

Quasi-transitive mixed graphs and undirected squares of oriented graphs

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Abstract

By generalising the notion of quasi-transitive directed graphs to mixed graphs, we study the problem of classifying those graphs that arise as an undirected square of an oriented graph. We fully classify those graphs of maximum degree 3 and graphs of girth at least 4 that arise as an undirected square of an oriented graph. In contrast to the recognition problem for graphs that admit a quasi-transitive orientation, we find it is NP-complete to decide whether a graph admits a partial orientation as a quasi-transitive mixed graph. We prove the problem is Polynomial when restricted to inputs of maximum degree 3, but it remains NP-complete when restricted to inputs with maximum degree at least 5. Our proof further implies that for fixed $k \geq 3$, it is NP-complete to decide whether a graph arises as an undirected square of an orientation of a graph with $\Delta = k$.

1 Introduction and Background

A simple graph Γ is a *graph square* when there exists a graph Σ such that $\Gamma = \Sigma^2$, where Σ^2 is the graph formed from Σ by adding an edge between any pair of vertices at distance 2. We extend this notion to oriented graphs, that is, those graphs that arise from simple graphs by assigning each edge an orientation as an arc.

A simple graph Γ is an *oriented graph square* when there exists an oriented graph \vec{G} such that $\Gamma = U(\vec{G}^2)$, where $U(\cdot)$ denotes the simple graph underlying a mixed graph, and \vec{G}^2 is the mixed graph formed from the oriented graph \vec{G} by adding an edge between any pair of vertices at directed distance 2. Similar to the problem of finding square roots of graphs, given Γ it is not obvious how one may recover \vec{G} , nor whether \vec{G} is unique. Further, given an arbitrary graph Σ it is not clear how one can determine whether Σ is an oriented graph square. We consider these problems herein.

To study oriented graph squares, we reframe the problem as a graph orientation problem. Recall a *comparability graph* is a graph whose edges correspond to comparable pairs in a partial order. We say an oriented graph is *quasi-transitive* when it contains no induced directed path of length 2 (i.e., no 2-dipath).

Theorem 1.1. [12] *A graph Γ admits a quasi-transitive orientation if and only if Γ is a comparability graph.*

By definition, if \vec{G} is a quasi-transitive oriented graph, then $\vec{G}^2 = \vec{G}$ and so $U(\vec{G}) = U(\vec{G}^2)$. This implies that for every comparability graph Γ there exists an oriented graph \vec{G} such that $\Gamma = U(\vec{G}^2)$. However, there are non-comparability graphs for which there exists an oriented graph \vec{G} such that $\Gamma = U(\vec{G}^2)$. Consider the graphs given in Figure 1.

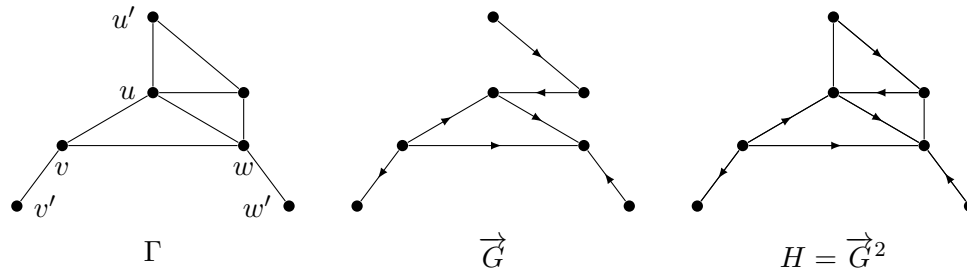


Figure 1: The oriented graph square Γ resulting from the oriented graph \vec{G} . H is the corresponding mixed graph.

The subgraph of Γ induced by $\{u, v, w, u', v', w'\}$ is a forbidden subgraph for the family of comparability graphs [11]. Therefore Γ does not admit an orientation as a quasi-transitive oriented graph. However, $\Gamma = U(\vec{G}^2)$ is an oriented graph square.

We generalise the notion of quasi-transitive oriented graphs to mixed graphs. Let H be a mixed graph. We say H is *quasi-transitive* when

- (1) H has no induced 2-dipath; and
- (2) for every edge uv in H there exists $w \in V(H)$ such that uwv or vwu is a 2-dipath.

Looking back at our example in Figure 1, we see that H is a quasi-transitive mixed graph.

Let H be a quasi-transitive mixed graph and let $\Gamma = U(H)$. Let \vec{G} be the oriented graph formed from H by removing all its edges. By the definition of quasi-transitive mixed graph we have $H = \vec{G}^2$ and thus $\Gamma = U(\vec{G}^2)$. And so we arrive at the following observation.

Observation 1.2. *Let Γ be a graph. There exists an oriented graph \vec{G} such that $\Gamma = U(\vec{G}^2)$ if and only if Γ admits a partial orientation as a quasi-transitive mixed graph.*

Observation 1.2 allows us to treat equivalently graphs that admit a partial orientation as a quasi-transitive mixed graph and graphs that arise as an undirected square of an oriented graph. Throughout the remainder of this work we treat these two concepts interchangeably, opting for the one which eases the path of the reader through a particular argument or which allows a direct comparison to previous work.

Quasi-transitive directed graphs were first studied by Ghouilà-Houri [12] and Gallai [8] in the 1960s. Bang-Jensen and Huang proved that quasi-transitive digraphs have similar structural properties to tournaments [4]. Continued interest in the study of quasi-transitive directed graphs follows from their connection to a variety of other concepts in the study of directed graphs such as local tournaments [14] and semi-complete digraphs [2]. One can decide in polynomial time whether a graph admits a quasi-transitive orientation [10] and whether a partial orientation of a graph can be completed to be a quasi-transitive orientation [3].

The study of graph squares and related concepts are a mainstay of research in structural graph theory. Graph squares were fully classified in 1967 by Mukhopadhyay [16]. Classifying the complexity of the recognition problem for graph squares remained open until 1994 when Motwani and Sudan proved the problem is NP-complete [17]. There is a full characterisation of tree squares (that is, graphs that arise as a square of a tree), which leads to a polynomial time algorithm for their recognition [15, 19]. More recently Farzad et al. presented a full dichotomy theorem for the problem of deciding whether a graph is a graph square of a graph with fixed girth. It is NP-complete to decide whether a graph is a graph square of graph with fixed girth at most 5 and Polynomial to decide whether a graph is a graph square of a graph with fixed girth $g \geq 6$ [1, 7]. Despite the abundance of research into graph squares, relatively little work has been devoted to the study of squares of directed graphs. In [13] Geller gives a classification of digraph squares from which Mukhopadhyay's result follows as a corollary.

The remainder of this work proceeds as follows. In the subsequent section we develop necessary lemmas and preliminary results required for the main results in the subsequent sections. In doing so we immediately arrive at a classification of oriented graph squares of girth 4. In Section 3 we classify graphs with maximum degree 3 that admit a partial orientation as a quasi-transitive mixed graph and thus classify the family of oriented graph squares of maximum degree 3. In Section 4 we study the computational complexity of the recognition problem for undirected squares and graphs admitting a partial orientation as a quasi-transitive mixed graph. We prove the problem of deciding whether a graph is an oriented graph square is NP-complete. Finally in Section 5 we provide additional discussion that relates the problem under study herein to the study of the oriented chromatic number and highlights potential areas of future study.

