced by the edges of colors red and blue, respectively. Let G and G_1, G_2 be given graphs. By $G \rightarrow (G_1, G_2)$ we mean if the edges of G are arbitrarily colored by red and blue, then either $G_1 \subseteq G^r$ or $G_2 \subseteq G^b$. If $G \nrightarrow (G_1, G_2)$, then we say that G has a (G_1, G_2)-free coloring. The *Ramsey number* $R(G_1, G_2)$ is defined as the smallest positive integer n such that $K_n \rightarrow (G_1, G_2)$. The existence of such a positive integer is guaranteed by Ramsey's classical result [11]. In recent years there have been several activities in the determination of Ramsey numbers of certain pairs of specific graphs [10]. One of the known results is due to Chavátal [3] who showed that $R(T, K_m) = (n-1)(m-1)+1$, where T is a tree of order n. Chavátal's theorem was extended by Stahl [12] who showed that if F is a forest, then

$$R(F, K_m) = \max_{j \in l(F)} \{(j-1)(n-2) + \sum_{i=j}^{l(F)} ik_i(F)\}.$$

Finally, the Ramsey number of any forest of order at least 3 versus any graph G of order n, $n \geq 4$, having clique number $n-1$ is computed exactly in [4].

For given graphs G_1 and G_2, instead of the complete graph $K_{R(G_1,G_2)}$, we are interested in subgraphs $H \subseteq K_{R(G_1,G_2)}$ such that $H \rightarrow (G_1, G_2)$. In this direction, we consider the *star-critical Ramsey number* as an extension of the Ramsey number which was first defined by Hook and Isaak in [8]. The *star-critical Ramsey number* $R_*(G_1, G_2)$ is defined as $\min\{\delta(H) : \ H \subseteq K_{R(G_1,G_2)},\ H \rightarrow (G_1, G_2)\}$. The *size Ramsey number* $\hat{R}(G_1, G_2)$ is defined as the minimum number of edges of a graph H such that $H \rightarrow (G_1, G_2)$. For results and related problems in this area we refer the reader to [7, 9, 13].

Let G be an n-vertex graph, $n \geq 4$, with clique number $n-1$ and let T be an arbitrary tree with at least three vertices. In this paper, all (T, G)-free colorings of $K_{R(T,G)-1}$ will be classified and consequently, the star-critical Ramsey number $R_*(T, G)$ will be computed exactly. This article further provides the exact value of the star-critical Ramsey number of every forest versus an n-vertex graph G, $n \geq 4$, with clique number $n-1$. In addition, a sharp bound is given for the size Ramsey number of forests versus near-complete graphs in terms of their star-critical Ramsey numbers.

2 Trees versus near-complete graphs

In this section, the star-critical Ramsey number and also the size Ramsey number of a tree versus a graph G of order $n \geq 4$ having clique number $n-1$ will be discussed. It is worth noticing that the Ramsey number of a tree versus $K_n - e$ is computed in

[4] as follows.

Theorem 2.1. *[4] If $n \geq 3$ and T_m is any tree of order $m \geq 3$, then*

$$R(T_m, K_n - e) = \begin{cases} m+1, & \text{if } n = 3,\ m \text{ is even and } T_m \text{ is a star,} \\ (m-1)(n-2)+1, & \text{otherwise.} \end{cases}$$

To determine the star-critical Ramsey number $R_*(T_m, K_n - e)$, for $m \geq 3$ and $n \geq 4$, we first characterize all $(T_m, K_n - e)$-free colorings of $K_{R(T_m,K_n-e)-1}$. Before that we give a result from [1], which will be used in the sequel.

Corollary 2.2. *[1] Any graph G with a connected component of minimum degree $\delta(G) \geq m-2$ and maximum degree $\Delta(G) \geq m-1$ contains every tree of order m.*

Theorem 2.3. *Let $m \geq 3$ and $n \geq 4$ be given and $r = R(T_m, K_n - e) = (m-1)(n-2)+1$. If c is an arbitrary $(T_m, K_n - e)$-free coloring of $F = K_{r-1}$ without a red copy of T_m and a blue copy of $K_n - e$, then the resulting graph must admit a red/blue coloring of F described as follows.*

$$\begin{aligned} F^r &\cong (n-2)K_{m-1}, \\ F^b &\cong K_{n-2}(m-1, m-1, \ldots, m-1). \end{aligned} \tag{1}$$

If $n = m = 4$ and $T_m = K_{1,3}$, the graph F could also have the following red/blue coloring.

$$F^r \cong C_6, \quad F^b \cong K_6 - C_6. \tag{2}$$

Proof. Assume that the statement of the theorem is not correct and suppose a counterexample exists. Therefore, there are some positive integers $m \geq 3$, $n \geq 4$ and a $(T_m, K_n - e)$-free coloring of F such that F^r or F^b are not isomorphic to the one's described in the theorem. Among all counterexamples, consider the one with the smallest possible value of n.

If there is a vertex $u \in V(F)$ such that $\deg_{F^b}(u) \geq R(T_m, K_{n-1} - e)$, then the subgraph of F induced by $N_{F^b}(u) \cup \{u\}$ contains a blue copy of $K_n - e$, a contradiction. Thus, for every vertex $u \in V(F)$, $\deg_{F^b}(u) < R(T_m, K_{n-1} - e)$ and so by Theorem 2.1, for each vertex $u \in V(F)$,

$$\deg_{F^r}(u) \geq \begin{cases} m-3, & \text{if } n = 4,\ m \text{ is even and } T_m \text{ is a star} \\ m-2, & \text{otherwise.} \end{cases}$$

First, let $n = 4$, m be even and $T_m \cong K_{1,m-1}$. In this case, for each vertex $u \in V(F)$, $\deg_{F^r}(u) \geq m-3$ and $\deg_{F^b}(u) \leq m$. Since c is a $(T_m, K_n - e)$-free coloring of F and $T_m \nsubseteq F^r$, then for each vertex $u \in V(F)$, $\deg_{F^r}(u) \in \{m-2, m-3\}$ and $\deg_{F^b}(u) \in \{m, m-1\}$. Let $z \in V(F)$ be a vertex of maximum degree in F^b and $\deg_{F^b}(z) = m - \epsilon$, where $\epsilon = 0$ or $\epsilon = 1$. We may assume that there are vertices $x, y \in N_{F^b}(z)$ such that xy is blue, otherwise since c is a $(T_m, K_n - e)$-free coloring of F, then $\epsilon = 1$, $E[N_{F^b}(z), N_{F^r}(z)] \subseteq E(F^b)$ and $F[N_{F^r}(z)] \subseteq F^r$. Therefore, $F^r \cong 2K_{m-1}$ and $F^b \cong K_2(m-1, m-1)$, as described in (1), a contradiction.

Thus, let $x, y \in N_{F^b}(z)$ and xy be blue. Now, if w is an arbitrary vertex in $N_{F^b}(u) \setminus \{x, y\}$, to avoid a blue copy of $K_4 - e$, the edge wx (and also wy) should be red. Since $\deg_{F^r}(x) \geq m-1$ and $\deg_{F^r}(x) \geq m-1$, then each of x and y has at least $m-3$ blue neighbors in $N_{F^r}(z)$ and since $K_4 - e \nsubseteq F^b$, all of the $2m-6$ neighbours of x and y in $N_{F^r}(z)$ are distinct. Since $|N_{F^r}(z)| = m + \epsilon - 3$ and x and y have at least $2m-6$ distinct neighbors in $N_{F^r}(z)$, then $2m - 6 \leq m + \epsilon - 3$, means that $m \leq 3 + \epsilon$. Since $m \geq 3$ is even, thus $\epsilon = 1$ and $m = 4$. Since z is a vertex with maximum degree in F^b and $\epsilon = 1$, then $\deg_{F^b}(v) = 3$, for every vertex $v \in V(F)$. In this case, one can easily see that $F^r \cong C_6$ and $F^b \cong K_6 - C_6$, as described in (1), a contradiction.

Now, let $n > 4$. In this case, for each vertex $u \in V(F)$, $\deg_{F^r}(u) \geq m-2$ and $\deg_{F^b} \leq (m-1)(n-3)$. If there is a vertex with degree at least $m-1$ in F^r, then by Corollary 2.2, F^r contains a copy of T_m, a contradiction. Hence, each vertex in F^r has degree $m-2$ and each vertex in F^b has degree $(m-1)(n-3)$. Let z be a vertex of F and H be the subgraph of F induced by $N_{F^b}(z)$. Since $(m-1)(n-3) = R(T_m, K_{n-1} - e) - 1$ and c induced a $(T_m, K_{n-1} - e)$-free coloring of H, the minimality of n implies that H has a red/blue coloring described in (1) or $n = 5$, $m = 4$, $T_m \cong K_{1,3}$ and H has the $(K_{1,3}, K_4 - e)$-free coloring described in (2). If H has a red/blue coloring described in (1), then

$$H^r = (n-3)K_{m-1} \quad \text{and} \quad H^b = K_{n-3}(m-1, m-1, \ldots, m-1).$$