At times throughout we will be interested in the graph formed from a subset of

the edges of a graph. Let Γ be a graph and let $X \subseteq E(\Gamma)$. The graph *formed from* X is the graph whose vertex set is those vertices that appear as an endpoint of an edge in X and whose edge set is X . Acknowledging the slight abuse of notation, we refer to this graph as $\Gamma[X]$. We extend this convention to subsets of arcs and edges in mixed and oriented graphs.

Throughout this work we will be interested in various properties and parameters of graphs, mixed graphs and oriented graphs. For ease of communication we may refer to the undirected property or parameter of a mixed or oriented graph to mean that of its underlying simple graph. For example, when we say that a vertex has degree k in \vec{G} we mean it has degree k in $U(\vec{G})$. To avoid needing to consider edge cases in which a mixed graph has no edges, we consider an oriented graph to be a partial orientation of a graph, that is, a mixed graph. We use $E(\cdot)$ and $A(\cdot)$ to respectively refer to the edge set and the arc set of a mixed graph. And so we may consider an oriented graph as a mixed graph H with $E(H) = \emptyset$. We say that a vertex v is a source (respectively, a sink) in a mixed graph H when v is a source (respectively sink) in $H[A(H)]$. For other graph theoretic notation not defined herein we refer the reader to [6].

2 Preliminaries

The classification of quasi-transitive oriented graphs as arising from comparability graphs gives a forbidden subgraph classification for the family of graphs that admit such an orientation [11]. We begin by proving no such classification exists for quasi-transitive mixed graphs.

Proposition 2.1. *For every graph Γ there exists a graph Γ' such that Γ' admits a partial orientation as a quasi-transitive mixed graph and Γ is an ‘induced subgraph of Γ' ’.*

Proof. Let Γ be a graph. Let \vec{G} be the oriented graph formed from Γ by arbitrarily orienting each edge into an arc, and then bisecting each arc to be a 2-dipath. Let $\Gamma' = U(\vec{G}^2)$. By construction, $\Gamma'[V(\Gamma)] = \Gamma$. By Observation 1.2, Γ' admits a partial orientation as a quasi-transitive mixed graph. \square

Though in general no forbidden subgraph characterisation exists, we do find such characterisations when we consider various restrictions on Γ . Below we prove such a characterisation exists when Γ has girth at least 4. In the subsequent section we prove such a characterisation exists when Γ has maximum degree 3.

Throughout the remainder of this work we apply the following structural lemma.

Lemma 2.2. *Let Γ be an oriented graph square. Let $e \in E(\Gamma)$ be contained in no copy of K_3 in Γ . In every partial orientation G of Γ as a mixed quasi-transitive graph,*

- (1) *e is oriented as an arc;*

- (2) *the ends of e are respectively a source and a sink; and*
- (3) *the graph formed from the set of edges contained in no copy of K_3 in Γ contains no odd cycle.*

Proof. Let Γ be a graph arising as the square of an oriented graph. Let G be a partial orientation of Γ as a quasi-transitive mixed graph. Let X be the set of edges not contained in a copy of K_3 in Γ . Consider $e \in X$ with $e = xy$. Since there is no vertex z such that $zx, yz \in E(\Gamma)$, there is no 2-dipath in G with ends at x and y . Therefore e is oriented, in some direction, as an arc in G .

Consider a pair of distinct edges $e_1, e_2 \in X$ with a common end-point. Let $e_1 = uv$ and $e_2 = vw$. By construction of X , $wu \notin E(\Gamma)$. Since G is a quasi-transitive mixed graph, uvw is not an induced 2-dipath in G . By construction, uvw is not an induced 2-dipath in $G[X]$.

Therefore $G[X]$ contains no induced 2-dipath. And so $G[X]$ is a quasi-transitive oriented graph. Therefore $U(G[X])$ is a comparability graph. Recall that every comparability graph is perfect (see [9]) By construction, $U(G[X])$ contains no copy of K_3 . Since $U(G[X])$ is perfect and contains no copy of K_3 , necessarily $\chi(\Gamma) \leq 2$. And so it follows $U(G[X])$ is bipartite. \square

Using this lemma we fully classify those graphs with girth at least 4 that admit a partial orientation as a quasi-transitive mixed graph.

Theorem 2.3. *Let Γ be a graph with girth $g \geq 4$. The graph Γ admits a partial orientation as a quasi-transitive mixed graph if and only if Γ has no odd cycle.*

Proof. Let Γ be a graph with girth $g \geq 4$. If Γ admits a partial orientation as a quasi-transitive mixed graph, then by Lemma 2.2, Γ is bipartite and thus has no odd cycle.

Let Γ be bipartite with partition $\{V_1, V_2\}$. We find a partial orientation of Γ as a quasi-transitive mixed graph G by orienting all arcs to have their tail in V_1 and their head in V_2 . In such an orientation every vertex is either a source vertex or a sink vertex. The mixed graph G has no induced 2-dipath and no edges. Therefore G is a quasi-transitive mixed graph. \square

Let Γ be a graph that admits a partial orientation as a quasi-transitive mixed graph and consider the existence of a cut vertex v . Let $\{V_1, V_2\}$ be a partition of $V(\Gamma) \setminus \{v\}$ such that v has a neighbour in both V_1 and V_2 and no vertex of V_1 is adjacent to a vertex of V_2 . Let G be a partial orientation of Γ as a quasi-transitive mixed graph.

By construction, v cannot be the centre of a 2-dipath with an end in V_1 and an end in V_2 . Furthermore, if v is the centre of a 2-dipath with both ends in V_1 , then no edge between v and a vertex of V_2 can be oriented as an arc in G . However this cannot be; any vertex in a quasi-transitive mixed graph that is incident with an edge must also be an end-point of a 2-dipath whose end is at the other end of the edge.

And so v cannot be the centre of a 2-dipath with both ends in V_1 . Therefore v is a source or a sink in G .

We extend this argument to vertex cuts consisting of independent sets. Let Γ be a graph and let $I \subset V(\Gamma)$ be an independent set. We say I is an *independent vertex cut* when there exists a partition $\{V_1, V_2, I\}$ of $V(\Gamma)$ such that $\Gamma - I$ is not connected, every vertex in I is adjacent to at least one vertex in V_1 and one in V_2 and no vertex of V_1 is adjacent to a vertex of V_2 . We can extend partial orientations of $\Gamma[V_1 \cup I]$ and $\Gamma[V_2 \cup I]$ as quasi-transitive mixed graphs to one to Γ provided each vertex in the independent set is respectively a source or a sink in both the partial orientation of $\Gamma[V_1 \cup I]$ and of $\Gamma[V_2 \cup I]$.

Lemma 2.4 (The Vertex Cut Lemma). *Let Γ be a graph and let $I \subset V(\Gamma)$ be an independent vertex cut with $I = \{v_1, v_2, \dots, v_k\}$. The graph Γ admits a partial orientation as a quasi-transitive mixed graph if and only if there exists partial orientations of $\Gamma[V_1 \cup I]$ and $\Gamma[V_2 \cup I]$ as quasi-transitive mixed graphs G_1 and G_2 such that for each $1 \leq i \leq k$, v_i is a source (a sink) in both G_1 and G_2 .*

Proof. Let Γ be a graph and let $I \subset V(\Gamma)$ be an independent vertex cut with $I = \{v_1, v_2, \dots, v_k\}$.

Let G be a partial orientation of Γ as a quasi-transitive mixed graph. Consider $v \in I$. We claim v is a source or a sink. We prove v is not the centre vertex of a 2-dipath in G . Let u and u' be incident with v and such that u and u' are, without loss of generality, contained in V_1 .

By hypothesis, there exists $w \in V_2$ such that w is adjacent to v . Further we may assume vw is an arc (in some direction) in G , as otherwise there exists a 2-dipath $ww'v$ such that w' is in the same component as w in $G - v_i$. In this case we may swap the roles of w and w' .

If uvu' is a 2-dipath (in some direction), then one of u and u' must be adjacent to w , which contradicts our choice of V_1 and V_2 . Therefore v is not the centre of a 2-dipath whose ends are in V_1 . Therefore v is a source or a sink vertex in $G[V_1 \cup I]$.

By choice of v , it then follows that for all $v \in I$, vertex v is a source or a sink vertex in $G[V_1 \cup I]$. Therefore there is no 2-dipath in G with an end in V_1 and an end in V_2 . And so $G[V_1 \cup I]$ is partial orientation of $\Gamma[V_1 \cup I]$ as a quasi-transitive mixed graph. Similarly, $G[V_2 \cup I]$ is partial orientation of $\Gamma[V_2 \cup I]$ as a quasi-transitive mixed graph.

Assume now that there exist partial orientations of $\Gamma[V_1 \cup I]$ and $\Gamma[V_2 \cup I]$ as quasi-transitive mixed graphs G_1 and G_2 such that for each $1 \leq i \leq k$, v_i is a source in both G_1 and G_2 . Let G be the mixed graph formed from G_1 and G_2 by identifying corresponding elements of I in G_1 and G_2 . Since all identified vertices are source vertices, this process neither creates nor removes 2-dipaths. Similarly, this process neither creates nor removes edges. Since this construction preserves the existence of all 2-dipaths and edges, G is a partial orientation of Γ as a quasi-transitive mixed graph. \square

Corollary 2.5. *Let Γ be a graph arising as an oriented graph square. If $v \in V(\Gamma)$ is a cut vertex, then v is a source or a sink in every oriented graph \vec{G} such that $\Gamma = U(\vec{G}^2)$.*

3 Quasi-Transitive Mixed Graphs with Maximum Degree 3

We turn now to the problem of classifying graphs with maximum degree 3 that admit a partial orientation as a quasi-transitive mixed graph. We begin by identifying a reduction that can be applied to graphs with maximum degree 3 that preserves the property of (not) admitting such a partial orientation.

Let Γ be a graph with maximum degree 3 and let $u \in V(\Gamma)$. We say u is *removable* when it has degree 2 and its neighbours are adjacent, have degree 3, and have only one common neighbour (see Figure 2).

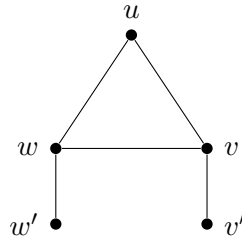


Figure 2: A removable vertex, u , in a graph with maximum degree 3

Lemma 3.1. *Let Γ be a graph with maximum degree 3 and let u be a removable vertex of Γ . The graph Γ admits a partial orientation as a quasi-transitive mixed graph if and only if $\Gamma - u$ admits a partial orientation as a quasi-transitive mixed graph.*

Proof. Let Γ be a graph with maximum degree 3 and let u be a removable vertex in Γ with neighbours v and w . Let v' and w' respectively be the other neighbour of v and w . Since Γ has maximum degree 3, edges ww' and vv' are contained in no copy of K_3 .

Let G be a partial orientation of Γ as a quasi-transitive mixed graph. By Lemma 2.2, v and w are each a source or a sink. Therefore neither v nor w is the centre vertex of a 2-dipath. Since v is not the centre of a 2-dipath, the edge uw is oriented in some direction in G . Similarly, the edge uv is oriented in some direction in G .

If both of these arcs are oriented to have their head at u , then both v and w are source vertices. However, in this case notice that the edge vw must be an edge in G , but there is no 2-dipath between v and w .

Therefore, without loss of generality, vuw is a 2-dipath in G , which then implies v is a source and w is a sink. If vw is an arc in G , then it must have its head at w . Furthermore, if vw is an edge in G it can be replaced by the arc vw . And so we may

assume vw is an arc in G . Removing u from G yields a partial orientation of $\Gamma - u$ as quasi-transitive mixed graph.

Let H be a partial orientation of $\Gamma - u$ as a quasi-transitive mixed graph. Since none of vv' , vw or ww' is contained in a copy of K_3 in $\Gamma - u$, then, by Lemma 2.2, each of these edges is oriented as an arc in H . The path $v'vw w'$ must be oriented to have no induced 2-dipath. Thus it is oriented as an alternating path. Without loss of generality, we assume $v'v, wv, ww' \in A(H)$. We extend the partial orientation of $\Gamma - u$ to one of Γ by orienting the edge uv to have its head at v and the edge uw to have its head at u . \square

This lemma implies that removing a removable vertex from a graph does not affect whether it admits a partial orientation as a quasi-transitive oriented graph. The definition of removable implies directly that removing a removable vertex does not affect the set of removable vertices.

Lemma 3.2. *Let Γ be a graph with maximum degree 3 and let R be the set of removable vertices Γ . For every $u \in R$ the set of removable vertices in $\Gamma - u$ is $R \setminus u$.*

Together Lemmas 3.1 and 3.2 imply that removing the set of removable vertices of Γ produces a graph with no removable vertices, Γ^R , with the property that Γ admits a partial orientation as a quasi-transitive mixed graph if and only if Γ^R admits a partial orientation as a quasi-transitive mixed graph.

Theorem 3.3. *Let Γ be a graph with maximum degree 3. Let R be the set of removable vertices of Γ . The graph Γ admits a partial orientation as a quasi-transitive mixed graph if and only if $\Gamma - R$ admits a partial orientation as a quasi-transitive mixed graph.*

Proof. The result follows directly by induction on $|R|$ by applying Lemmas 3.1 and 3.2. \square

We say a graph Γ with maximum degree 3 is *reduced* when it contains no removable vertices. We classify those reduced graphs with maximum degree 3 that arise as an oriented square by way of forbidden subgraphs.

Let Π be the split graph on six vertices consisting of a copy of K_3 with a pendant edge on each vertex. Recall Π is a forbidden subgraph for the family of graphs that admit a quasi-transitive orientation [11].

Lemma 3.4. *Let Γ be a graph with maximum degree 3. If Π is a subgraph of Γ , then Γ does not admit a partial orientation as a quasi-transitive mixed graph.*

Proof. Let Γ be a graph with maximum degree 3 containing Π , as labelled in Figure 3. We proceed by contradiction. Let G be a partial orientation of Γ as a quasi-transitive mixed graph.

Since Γ has maximum degree 3, none of uu' , vv' and ww' is contained in a copy of K_3 in Γ . And so by Lemma 2.2, each is oriented as an arc in Γ and further each of u, v and w is a source or a sink.