Since every vertex in F^r has degree $m-2$, then all edges between $N_{F^r}(z)$ and $V(H)$ must be blue and so, all edges contained in $N_{F^r}(z)$ are colored red. Thus, we have the red/blue coloring of F described in (1), a contradiction. Note that if $n = 5$, $m = 4$, $T_m \cong K_{1,3}$ and the $(T_m, K_4 - e)$-free coloring induced on H is the ones described in (2), then considering three vertices in $N_{F^b}(z)$ forming a triangle together with the two vertices in $N_{F^r}(z)$, we have a blue copy of $K_5 - e$, contradicting the fact that c is a $(T_m, K_5 - e)$-free coloring of F. This contradiction shows that there is no counterexample to the theorem and for every $m \geq 3$ and $n \geq 4$, any $(T_m, K_n - e)$-free coloring of F is isomorphic to the ones described in (1) or (2). $\square$

Now, in the sequel we prove that if $r = R(T_m, K_n - e) = (m-1)(n-2) + 1$ and H is a subgraph of K_r such that $H \to (T_m, K_n - e)$, then $\delta(H) \geq (m-1)(n-3) + 1$. For this purpose, we prove a more general result by determining the exact value of the star-critical Ramsey number of any tree of order at least three versus $K_n - e$, for $n \geq 4$.

Lemma 2.4. *If $n \geq 4$ and T_m is any tree of order $m, m \geq 3$, then*

$$R_*(T_m, K_n - e) = (m-1)(n-3) + 1.$$

Proof. Let $r = R(T_m, K_n - e) = (m-1)(n-2) + 1$ and r_* be the claimed number for $R_*(T_m, K_n - e)$. To see r_* is a lower bound for $R_*(T_m, K_n - e)$, let H be a subgraph of K_r such that $H \to (T_m, K_n - e)$. We prove that $\delta(H) \geq r_*$. On the contrary, let v be the vertex of H with degree at most $r_* - 1$. Partition the vertices of K_{r-1} into sets $V_1, V_2, \ldots, V_{n-2}$ such that for every i, $1 \leq i \leq n-2$, $|V_i| = m-1$ and $N(v) \subseteq \bigcup_{i=2}^{n-2} V_i$. Color all edges contained in V_i, $1 \leq i \leq n-2$, by red and the rest by blue. Also, color all edges incident with v by blue. Since the clique number of the subgraph of H spanned by the blue edges is $n-2$, then H^b does not contain $K_n - e$ as a subgraph. Also, H^r does not contain T_m, because the largest component in H^r has size $(m-1)$. Thus, we have a $(T_m, K_n - e)$-free coloring of H, contradicting that $H \to (T_m, K_n - e)$.

For the upper bound, we construct a subgraph H of K_r such that $\delta(H) = r_*$ and $H \to (T_m, K_n - e)$. For this purpose, let H be a subgraph of K_r constructed from K_{r-1} by adding a vertex adjacent to exactly r_* vertices of K_{r-1}. We prove that $H \to (T_m, K_n - e)$. On the contrary, assume that $H \not\to (T_m, K_n - e)$ and consider a $(T_m, K_n - e)$-free coloring of H. This coloring induces a $(T_m, K_n - e)$-free coloring of $H \setminus \{v\} \cong K_{r-1}$ and so, by Theorem 2.3, this coloring is unique as described in the theorem. If we have the red/blue coloring described in (1), then the red graph is isomorphic to the $(n-2)$-partite graph $K_{n-2}(m-1, \ldots, m-1)$. Let $V_1, V_2, \ldots, V_{n-2}$ be the partite sets of the red subgraph of $H \setminus \{v\}$. Thus, all edges incident with v must be blue, otherwise, H^r contains a copy of T_m, a contradiction. Now, $\deg(v) = r_*$ and so by the Pigeonhole principle, v has at least one neighbor in each V_i, $1 \leq i \leq n-2$ and there are some j, $1 \leq j \leq n-2$, such that v has at least two neighbors in V_j, means that H^b contains a copy of $K_n - e$, a contradiction. Now, let us have the red/blue coloring described in (2), i.e. let $K_{r-1}^r = C_6$ and $K_{r-1}^b = K_6 - C_6$. By a similar argument, if there is a red edge between v and $H \setminus \{v\}$, we have a red copy of $K_{1,3}$. Thus, we may assume that all four edges between v and $H \setminus \{v\}$ are blue, which form a blue copy of $K_4 - e$, a contradiction. This contradiction shows that r_* is an upper bound for $R_*(T_m, K_n - e)$, completing the proof of lemma. □