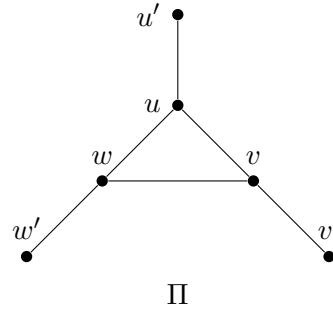


Figure 3: a forbidden subgraph for quasi-transitive mixed partial orientability of graphs with maximum degree 3

Without loss of generality, assume u and w are source vertices. Therefore wu is an edge in G . Since Γ has degree 3, the only copy of K_3 in Γ that contains wu is the one with vertex set $\{u, v, w\}$. Therefore uvw is a 2-dipath (in some direction). Hence v is not a source nor a sink, a contradiction. \square

Lemma 3.4 restricts induced subgraphs containing copies of K_3 in reduced graphs with maximum degree 3 that admit a partial orientation as a quasi-transitive mixed graph. The subsequent lemma characterises these subgraphs as those appearing in Figure 4.

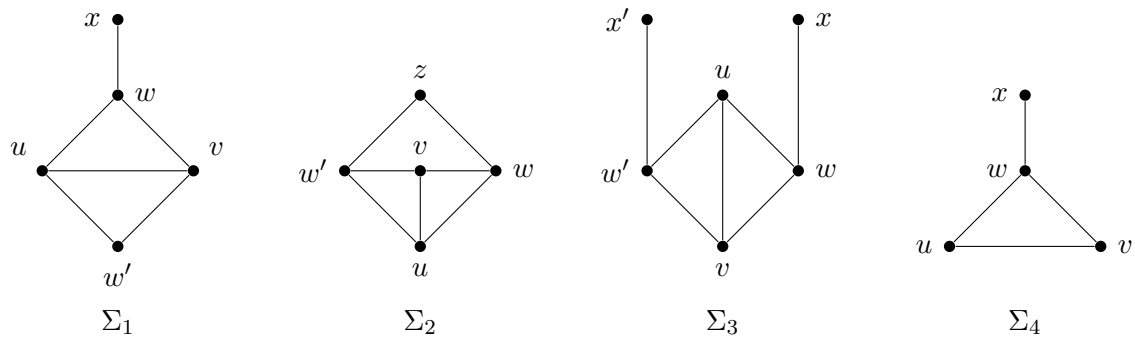


Figure 4: Configurations of K_3 in graphs with maximum degree 3 that admit a partial orientation as a quasi-transitive mixed graph.

Lemma 3.5. *Let Γ be a reduced graph with maximum degree 3 such that $\Gamma \neq K_3, K_4, K_4 - e$. Consider a copy of K_3 in Γ with vertices u, v and w . If Γ is Π -free, then u, v and w are contained in an induced subgraph as shown in Figure 4.*

Proof. We proceed based on the degree of each of u, v and w in Γ .

If u and v have degree 2, then since $\Gamma \neq K_3$, w must have degree 3. And so u, v and w are configured as in Σ_4 .

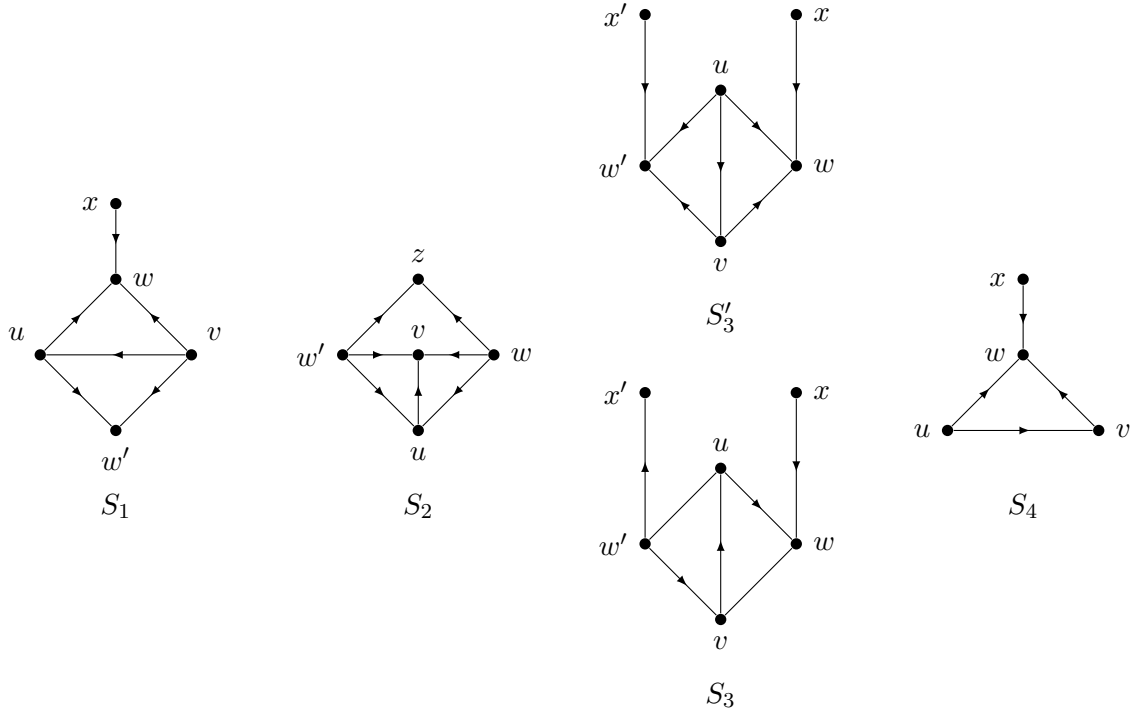


Figure 5: Partial orientations of graphs in Figure 4 as quasi-transitive mixed graphs.

Consider now the case where each of u, v have degree 3. Since Γ is reduced and Π -free, u and v have a second common neighbour, $w' \neq w$. Since $\Gamma \neq K_4$, w is not adjacent to w' .

- If $\deg(w) = 3$ and $\deg(w') = 2$, then u, v and w are configured as in Σ_1 .
- If $\deg(w) = \deg(w') = 3$ and w and w' have a common neighbour other than u and v , then u, v and w are configured as in Σ_2 .
- If $\deg(w) = \deg(w') = 3$ and w and w' have no common neighbour other than u and v then u, v and w are configured as in Σ_3 .

One observes that the case $\deg(w) = 2$ and $\deg(w') = 3$ is symmetric with the case $\deg(w) = 3$ and $\deg(w') = 2$ by swapping the label of w and w' . \square

If $\Sigma \in \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$ is an induced subgraph of Γ , then Γ contains an independent vertex cut. Thus in determining if a graph containing a copy of Σ_i admits a partial orientation as a quasi-transitive mixed graph we may appeal to the Vertex Cut Lemma. With this strategy in mind, we observe in Figure 5 a partial orientation as a quasi-transitive mixed graph for each of the graphs in Figure 4.

Theorem 3.6. *Let Γ be a reduced graph with maximum degree 3. The graph Γ admits a partial orientation as a quasi-transitive mixed graph if and only if Γ contains no copy of Π and the subgraph formed by the set of edges of Γ that are not contained in a copy of K_3 contains no odd cycle.*

Proof. Let Γ be a reduced graph with maximum degree 3.

Assume Γ admits a partial orientation as a quasi-transitive mixed graph. By Lemma 3.4, Γ is Π -free. By Lemma 2.2, the set of edges of Γ that are not contained in a copy of K_3 is bipartite.

To prove the converse we proceed by disproving the existence of a minimum counter example. Let Γ be a reduced graph with maximum degree 3 that is Π -free and such that the subgraph formed by the set of edges of Γ that are not contained in a copy of K_3 is bipartite and Γ does not admit a partial orientation as a quasi-transitive mixed graph. Among all choices for Γ , choose one with the fewest number of vertices.

Notice that each of K_3, K_4 and $\Gamma = K_4 - e$ are comparability graphs, and thus admit a partial orientation as a quasi-transitive mixed graph. Therefore $\Gamma \neq K_3, K_4, K_4 - e$.

Let Y be the set of edges contained in no copy of K_3 in Γ . Notice $Y \neq \emptyset$, as otherwise, by Theorem 2.3, Γ admits a partial orientation as a quasi-transitive mixed graph. Since $Y \neq \emptyset$, by Lemma 3.5, Γ contains one of $\Sigma_1, \Sigma_2, \Sigma_3$ or Σ_4 as an induced subgraph.

Let Σ' be an induced subgraph of Γ such that $\Sigma' \in \{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$. Consider the graph $\Gamma - X$ where

- if $\Sigma' = \Sigma_1$, then $X = \{u, v, w'\}$;
- if $\Sigma' = \Sigma_2$, then $X = \{u, v, w, w'\}$; and
- if $\Sigma' = \Sigma_3, \Sigma_4$, then $X = \{u, v\}$.

By minimality of Γ and Lemma 3.2, $\Gamma - X$ admits a partial orientation G as a quasi-transitive mixed graph.

We extend this partial orientation of $\Gamma - X$ to one of Γ as follows:

- If $\Sigma' = \Sigma_1$ and $xw \in A(G)$, then orient the edges $uw, wv, vw', w'u$ as in S_1 in Figure 5. If $wx \in A(G)$, then orient the edges as in the converse of S_1 in Figure 5.
- If $\Sigma' = \Sigma_2$ and z is the head of an arc in $\Gamma - X$, then orient the edges $uw, wz, zw', w'u, uv, vw$ and vw' as in S_2 in Figure 5.
- If $\Sigma' = \Sigma_2$ and z is not the head of an arc in $\Gamma - X$, then orient the edges $uw, wz, zw', w'u, uv, vw$ and vw' as in the converse of S_2 in Figure 5.
- If $\Sigma' = \Sigma_3$ and $w'x', xw \in A(G)$, then orient the edges $w'v, vu$ and uv as in S_3 in Figure 5. If $x'w', wx \in A(G)$, then orient the edges as in the converse of S_3 in Figure 5.
- If $\Sigma' = \Sigma_3$ and $x'w', xw \in A(G)$, then orient the edges $w'v, vw, wu, uw$ and uv as in S'_3 in Figure 5. If $w'x', wx \in A(G)$, then orient the edges as in the converse of S_3 in Figure 5.
- If $\Sigma' = \Sigma_4$ and $xw \in A(G)$, then orient the edges uv, vw and wu as in S_4 in Figure 5. If $wx \in A(G)$, then orient the edges as in the converse of S_4 in Figure 5.

This construction contradicts the fact that Γ admits no partial orientation as a quasi-transitive mixed graph, which completes the proof. \square

By noticing that a cubic graph is necessarily reduced, we arrive at the following classification of cubic graphs that admit a partial orientation as a quasi-transitive mixed graph

Corollary 3.7. *A cubic graph Γ admits a partial orientation as a quasi-transitive mixed graph if and only if Γ is Π -free and the subgraph of Γ formed by the set of edges of Γ that are not contained in a copy of K_3 contains no odd cycle.*

As the collection of edges that are contained in no copy of K_3 and removable vertices can be found in polynomial time, Theorem 3.6 implies that it is Polynomial to decide whether a graph with maximum degree 3 is an oriented graph square.

4 Decision Problems and Computational Complexity

Recall that a boolean formula is monotone when it has no negated literals. We prove the problem of deciding whether a graph is an oriented graph square is NP-complete via reduction from the problem of monotone not-all-equal 3-satisfiability.

Monotone NAE3SAT

Instance: A monotone boolean formula $Y = (L, C)$ in conjunctive normal form with three literals in each clause

Question: Does there exist a not-all-equal satisfying assignment for the elements of L ?

Theorem 4.1. [20] *Monotone NAE3SAT is NP-complete.*

MIXEDQT

Instance: A graph Γ

Question Does Γ admit a partial orientation as a mixed quasi-transitive graph?

UNDIRSQUARE

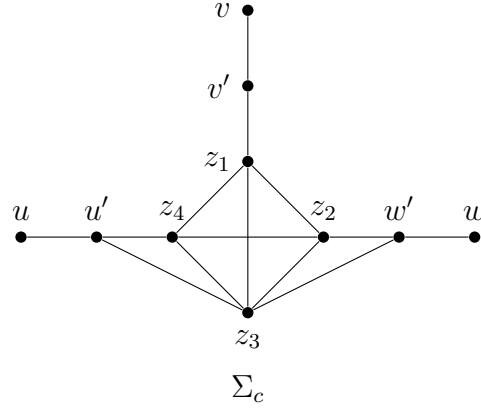
Instance: A graph Γ

Question Does Γ arise as an undirected square of an oriented graph?

Our proof proceeds by constructing a gadget graph for each clause such that the truth value of a literal in the clause corresponds to whether a specified vertex is a source or a sink in the gadget graph.

Let $Y = (L, C)$ be an instance of Monotone NAE3SAT. Let $c \in C$ and let Σ_c be the graph shown in Figure 6. We refer to Σ_c as a *clause graph*.

Let S_c be a partial orientation of Σ_c in which the edges uu' , vv' , ww' are all oriented as arcs in some direction. We associate with S_c a vector $\sigma_c = (\sigma_{u,c}, \sigma_{v,c}, \sigma_{w,c}) \in \{-, +\}^3$ where $\sigma_{u,c}$ (respectively $\sigma_{v,c}, \sigma_{w,c}$) is $+$ when u (respectively v, w) is a source in S_c and $-$ when u (respectively v, w) is a sink in S_c . We call σ_c the *signature* of S_c . See Figure 7 for an example. We will use the signature of a partial orientation

Figure 6: The clause graph, Σ_c , for $c = u \vee v \vee w$

of a clause graph to denote the truth values of the literals in the clause graph. The literal u (respectively v and w) will be TRUE in C if and only if $\sigma_{u,c}$ (respectively, $\sigma_{v,c}$ and $\sigma_{w,c}$) is $+$.

Previewing our approach to an NP-completeness proof, we take a moment to justify collapsing discussion of not-all-equal cases to one where we may restrict our consideration to satisfying assignments of a YES instance of not-all-equal satisfiability where we may assume without loss of generality for $c = u \vee v \vee w$ we have that u and v are FALSE and w is TRUE, or u and w are FALSE and v is TRUE.

Consider the two partial orientations given in Figure 7. Reversing the orientation of each of the arcs yields partial orientations with $\sigma_c = (+, +, -)$ and $\sigma_c = (+, -, +)$. Moreover, notice the existence of an automorphism of Σ_c that swaps u and w . Applying this automorphism to the partial orientation on the left permits us to generate partial orientations with $\sigma_c = (-, +, +)$ and $\sigma_c = (+, -, -)$. Thus there are partial orientations of Σ_c as a quasi-transitive mixed graphs for each signature in $\{-, +\}^3 \setminus \{(-, -, -), (+, +, +)\}$. The signature of a clause graph for $c = u \vee v \vee w$ will give the truth value of the literals u , v and w in c .