Hook and Issak [8] determined the star-critical Ramsey number of a tree versus a complete graph and proved that $R_*(T_m, K_{n-1}) = (m-1)(n-3) + 1$. Since for every graph G of order n and having clique number $n-1$, we have $K_{n-1} \subset G \subset K_n - e$, then $R_*(T_m, K_{n-1}) \leq R_*(T_m, G) \leq R_*(T_m, K_n - e)$. Having applied Lemma 2.4, we obtain the following corollary.

Corollary 2.5. *If T_m is any tree of order $m \geq 3$ and G is any graph of order $n \geq 4$ having clique number $n-1$, then $R_*(T_m, G) = (m-1)(n-3) + 1$. In particular, if $H \to (T_m, G)$ and $|V(H)| = (m-1)(n-2) + 1$, then $\delta(H) \geq (m-1)(n-3) + 1$.*

3 Forests versus near-complete graphs

The aim of this section is to determine the exact value of the star-critical Ramsey number of a forest versus a near complete graph and finally provide a sharp bound for their size Ramsey numbers. For this purpose, we consider a more general case and the star-critical Ramsey number of a forest versus a near complete graph will be obtained consequently. Let G and H be given connected graphs. G is called *H-good* if $R(G,H) = (\chi(H)-1)(|V(G)|-1) + s(H)$, where $\chi(H)$ is the chromatic number of H and $s(H)$ is the chromatic surplus of H, i.e., the cardinality of a minimum color class taken over all proper colorings of H with $\chi(H)$ colors. Also, we say G is *H-star-good*, if G is H-good and $R_*(G,H) = (\chi(H)-2)(|V(G)|-1) + s(H)$. For example, it is proved that [8, 12] every tree is K_m-good and also K_m-star-good, for every $m \geq 2$. In [2], Bielak proved that for given graph G with $\chi(G) \geq 2$ and chromatic surplus $s(G)$, if H is a disjoint union of G-good graphs, then

$$R(H,G) = \max_{j \in C(H)} \left\{ (j-1)(\chi(G)-2) + \sum_{i=j}^{n(H)} ik_i(H) \right\} + s(G) - 1.$$

Also, the following simple proposition is proved by Bielak in [2].

Proposition 3.1. ([2]) *Let G be a graph with $\chi(G) \geq 2$ and chromatic surplus $s(G)$. If H is a G-good graph, then $|V(H)| \geq s(G)+1$.*

To determine the exact value of the star-critical Ramsey number of a forest versus a near complete graph, we start with the following theorem, generalizing the result of Bielak [2]. Hereafter, let $K_n \sqcup K_{1,k}$ be the graph obtained from K_n by adding a new vertex v adjacent to k vertices of K_n.

Theorem 3.2. *For given graph G with $\chi(G) = n \geq 3$ and chromatic surplus $s(G)$, let H be the disjoint union of components which are G-star-good. If j_0 is the smallest positive integer such that $\max_{j\in C(H)}\{(j-1)(n-2) + \sum_{i=j}^{l(H)} ik_i(H)\}$ happens, then,*