Lemma 4.2. *There exists a partial orientation S_c of Σ_c as a quasi-transitive mixed graph if and only if $\sigma_c \in \{-, +\}^3 \setminus \{(-, -, -), (+, +, +)\}$*

Proof. Consider $\sigma_c \in \{-, +\}^3 \setminus \{(-, -, -), (+, +, +)\}$. Figure 7 gives a partial orientation of S_c of Σ_c as a quasi-transitive mixed graph for $\sigma_c \in \{(-, -, +), (-, +, -)\}$. The other cases are obtained by applying the automorphism of Σ_c that swaps u and w and/or reversing the orientation of every arc.

To complete the proof it suffices to prove there is no partial orientation of Σ_c as a quasi-transitive mixed graph with $\sigma_c = (-, -, -)$ or $\sigma_c = (+, +, +)$. By computer search (see Appendix) no such partial orientation exists. \square

To prove MIXEDQT is NP-hard, given an instance of $Y = (L, C)$ of Monotone

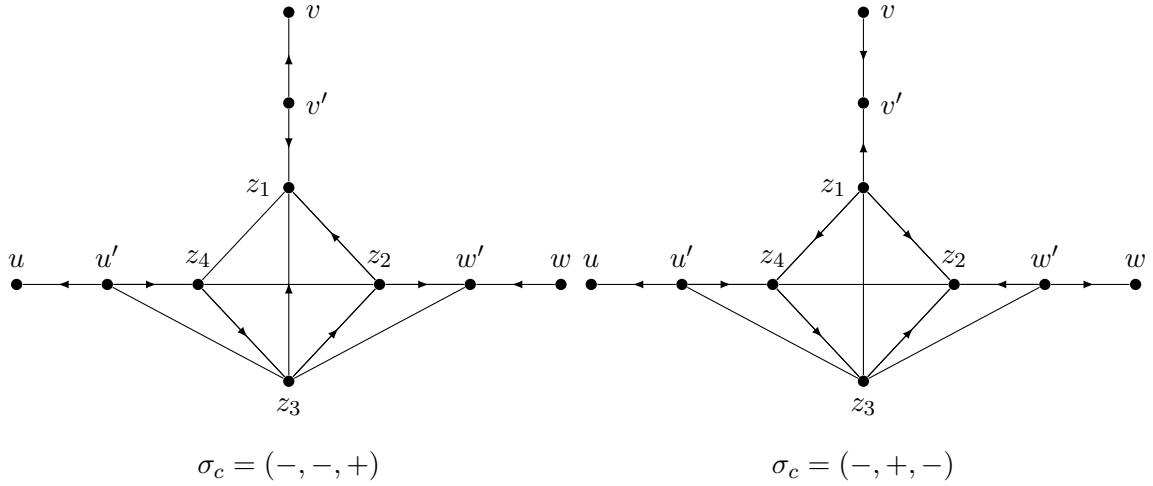


Figure 7: Partial orientations of a clause graph with signatures $\sigma_c = (-, -, +)$ and $\sigma_c = (-, +, -)$

NAE3SAT we construct a graph, Γ_Y , that admits a partial orientation as a quasi-transitive mixed graph if and only if Y is a YES instance of Monotone NAE3SAT.

We construct Γ_Y from the disjoint union of paths and copies of Σ_c by taking

- for each $c \in C$, a copy of Σ_c ; and
- for each $x \in L$, a path P_x with $2|C| + 2$ vertices: $x_0, x_1, \dots, x_{2|C|+2}$

and by identifying vertices as follows.

For every $1 \leq k \leq |C|$ with $c_k = u \vee v \vee w$, identify vertex u_{2k} (respectively v_{2k} and w_{2k}) in P_u (respectively P_v and P_w) with vertex u (respectively v and w) in Σ_{c_k} (see Figure 8). In other words, if $x \in L$ appears in clause $c_k \in C$, then vertex x in Σ_k is identified with vertex x_{2k} in P_x .

Notice that for every variable $x \in L$, no edge in P_x is contained in a copy of K_3 in Γ_Y . Therefore by Lemma 2.2, in every partial orientation of Γ_Y as a quasi-transitive mixed graph, every such edge is oriented as an arc. And so in every every partial orientation of Γ_Y as a quasi-transitive mixed graph, every path P_x is oriented as alternating path where every vertex of the form x_{2k} is a source (respectively, a sink) if and only if x_0 is a source (respectively, a sink).

Lemma 4.3. *Let $Y = (L, C)$ be an instance of Monotone NAE3SAT and consider $x \in L$. In every partial orientation of Γ_Y as a quasi-transitive mixed graph*

- *the path P_x is oriented as an alternating path; and*
- *for every $1 \leq k \leq |C|/2$, x_{2k} is a source if and only if x_0 is a source.*

Proof. Let $Y = (L, C)$ be an instance of Monotone NAE3SAT and let G be a partial orientation of Γ_Y as a quasi-transitive mixed graph. Consider $x \in L$. By construction, no vertex of P_x is contained in a copy of K_3 . And so by Lemma 2.2, every edge

of this path is oriented as an arc in G and every vertex of this path is either a source or a sink. Noting that there cannot be an arc with both ends at a source or sink vertex, it then follows that P_x is oriented as an alternating path in G . From this it follows directly that for every $1 \leq k \leq |C|/2$, x_{2k} is a source if and only if x_0 is a source. \square

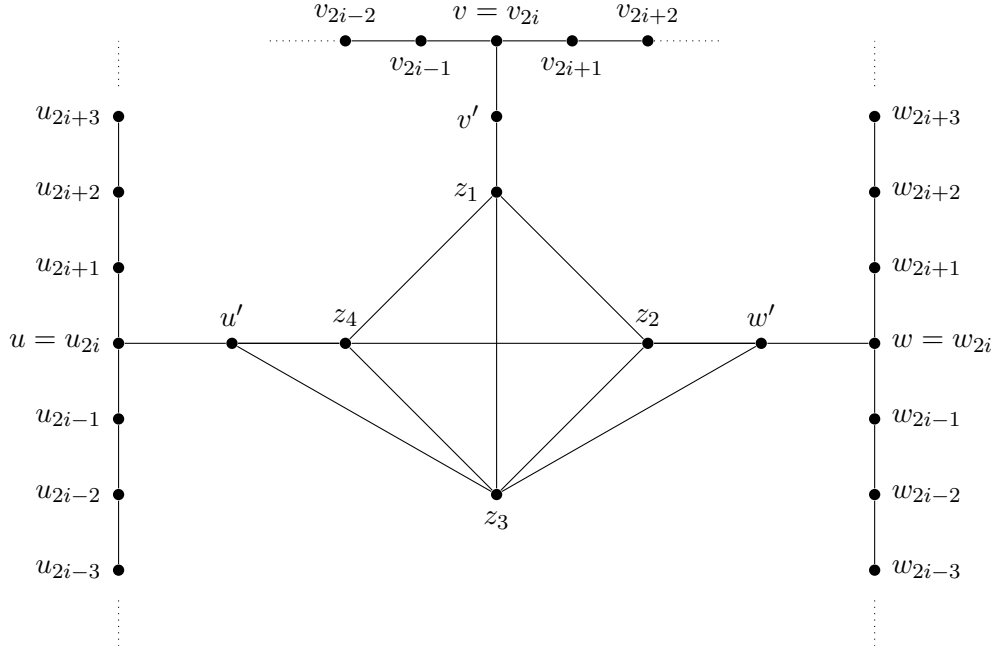


Figure 8: A clause subgraph in the construction of Γ_Y for the clause $c_i = u \vee v \vee w$

In our mapping between YES instances of Monotone NAE3SAT and partial orientations of Γ_Y as a quasi-transitive mixed graph, x_0 as a source (respectively, a sink) corresponds to x being TRUE (respectively, FALSE) in a satisfying assignment of the instance of Monotone NAE3SAT.

Lemma 4.4. *An instance $Y = (L, C)$ of Monotone NAE3SAT is a YES instance if and only if Γ_Y admits a partial orientation as a mixed quasi-transitive graph.*

Proof. Let $Y = (L, C)$ be a YES instance of Monotone NAE3SAT. Consider the following partial orientation of G of Γ_Y :

- for each $x \in L$, if x is true, then orient P_x as an alternating path where x_0 is a source;
- for each $x \in L$, if x is false, then orient P_x as an alternating path where x_0 is a sink; and
- for each clause $c = u \vee v \vee w$, partially orient Σ_c as a quasi-transitive mixed oriented graph whose signature matches the truth values of the literals u, v and w .

Note that by Lemmas 4.2 and 4.3 such a partial orientation necessarily exists. (See Figure 9 for the case $c_i = u \vee v \vee w$ where u , v and w are respectively FALSE, FALSE and TRUE.)

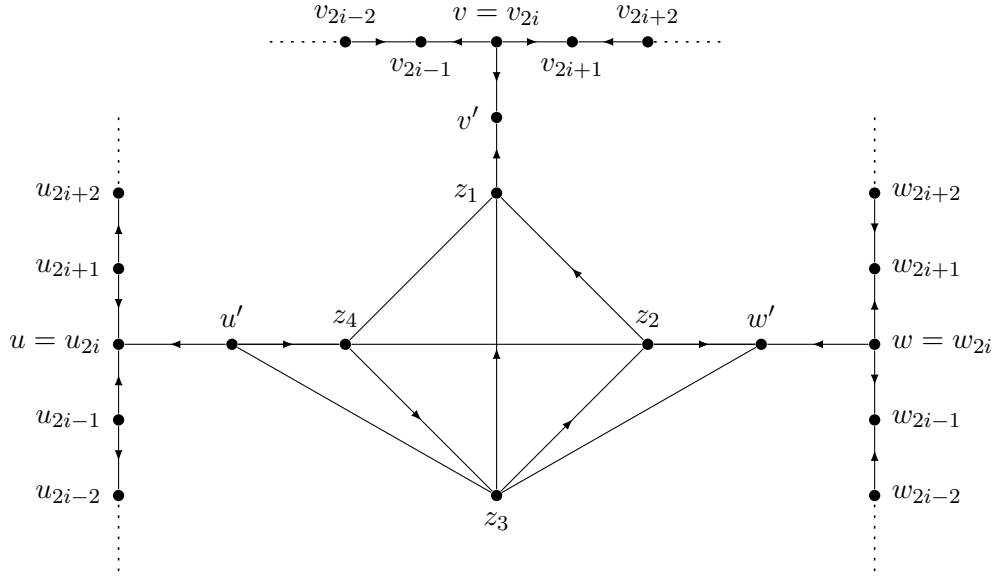


Figure 9: A clause subgraph in the partial orientation of G in the proof of Theorem 4.4 where u , v and w are respectively FALSE, FALSE and TRUE.

We claim G is a quasi-transitive mixed graph. We proceed by appealing to the definition of quasi-transitive mixed graph.

Let $y \in V(\Gamma)$. We prove y is not the centre vertex of any induced 2-dipath. We proceed in cases based on whether y is contained in a clause graph.

If there exists $c \in C$ such that $y \in V(\Sigma_c)$, then, without loss of generality, y is configured as one of the vertices in Figure 7. If $y \neq u, v, w$, then by observing the partial orientations in vertices in Figure 7, we see that y is not the centre vertex of an induced 2-dipath. If $y \in \{u, v, w\}$, then by construction of Γ_Y there exists $1 \leq i \leq |C|/2$ such that $y = u_{2i}$, $y = v_{2i}$ or $y = w_{2i}$. By the Vertex Cut Lemma with $I = \{u_{2i}, v_{2i}, w_{2i}\}$, y is a source or a sink. Therefore y is not the centre vertex of an induced 2-dipath in G .

If y is contained in no clause graph, then there exists $x \in L$ such that $y = x_0$ or $y = x_{2i+1}$ for $1 \leq i \leq |C|/2$. In either case, we note that by Lemma 4.3, y is a source or a sink in G . And so we conclude y is not the centre vertex of an induced 2-dipath in G .

Consider $e \in E(G)$. By construction, there exists $c \in C$ such that $e \in E(\Sigma_c)$. Without loss of generality, assume $\sigma_c \in \{(-, -, +), (-, +, -)\}$. Therefore Σ_c is partially oriented as one of the mixed graphs in Figure 7. We observe that in either case the end points of every edge are also the endpoints of a 2-dipath. Therefore G is a quasi-transitive mixed graph.

Let G be a partial orientation of Γ as a quasi-transitive mixed graph. We assign the truth value of the variable x based on whether $x_0 \in V(G)$ is a source or sink. Let $f : L \rightarrow \{TRUE, FALSE\}$ such that $f(x) = TRUE$ if and only if x_0 is a source in G . Notice that since x_0 has degree 1 in Γ , x_0 is necessarily a source or a sink in G . And so $f(x) = FALSE$ if and only if x_0 is a sink in G .

Notice that for any clause $c_i = u \vee v \vee w$, the set $I = \{u_{2i}, v_{2i}, w_{2i}\}$ satisfies the hypothesis of the Vertex Cut Lemma. Therefore for each $x \in L$ and each $1 \leq i \leq |C|/2$, x_{2i} is a source or a sink. Note that if x is not contained in clause c_i , then x_{2i} is a source or a sink by Lemma 4.3. By Lemma 4.3, x_{2i} is a source if and only if x_0 is a source. Therefore for every $c \in C$ with $c = u \vee v \vee w$, we have $\sigma_{c,u} = +$ if and only if u_0 (respectively v_0 and w_0) is a source. By Lemma 4.2, $\sigma_c \notin \{(-, -, -), (+, +, +)\}$. Therefore f is a satisfying assignment for Y . \square

Theorem 4.5. *The decision problem MIXEDQT is NP-complete. The problem is Polynomial when restricted to inputs of maximum degree 3, but remains NP-complete when restricted to inputs of maximum degree k for all $k \geq 5$.*

Proof. The reduction is from Monotone NAE-3SAT, noting that given an instance Y of Monotone NAE3SAT, Γ_Y can be constructed in polynomial time. By Lemma 4.4, Γ_Y admits a partial orientation as a quasi-transitive mixed graph if and only if Y is a YES instance. Therefore the problem is NP-complete.