$$R_*(H,G) = (j_0-1)(n-3) + \sum_{i=j_0}^{l(H)} ik_i(H) + s(G) - 1.$$

Proof. Let $r = R(H,G) = (j_0-1)(n-2) + \sum_{i=j_0}^{l(H)} ik_i(H) + s(G) - 1$ and r_* be the claimed number for $R_*(H,G)$. During the proof, for simplicity, we use $\sum_j^l$ to denote $\sum_{i=j}^{l(H)} ik_i(H)$ and briefly, we use k_i to denote $k_i(H)$. For the lower bound, let $F = K_{r-1} \sqcup K_{1,r_*-1}$ and we show that $F \nrightarrow (H,G)$. Let v be the vertex of degree r_*-1 in F and let V_1 be the set of non-neighbors of v in F. Clearly, $|V_1| = j_0 - 1$. Now, let $V_2, \ldots, V_n$ be a partition of neighbors of v such that for every i, $2 \leq i \leq n-2$, $|V_i| = j_0 - 1$, $|V_{n-1}| = (\sum_{j_0}^l) - 1$ and $|V_n| = s(G) - 1$. Color all edges with both ends in V_i, $1 \leq i \leq n$, by red and the rest by blue. Clearly, F^r does not contain H, because in part V_{n-1}, some components of H of order j_0 or larger are missed and by Proposition 3.1, $s(G)-1 < j_0 - 1$ and thus, in other parts there is no red component of order j_0. On the other hand, if $s(G) = 1$, then $\chi(F^b) = n-1$ and if $s(G) > 1$, then $\chi(F^b) = n$, but the smallest color class of F^b contains $s(G)-1$ vertices. Therefore, $G \nsubseteq F^b$ and we have a (H,G)-free coloring of F, means that $F \nrightarrow (H,G)$.

For the upper bound, let $Q = K_{R(H,G)-1} \sqcup K_{1,r_*}$ and v be the vertex of degree r_* in Q. We will show that $Q \rightarrow (H,G)$. Let s denote the number of vertices of the smallest component of H, $H = \cup_{i=s}^{l(H)} k_i H_i$ and $t = \sum_{i=s}^{l(H)} k_i$ be the number of components of H. Now, on the contrary, assume that the statement of theorem is not correct and suppose that a counterexample exists. Therefore, for a given graph G, there are some graphs H containing disjoint union of G-star-good components such that $Q \nrightarrow (H, G)$. Among all counterexamples, let H be the one having the minimum t i.e., the minimum number of components. Since each component of H is G-star-good, it follows that $t \geq 2$. Set

$$Q_s = \begin{cases} H_s & q(H) = 1 \\ & \\ k_s H_s & q(H) \neq 1. \end{cases}$$

Let $H' = H - Q_s$ and j_0' be the smallest value of j that realizes $\max_{j \in C(H')}\{(j-1)(n-2) + \sum_j^l\}$. Since $C(H) \setminus C(H') = \{s\}$, we have $j_0 \leq j_0'$. Also, since H is the counterexample with minimum t, then H' could not be a counterexample and so $R_*(H', G) \leq r_*'$, where $r_*' = (j_0' - 1)(n-3) + \sum_{j_0'}^l + s(G) - 1$. Since $C(H') \subseteq C(H)$, then $(j_0 - 1)(n-2) + \sum_{j_0}^l \geq (j_0' - 1)(n-2) + \sum_{j_0'}^l$, and using $j_0 \leq j_0'$, we conclude that

$$\begin{aligned} r_* &= (j_0 - 1)(n-3) + s(G) - 1 + \textstyle\sum_{j_0}^l \\ &\geq (j_0' - 1)(n-2) + s(G) - 1 - (j_0 - 1) + \textstyle\sum_{j_0'}^l \\ &\geq (j_0' - 1)(n-2) + s(G) - 1 - (j_0' - 1) + \textstyle\sum_{j_0'}^l \\ &= (j_0' - 1)(n-3) + s(G) - 1 + \textstyle\sum_{j_0'}^l = r_*'. \end{aligned}$$

Therefore, $\deg(v) \geq r_* \geq r_*' \geq R_*(H', G)$ and since $H' \subseteq H$, then $R(H, G) \geq R(H', G)$. Set $r' = R(H', G)$ and choose $r' - 1$ vertices from $Q - \{v\} \simeq K_{r-1}$ containing r_*' vertices from $N(v)$ and let D be the subgraph of Q spanned by v and the chosen $r' - 1$ vertices. Clearly, $D \simeq K_{r'-1} \sqcup K_{1,r_*'} \subseteq Q$ and since $R_*(H', G) \leq r_*'$, then $D \rightarrow (H', G)$. As $D \subseteq Q$ and Q does not contain a blue monochromatic copy of G, we obtain that $H' \subseteq Q^r$. Discard the vertices of such a copy of H' from Q and let Q' be the resulting graph. In the sequel, we prove that there is a red monochromatic copy of Q_s in Q'. Since $Q_s \subset H$, then $R_*(Q_s, G) \leq (s-1)(m-3) + sp + s(G) - 1$, where $p = 1$ if $q(H) = 1$ and $p = k_s$, otherwise. Note that $|V(H')| = s(k_s - 1)$ if $q(H) = 1$ and $|V(H')| = \sum_{s_{H'}}^l$, otherwise. Now, we consider the following cases.