For every instance Y , Γ_Y has maximum degree 5, thus problem remains NP-complete when restricted to inputs with maximum degree 5. Further, Γ_Y can be modified by adding arbitrarily many pendants to a vertex x_0 without changing the correspondence between YES instances and partial orientations as a quasi-transitive mixed graphs. Therefore the problem remains NP-complete when restricted to inputs with maximum degree k for any $k \geq 5$. By Theorem 3.6, the problem is Polynomial when restricted to inputs with maximum degree 3. \square

Corollary 4.6. *The decision problem UNDIRSQUARE is NP-complete. The problem is Polynomial when restricted to inputs of maximum degree 3, but remains NP-complete when restricted to inputs of maximum degree k for all $k \geq 5$.*

We notice here the gap in the analysis for the case $\Delta = 4$. It is possible that the analysis done in Section 3 can be undertaken for graphs with maximum degree 4. However in this case Π is not a forbidden subgraph, as demonstrated by the example in Figure 1.

Let \mathcal{F} be a family of oriented graphs. Consider the following decision problem.

\mathcal{F} -UNDIRSQUARE

Instance: A graph Γ .

Question Does Γ arise as an undirected square of an oriented graph $G \in \mathcal{F}$?

Using the proof of Theorem 4.5, we consider \mathcal{F}_k -UNDIRSQUARE where \mathcal{F}_k is the family of orientations of graphs with maximum degree k .

Theorem 4.7. *Let k be a fixed positive integer and let \mathcal{F}_k be the family of orientations of graphs with $\Delta = k$. For $k \geq 3$ the decision problem \mathcal{F}_k -UNDIRSQUARE is NP-complete.*

Proof. The reduction is from Monotone NAE3SAT, noting that given an instance Y of Monotone NAE3SAT, Γ_Y can be constructed in polynomial time.

In any YES instance Y of Monotone NAE-3SAT, there is a corresponding partial orientation of Γ_Y as a quasi-transitive mixed graph in which every vertex is incident with at most 3 arcs (see Figure 5). Therefore Y is a YES instance if and only if there exists an oriented graph \vec{H} with maximum degree 3 such that $\Gamma_Y = U(\vec{H}^2)$. And so it is NP-complete to decide whether a graph is an oriented graph square of an oriented graph with maximum degree 3.

For fixed $k > 3$, as in the proof of Theorem 4.5, we can modify the construction of Γ_Y by adding arbitrarily many pendants to a vertex x_0 without changing the correspondence between YES instances and partial orientations as a quasi-transitive mixed graphs. In these partial orientations every edge incident with x_0 is oriented as an arc. Therefore Y is a YES instance if and only if there exists an oriented graph \vec{H} with maximum degree k such that $\Gamma_Y = U(\vec{H}^2)$. And so it is NP-complete to decide whether a graph is an oriented graph square of an oriented graph with maximum degree k for any fixed $k \geq 3$. \square

5 Discussion and Future Work

As with other extensions of graph theoretic concepts to the study of oriented graphs, the landscape of open areas for investigation on this topic is vast. For example, one may consider graphs arising from orientations of trees. As with the case for graphs, we expect there to be a classification that yields a polynomial time algorithm for identifying graphs that arise as undirected squares of orientations of trees.

Recall that one may extend the definition of graph colouring to oriented graphs by way of oriented graph homomorphism (see [21]). In this context, an oriented k -colouring of an oriented graph \vec{G} is a homomorphism to a tournament on k vertices. The oriented chromatic number, denoted $\chi_o(\vec{G})$, is the least integer k such that \vec{G} admits an oriented k -colouring. In every oriented colouring vertices at directed distance 2 necessarily are assigned different colours. Hence $\chi(U(\vec{G}^2)) \leq \chi_o(\vec{G})$. And so the oriented chromatic number of the underlying oriented graph gives an upper bound for the chromatic number of undirected square.

For many fundamental families of oriented graph, the problem of computing the oriented chromatic number remains open. For example, the best known upper bound on the chromatic number of the family of orientations of planar graphs is 80 [18]. This bound is not expected to be tight; there are no known constructions for oriented planar graphs for which the chromatic number exceeds 16 [5]. The example of an orientation of a planar graph \vec{G} with oriented chromatic number 16 has 16 vertices

and thus satisfies $U(\vec{G}^2) = K_{16}$. Hence $\chi(U(\vec{G}^2)) = 16 = \chi_o(\vec{G})$.

Oriented cliques (i.e., oriented graphs for which $\chi_o(G) = |V(G)|$) present an example of pairs of non-isomorphic oriented graphs that have the same undirected square. For example, K_5 is an oriented graph square of both the directed 5-cycle and the transitive tournament on five vertices. And so K_5 does not have a unique square root with respect to oriented graphs. Graphs with a unique square root (with respect to graphs) are known to exist. For example, tree squares are unique and, more generally, so are squares of graphs with girth at least 6 [1, 19]. The former of these results does not extend to graphs arising as undirected squares of orientations of trees. The directed path on 3 vertices and an orientation of $K_{1,3}$ in which the vertex of degree 3 is not a source nor a sink generates the same undirected square.

Appendix

This code checks that Σ has no partial orientation as a quasi-transitive graph with $\sigma_c = (+, +, +)$ or $\sigma = (-, -, -)$. We proceed by generating all partial orientations as a quasi-transitive graph via brute force search of all mixed graphs that can be formed from Σ .

We generate mixed graphs as follows: for every subset of the edge set of Σ we assign these edges to remain edges in the mixed graphs and then iterate over all orientations of the remaining edges, testing for each orientation if the constructed mixed graph is quasi-transitive. Those that are valid are stored. We then iterate over all of the valid partial orientations and check that none has $\sigma_c = (+, +, +)$ or $\sigma_c = (-, -, -)$. The expected output is 0,0.

```

sigma = graphs.PathGraph(6)
D = Graph(9)
G = Graph(9)

dict = [] # This list stores all possible partial orientations of
          \Sigma as a mQT
ttt = [] # This list stores all mQTs with \sigma_c = (+,+,+)
fff = [] # This list stores all mQTs with \sigma_c = (-,-,-)

sigma.add_edges([(2, 7), (7, 3), (6, 7), (1, 6), (6, 4), (2, 6),
                 (3, 6), (7, 8)])

# Generate all subsets of edges of \Sigma
edgeSet = sigma.edges()
subgraphs = [[]]
for e in edgeSet:
    for i in range(len(subgraphs)):
        subgraphs += [subgraphs[i] + [e]]

# Iterate over all choices of edges
for subEdge in subgraphs:
    D = Graph(9)
    G = Graph(9)
    for e in sigma.edges():
        G.add_edge(e)
    D.add_edges(subEdge)
    G.delete_edges(subEdge)

# Iterate over all orientations of the edges that were not chosen
allOrientations = D.orientations()
for J in allOrientations:
    isValid = isMQT(G, J)
    if isValid == True:
        dict.append((G, J))

# At this stage, dict holds all mQTs that can be formed from \Sigma.
# We now count the number that have \sigma_c = (+,+,+) or \sigma_c =
# (-,-,-).

for i in range(len(dict)):
    D = dict[i]
    edges = D[1].edges()
    if (0, 1, None) in edges and (7, 8, None) in edges and (5, 4, None)
        in edges:
        ttt.append(D[1])
    elif (1, 0, None) in edges and (8, 7, None) in edges and (4, 5,
        None) in edges:
        fff.append(D[1])

print(len(ttt), len(fff))

```

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