Case 1: $v \in V(H')$.

In this case, $Q' \subseteq K_{r-1}$ and $|V(Q')| = r - 1 - |V(H') \setminus \{v\}| \geq (s-1)(n-2) + sp + s(G) - 1$. Therefore, $|V(Q')| \geq (s-1)(n-2) + sp + s(G) - 1 \geq R(Q_s, G)$ and so, $Q' \rightarrow (Q_s, G)$, means that there is a monochromatic red copy of Q_s in Q'.

Case 2: $v \notin V(H')$.

In this case, $V(H') \subseteq K_{r-1}$ and $|V(Q') \setminus \{v\}| = |V(K_{r-1}) \setminus V(H')|$. Therefore,

$$\begin{aligned} |V(Q') \setminus \{v\}| &= r-1-|V(H')| \\ &= (j_0-1)(n-2)+\sum\nolimits_{j_0}^{l} + s(G)-2-|V(H')| \\ &\geq (s-1)(n-2)+\sum\nolimits_{s}^{l} + s(G)-2-|V(H')| \\ &\geq (s-1)(n-2)+sp+s(G)-2 = R(Q_s,G)-1. \end{aligned}$$

In addition,

$$\begin{aligned} \deg_{Q'}(v) \geq r_* - |V(H')| &= (j_0-1)(n-3)+\sum\nolimits_{j_0}^{l} + s(G)-1-|V(H')| \\ &\geq (s-1)(n-3)+\sum\nolimits_{s}^{l} + s(G)-1-|V(H')| \\ &= (s-1)(n-3)+sp+s(G)-1 \geq R_*(Q_s,G). \end{aligned}$$

Therefore, in this case $|V(Q') \setminus \{v\}| \geq R(Q_s,G)-1$ and $\deg_{Q'}(v) \geq R_*(Q_s,G)$ and thus, $Q' \to (Q_s,G)$. Since $Q' \subset Q$ and there is no blue copy of G in Q, then Q' contains a red monochromatic copy of Q_s.

Now, this red copy of Q_s from Q' with the deleted monochromatic red H', forms a monochromatic copy of H in Q^r, which means that $Q \to (H,G)$, a contradiction. This contradiction shows that such a counterexample does not exist and so the proof of theorem is completed. □

Since every tree is a K_n-star-good graph [5, 8], then for $n \geq 4$, the star-critical Ramsey number of a forest F containing trees of order at least three versus a complete graph will be obtained directly from Theorem 3.2. Also, combining Theorems 3.2 and 2.5, we have the following corollary.

Corollary 3.3. *Let F be a forest that is a disjoint union of trees of order at least 3 and let G be any graph of order $n \geq 4$ having clique number $n-1$. Also, let j_0 be the smallest value of j that realizes* $\max_{j \in C(F)}\{(j-1)(n-3)+\sum_{i=j}^{l(F)} ik_i(F)\}$. *Then,*

$$R_*(F,G) = (j_0-1)(n-4) + \sum_{i=j_0}^{l(F)} ik_i(F).$$

Let G and H be given graphs and $Q = K_{R(G,H)-1} \sqcup K_{1,R_*(G,H)}$. By the definition of the star-critical Ramsey number, $Q \to (G,H)$ and Q is a graph of size $\binom{R(F,G)-1}{2} + R_*(F,G)$. Therefore,

$$\hat{R}(F,G) \leq \binom{R(G,H)-1}{2} + R_*(G,H).$$

In particular, if $n \geq 4$ and F is a forest containing disjoint union of trees with order at least 3, then by Corollary 3.3, we conclude that

$$\hat{R}(F, K_n - e) \leq \binom{R(F, K_n - e)}{2} - (j_0 - 2). \tag{3}$$

If F is isomorphic to $2K_2$, P_3 or $K_{1,3}$, then by Eq. 3, $\hat{R}(P_3, K_4 - e) \leq 9$, $\hat{R}(K_{1,3}, K_4 - e) \leq 19$ and $\hat{R}(2K_2, K_4 - e) \leq 10$, respectively. On the other hand, Faudree and Sheehan in [6] proved that $\hat{R}(P_3, K_4 - e) = 9$, $\hat{R}(K_{1,3}, K_4 - e) = 19$ and $\hat{R}(2K_2, K_4 - e) = 10$, which means that the bound presented in (3) is best possible for these cases.

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